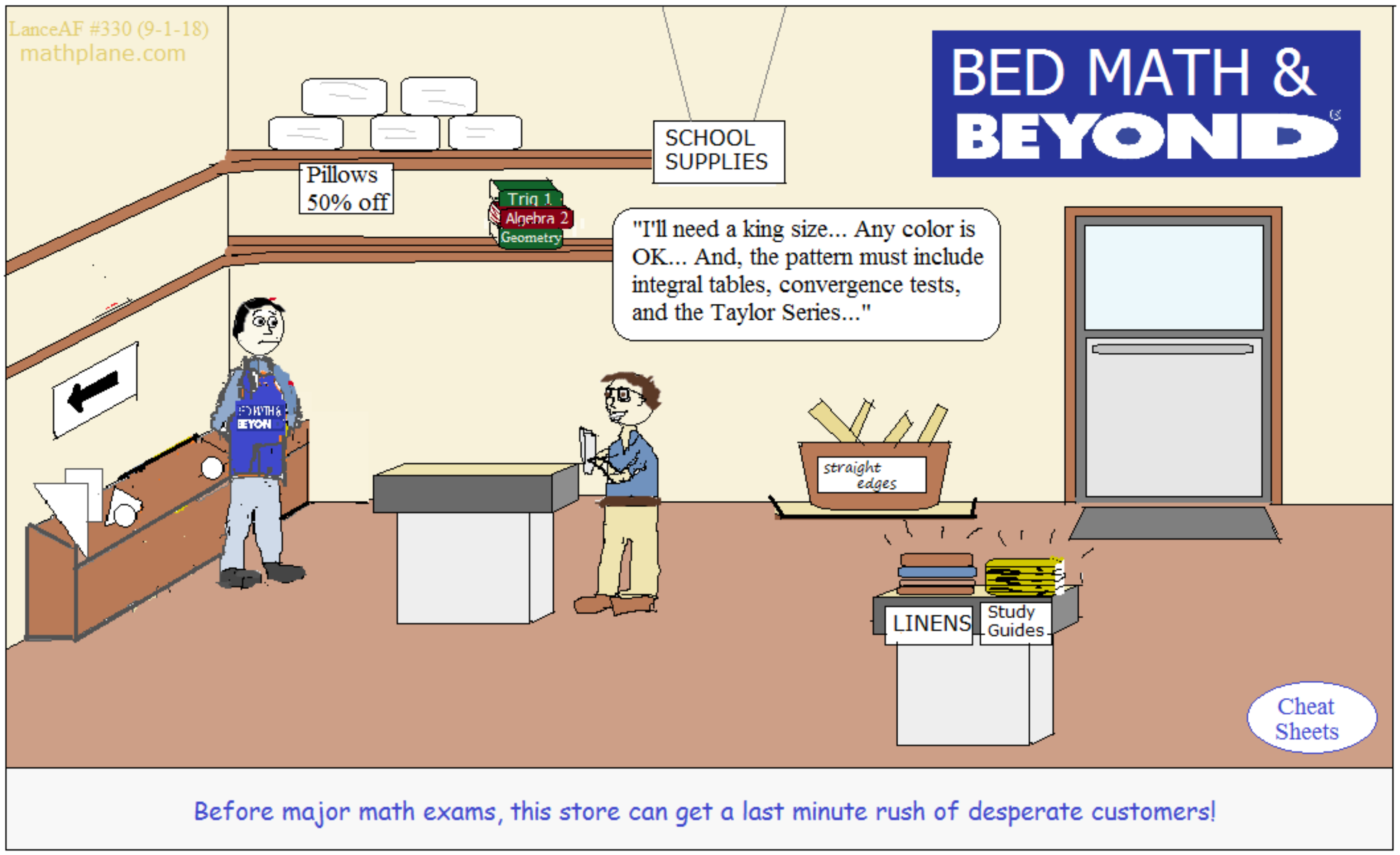
Calculus: Series Convergence and Divergence

Notes, Examples, and Practice Questions (with Solutions)



Topics include geometric, power, and p-series, ratio and root tests, sigma notation, taylor and maclaurin series, and more.

Mathplane.com

Geometric Series $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$

a = initial value
r = common ratio (growth factor)
("exponent increases; base is constant")

TEST: $|r| \geq 1$ diverges

$|r| < 1$ converges

If the series converges, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Examples: $\sum_{n=0}^{\infty} 8(\frac{1}{2})^n = 8 + 4 + 2 + \ldots$ converges

Since $\frac{1}{2} < 1$, it converges $\frac{a}{1-r} = \frac{8}{(1-1/2)} = 16$

$\sum_{n=0}^{\infty} .7(3)^n = .7 + 2.1 + 6.3 + \ldots$ diverges

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p-Series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} +$ ("exponent is constant; fraction is increasing")

TEST: $p \leq 1$ diverges

$p > 1$ converges and,

$$\frac{1}{p-1} < \sum_{n=1}^{\infty} \frac{1}{n^p} < 1 + \frac{1}{p-1}$$

Examples: $\sum_{n=1}^{\infty} \frac{5}{n^3} = 5 + \frac{5}{8} + \frac{5}{27} + \frac{5}{81} + \ldots$ converges

Since p = 3, it converges

$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3 + 2.12 + 1.73 + 1.5 + \ldots$ diverges

Since p = 1/2, it diverges

(note: the sequence is converging to 0, but the series is diverging...)

Harmonic Series ("a special p-series") $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$

Since p = 1, it diverges

Power Series (centered at a) $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ where the domain of f is the set of all x for which the power series converges.

c are the 'coefficients' of each term (constants)
a is a constant
x is a variable

TEST: $|x-a| < R$ converges

$|x-a| > R$ diverges

$|x-a| = R$ inconclusive

Example: $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+6)^n$ What is the interval of convergence?

Using the ratio test,

$$L = \lim_{n\to\infty} \frac{\frac{n+1}{4^{n+1}}(x+6)^{n+1}}{\frac{n}{4^n}(x+6)^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)(x+6)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n\ (x+6)^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)(x+6)}{4 \cdot n}$$

$$= |x+6| \lim_{n\to\infty} \frac{(n+1)}{4n}$$

$$L = |x+6|\frac{1}{4}$$

If $\frac{1}{4}|x+6| < 1$ converges

$|x+6| < 4$, then series converges

If $\frac{1}{4}|x+6| > 1$ diverges

$|x+6| > 4$, then series diverges

So, the radius of convergence R = 4

and, the interval of convergence is $-10 < x < -2$

TEST: Sequence Test If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty}$ DIVERGES

Example: Sequence 3, 6, 9, 12, ... is geometric

$a_n = 3(2)^{k-1}$ $a = 3$
$r = 2$ and, since $r > 2$, it diverges..

Therefore, the series $3 + 6 + 9 + 12 + \ldots$ is diverging...

TEST: Sequence Test If $\sum_{n=0}^{\infty}$ converges, then $\lim_{n\to\infty} a_n = 0$

Example: $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \ldots \ = 2$

and, $\lim_{n\to\infty} \frac{1}{2^n} = 0$

NOTE: Converse isn't true... i.e. if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty}$ converges OR diverges..

Example: Harmonic series...

$\sum_{n=1}^{\infty} \frac{1}{n}$ p = 1, so diverges Series P $= 1 + \frac{1}{2} + \frac{1}{3} + \ldots$

However, sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is obviously going to 0 converges

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TEST: Integral Test

If $\sum_{n=1}^{\infty} a_n$ converges, then $\int_1^{\infty} f(x)\,dx$ converges

Example: $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$ $\lim_{n\to\infty} \frac{2n}{n^2+1} = 0$ so, the series may converge OR diverge!

Using the integral test: $\int_1^{\infty} \frac{2x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{2x}{x^2+1}\,dx$

$= \lim_{b\to\infty} \ln(x^2+1)\Big|_1^b = \infty - \ln(2)$ DIVERGES

TEST: Comparison Test

Series convergence or divergence?

If $\sum_{n=1}^{\infty} U_n$ converges, and $a_n \leq U_n$ then a_n converges

Example: $\sum_{n=1}^{\infty} \frac{1}{2 + n^3}$

Since the integral test is difficult, we can try the comparison test. We'll choose the p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ because it is similar AND the terms will be greater than the terms in the main series

In this p-series, p > 3, so it converges...

$$\sum_{n=1}^{\infty} \frac{1}{2 + n^3} < \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{CONVERGES}$$

If $\sum_{n=1}^{\infty} U_n$ diverges, and $a_n \geq U_n$ then a_n diverges

Example: $\sum_{n=1}^{\infty} \frac{2}{\sqrt{5n-1}}$

If we use the comparison test, we can choose $\sum_{n=1}^{\infty} \frac{2}{\sqrt{5n}}$

$\frac{2}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is p-series where p = 1/2 so, it diverges

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{5n-1}} > \sum_{n=1}^{\infty} \frac{2}{\sqrt{5n}} \quad \text{DIVERGES}$$

(Note: the integral test could verify that this series diverges)

TEST: Limit Comparison Test

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite value (and non-zero), then

$\sum_{n=1}^{\infty} a_n$ AND $\sum_{n=1}^{\infty} b_n$ are either BOTH converging or BOTH diverging

Example: $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

$\frac{1}{2n+1} < \frac{1}{n}$ for all positive n

$\sum_{n=1}^{\infty} \frac{1}{2n+1} < \sum_{n=1}^{\infty} \frac{1}{n}$ We know the harmonic series diverges, so the comparison test doesn't help...

Comparison test is inconclusive... However, the Limit Comparison test succeeds!

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

There is a finite value, 1/2, and since $\frac{1}{n}$ is diverging, then

$\frac{1}{2n+1}$ must be diverging

Series convergence or divergence?

TEST: Ratio Test

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ (if it exists), then $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$

diverges if $L > 1$

is inconclusive if $L = 1$

Examples:

$$\sum_{n=1}^{\infty} \frac{7^n}{(-3)^{n+1}\cdot n} = \frac{7}{9} + \frac{49}{-54} + \frac{343}{243} + \ldots$$

$$\lim_{n\to\infty} \frac{\frac{7^{n+1}}{(-3)^{n+2}\cdot(n+1)}}{\frac{7^n}{(-3)^{n+1}\cdot n}} = \lim_{n\to\infty} \frac{7^{n+1}}{(-3)^{n+2}\cdot(n+1)} \cdot \frac{(-3)^{n+1}\cdot n}{7^n} = \lim_{n\to\infty} \frac{7^1\cdot n}{(-3)^1\cdot(n+1)} =$$

$$\left|\frac{7}{-3}\right| \lim_{n\to\infty} \frac{n}{n+1} = \frac{7}{3}$$

Since the limit L of the sequence > 1, the series DIVERGES

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} = 3 + \frac{9}{2} + \frac{27}{6} + \ldots$$

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3^1}{n+1} = 0$$

Since the limit L of the sequence < 1, the series CONVERGES

TEST: Nth Root Test

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$

diverges if $L > 1$

is inconclusive if $L = 1$

Examples:

$$\sum_{n=0}^{\infty} \frac{2^n 3^{2n}}{10^n}$$

Using the nth root test, $\lim_{n\to\infty} \sqrt[n]{\frac{2^n 3^{2n}}{10^n}} = \lim_{n\to\infty} \frac{2^1 3^2}{10^1} = \frac{18}{10}$

Since L = 9/5 > 1

the series DIVERGES

$$\sum_{n=0}^{\infty} \frac{2^n n^3}{5^n}$$

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{2^n n^3}{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2^n}\ \sqrt[n]{n^3}}{\sqrt[n]{5^n}} = \lim_{n\to\infty} \frac{2\cdot n^{\frac{3}{n}}}{5} = \frac{2\cdot 1}{5} = \frac{2}{5}$$

Since L = 2/5 < 1

the series CONVERGES

Series convergence or divergence?

TEST: Alternating Series Test   An alternating series converges if $\lim_{n\to\infty} a_n = 0$

AND $0 < a_{n+1} < a_n$ for all $n \geq 1$

Examples:

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{\ln(2n)} = \frac{-1}{\ln(2)} + \frac{2}{\ln(4)} + \frac{-3}{\ln(6)} + \ldots$$

Using L'Hopital's Rule $$\lim_{n\to\infty} \frac{n}{\ln(2n)} = \lim_{n\to\infty} \frac{1}{\frac{2}{2n}} = \lim_{n\to\infty} n = \infty$$

Since the limit $\neq 0$, the series DIVERGES

$$\sum_{n=1}^{\infty} \frac{n}{(-3)^{n-1}} = \frac{1}{1} + \frac{2}{-3} + \frac{3}{9} + \frac{4}{-27} + \ldots.$$

$$\lim_{n\to\infty} \frac{n}{(3)^{n-1}} = \frac{\infty}{\infty}$$ inconclusive, so we'll use L'Hopital's Rule

$$= \lim_{n\to\infty} \frac{1}{3^{n-1}\,(\ln 3)} = 0$$ So, the sequence converges and the series MAY converge....

check $0 < a_{n+1} < a_n$

$$0 < \frac{n+1}{3^n} < \frac{n}{(3)^{n-1}}$$ "cross-multiply"

this is satisfied if $(n+1)(3)^{n-1} < n3^n$ "divide by (n + 1)"

$$(3)^{n-1} < \frac{n3^n}{(n+1)}$$ "divide by 3^n "

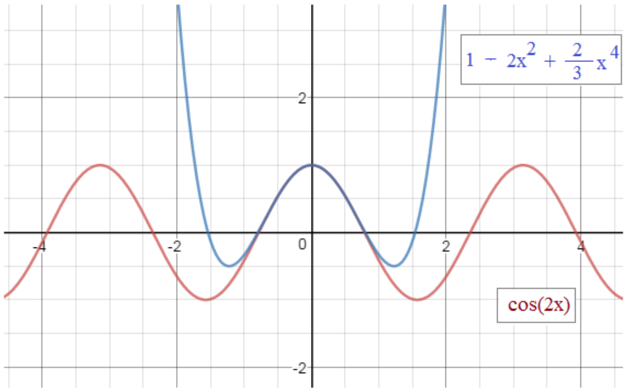
$$\frac{1}{3} < \frac{n}{(n+1)}$$ Since this is satisfied for $n \geq 1$, the series CONVERGES

Example: Find the Taylor (polynomial) series of the 4th order for the function $f(x) = \cos(2x)$

$f(x) = \cos(2x)$	$f(0) = 1$
$f'(x) = -2\sin(2x)$	$f'(0) = 0$
$f''(x) = -4\cos(2x)$	$f''(0) = -4$
$f'''(x) = 8\sin(2x)$	$f'''(0) = 0$
$f^{(4)}(x) = 16\cos(2x)$	$f^{(4)}(0) = 16$

TAYLOR SERIES

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^{n}$$



$$f(x) = \sum_{n=0}^{4} \frac{f^{n}(0)}{n!}(x)^{n}$$ is the series of the 4th order...

$$= \frac{1}{0!}(x)^0 + \frac{0}{1!}(x)^1 + \frac{-4}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{16}{4!}(x)^4$$

$$= 1 - 2x^2 + \frac{2}{3}x^4$$

Note the similarity of the graphs!

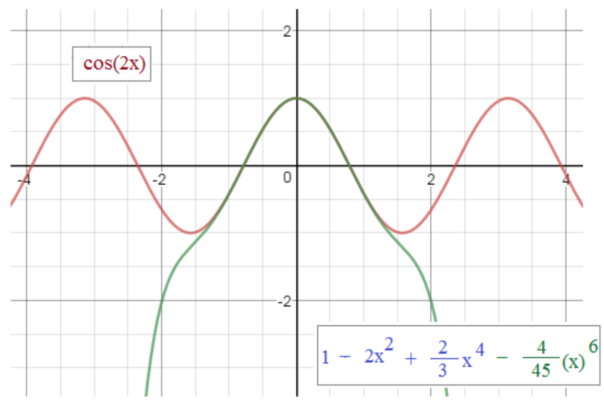
$f^{(5)}(x) = -32\sin(2x)$ $\quad f^{(5)}(0) = 0$

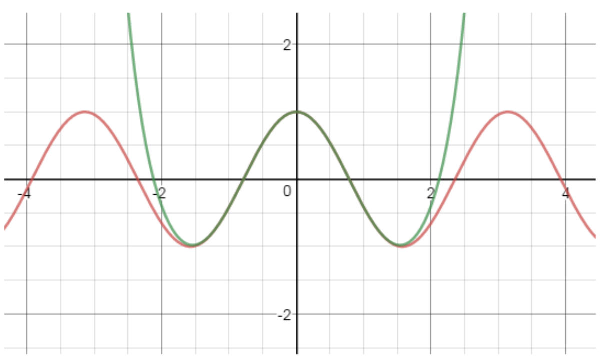
$f^{(6)}(x) = -64\cos(2x)$ $\quad f^{(6)}(0) = -64$

$$f(x) = \sum_{n=0}^{6} \frac{f^{n}(0)}{n!}(x)^{n}$$ is the series of the 6th order...

$$= \frac{1}{0!}(x)^0 + \frac{0}{1!}(x)^1 + \frac{-4}{2!}(x)^2 + \frac{0}{3!}(x)^3 + \frac{16}{4!}(x)^4 + \frac{0}{5!}(x)^5 + \frac{-64}{6!}(x)^6$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}(x)^6$$





$$f(x) = \sum_{n=0}^{8} \frac{f^{n}(0)}{n!}(x)^{n}$$ is the series of the 8th order...

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}(x)^6 + \frac{256}{8!}(x)^8$$

NOTE: This is a MacLaurin Series, a special version of the Taylor Series. It occurs when $a = 0$

"A Taylor series about $x = 0$" is a MacLaurin series for $f(x)$

Example: Find the 1st 5 non-zero terms in the Taylor Series generated by $f(x) = \sqrt{x+1}$ at $x = 0$

TAYLOR SERIES

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^{n}$$

$$f(x) = (x+1)^{\frac{1}{2}} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{4}(x+1)^{-\frac{3}{2}} \qquad f''(0) = \frac{-1}{4}$$

$$f'''(x) = \frac{3}{8}(x+1)^{-\frac{5}{2}} \qquad f'''(0) = \frac{3}{8}$$

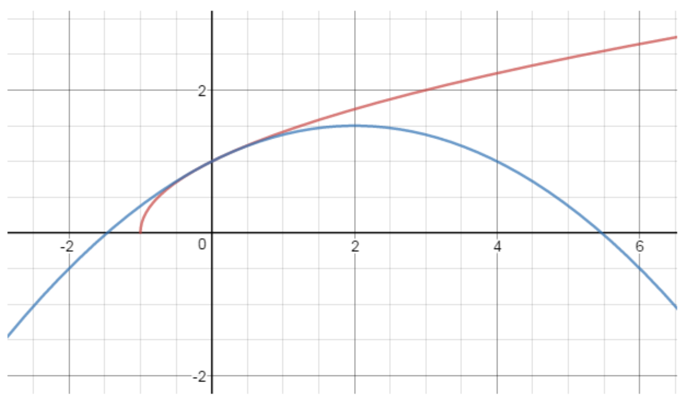
$$f^{(4)}(x) = \frac{-15}{16}(x+1)^{-\frac{7}{2}} \qquad f^{(4)}(0) = \frac{-15}{16}$$

Applying the formula.....

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!}(x)^{n} = \frac{1}{0!}(x-0)^{0} + \frac{\frac{1}{2}}{1!}(x-0)^{1} + \frac{\frac{-1}{4}}{2!}(x-0)^{2} + \frac{\frac{3}{8}}{3!}(x-0)^{3} + \frac{\frac{-15}{16}}{4!}(x-0)^{4} + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^{2} + \frac{1}{16}x^{3} - \frac{5}{128}x^{4}$$

First 5 terms...

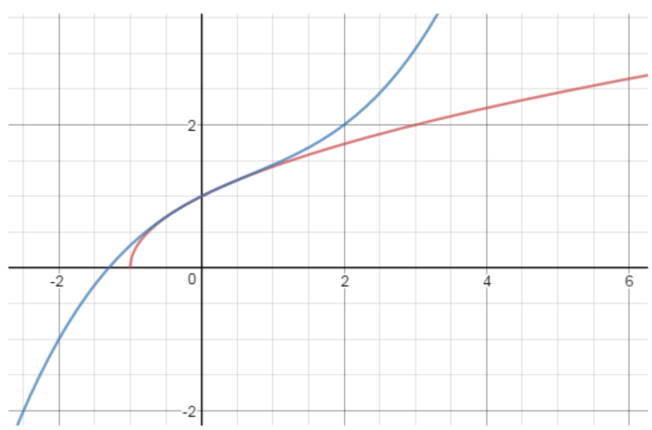


$\sqrt{x+1}$

$1 + \frac{1}{2}x - \frac{1}{8}x^{2}$

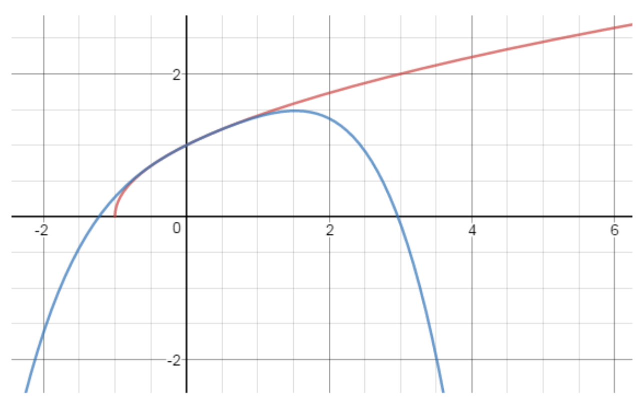
$\sqrt{x+1}$

$1 + \frac{1}{2}x - \frac{1}{8}x^{2} + \frac{1}{16}x^{3}$



$\sqrt{x+1}$

$1 + \frac{1}{2}x - \frac{1}{8}x^{2} + \frac{1}{16}x^{3} - \frac{5}{128}x^{4}$



Applications of Taylor Polynomials and Series

Example: $\int \frac{e^x - 1}{x}\, dx$

$\int \frac{1}{x} e^x \, dx - \int \frac{1}{x} \, dx$ (split the fraction into two parts)

We know the Taylor Polynomial:

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots \Rightarrow \frac{1}{x} e^x = \frac{1}{x} + 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \ldots$

$\int \frac{1}{x} e^x \, dx = \ln|x| + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \ldots \qquad \int \frac{1}{x} \, dx = \ln|x|$

$\int \frac{1}{x} e^x \, dx - \int \frac{1}{x} \, dx = x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \ldots$

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n!)}$$

(Note: the ln|x| cancelled out..)

Example: $\int_0^1 \sin(x^4)\, dx$ Evaluate the integral (to an accuracy of 5 decimal places)

Written as a trig function, this is a difficult equation to integrate..
However, if converted to a Taylor Polynomial, it's more manageable!

$\text{Sin}(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\text{Sin}(x^4) = \frac{x^4}{1!} - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \ldots$

$\int_0^1 \text{Sin}(x^4)\, dx = \frac{x^5}{5} - \frac{x^{13}}{78} + \frac{x^{21}}{2520} - \frac{x^{29}}{7! \cdot 29}$

.2 .0128 .0003968 .00000684 $\Rightarrow$ .18756947

within 5 decimal places...

The longer the polynomial, the closer we get to the true value..
(i.e. the remainder gets smaller and smaller...)

('true value': .18756954) ✓

Example: Find $\frac{1}{\sqrt[10]{e}}$ to five decimal places

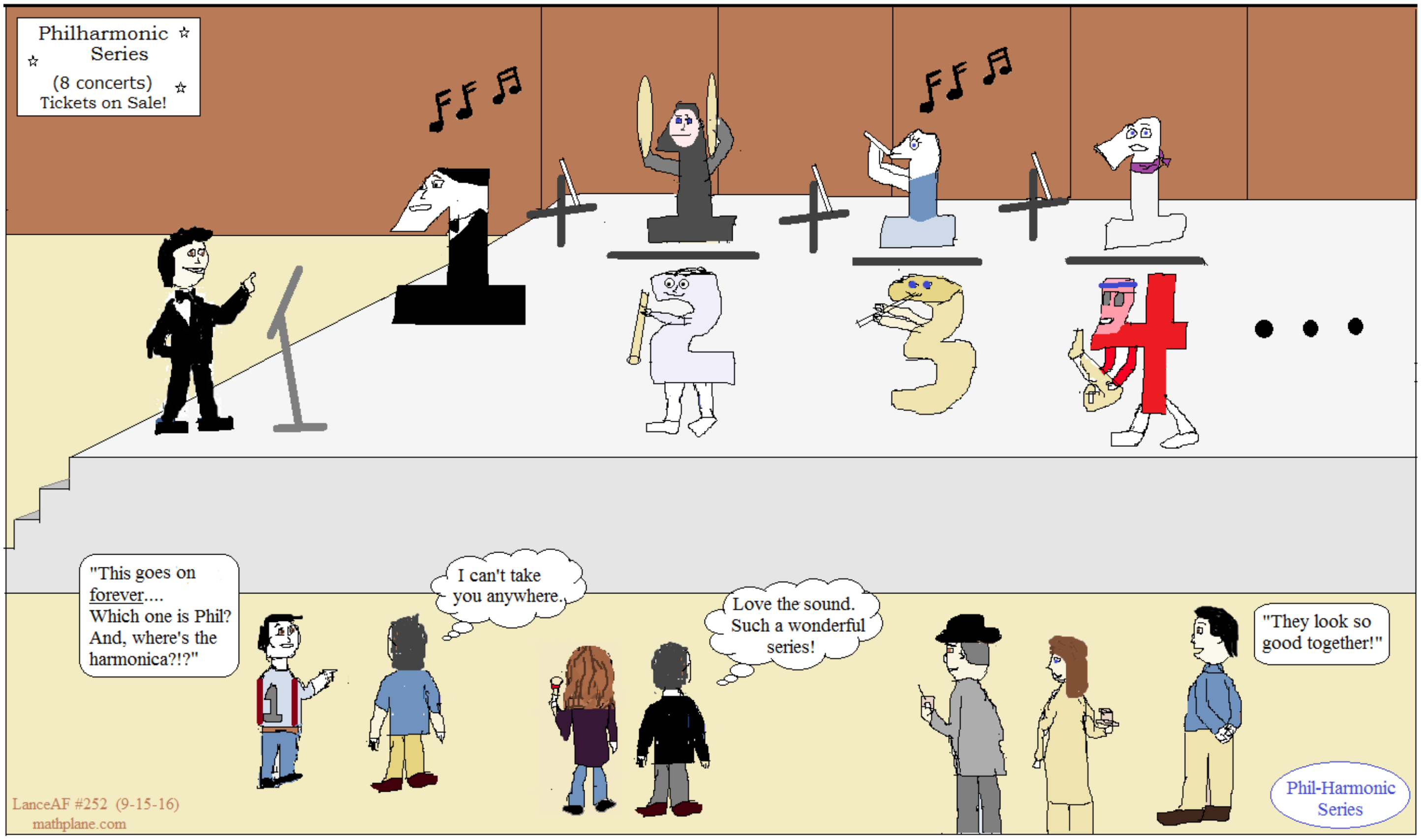
First, we'll rewrite the expression $\Rightarrow e^{-.1}$

Then, apply the Taylor Polynomial and substitute -.1 for x...

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

$e^{-.1} = 1 + (-.1) + \frac{(-.1)^2}{2} + \frac{(-.1)^3}{6} + \frac{(-.1)^4}{24} = .9048375$

('true value': .90483742) ✓



Practice Exercises -→

Determine if the following series converge or diverge
(using a suggested method listed at the right)

Series Convergence and Divergence

Suggested tests:

a) p-series

b) geometric series

c) comparison

d) nth root

e) integral

f) telescoping

g) alternate series

h) ratio

1) $$\sum_{n=1}^{\infty} \frac{1}{4^{n+1}}$$

2) $$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$$

3) $$\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

4) $$\sum_{n=1}^{\infty} \frac{n}{3^{n}}$$

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Series Convergence and Divergence

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d) nth root

e) integral

f) telescoping

g) alternate series

h) ratio

5) $\sum_{n=0}^{\infty} 8\left(\frac{-2}{5}\right)^n$

6) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

7) $\sum_{n=1}^{\infty} \frac{(n+1)!}{8^n}$

8) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

Determine if the following series converge or diverge.

Series Convergence and Divergence

9) $\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$

10) $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \ldots$

11) $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} + \ldots$

12) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

1) Find the MacLaurin Series of the 5th order for the function $f(x) = \sin(2x)$

2) Find the polynomial of order 4 at 0 for $f(x) = e^{-x}$
Use this to approximate $e^{(.5)}$

3) What is the coefficient of $(x-2)^3$ in the Taylor Series generated by ln(x) @ x = 2

4) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ Does the series converge or diverge?

5) $1 + \frac{1}{\sqrt[5]{2}^{\,2}} + \frac{1}{\sqrt[5]{3}^{\,2}} + \frac{1}{\sqrt[5]{4}^{\,2}} + \ldots$ Does the series converge or diverge?

6) $\sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \frac{1}{7^n} =$

7) Find the intervals of convergence.

a) $$\sum_{n=0}^{\infty} 3\left(\frac{x}{4}\right)^n$$

b) $$\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$$

8) $$\sum_{n=1}^{\infty} \frac{2}{n^2+n} =$$

9) Find the sum....

$$1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots + (-1)^n \frac{1}{4^n} + \ldots$$

Determine if the following series converge or diverge
(using a suggested method listed at the right)

SOLUTIONS

Series Convergence and Divergence

Suggested tests:

a) p-series
b) geometric series
c) comparison
d) nth root
e) integral
f) telescoping
g) alternate series
h) ratio

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^{n+1}}$

Use comparison test....

We know $\frac{1}{4^n}$ is always greater than $\frac{1}{4^{n+1}}$

$\lim_{n\to\infty} \frac{1}{4^n} = 0$ so, sequence converges...

then, we know $\frac{1}{4^n} = \left(\frac{1}{4}\right)^n$ is a geometric series..
since 1/4 < 1, it converges....

since this converges, the series $\frac{1}{4^{n+1}}$ converges!

$$\sum_{n=1}^{\infty} \frac{1}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{1}{4^n} \Longrightarrow \frac{1}{16} + \frac{1}{64} + \ldots \quad \frac{\frac{1}{16}}{1 - \frac{1}{4}} = \boxed{\frac{1}{12}}$$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

Use the p-series test...

$$\frac{n^{\frac{1}{3}}}{n} = \frac{1}{n^{\frac{2}{3}}}$$

since $p = \frac{2}{3} < 1$

it diverges

3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$

Use telescoping...

n = 1	n = 2	n = 3	n = 4	n = 5
$\frac{1}{2} - \frac{1}{4}$ +	$\frac{1}{3} - \frac{1}{5}$ +	$\frac{1}{4} - \frac{1}{6}$ +	$\frac{1}{5} - \frac{1}{7}$ +	$\frac{1}{6} - \frac{1}{8}$ . . .

$$\boxed{\frac{1}{2}} - \cancel{\frac{1}{4}} + \boxed{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{5}} - \frac{1}{7} + \cancel{\frac{1}{6}} - \frac{1}{8} \ldots$$

and, all remaining cancel each other out...

$$\frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

By noting the pattern, we can see this series converges...

4) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

Use the nth root test...

$$\lim_{n\to\infty} \sqrt[n]{\frac{n}{3^n}} = \lim_{n\to\infty} \left(\frac{n}{3^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{3} = \frac{1}{3}$$

since the limit $L = \frac{1}{3} < 1$, the series converges..

Determine if the following series converge or diverge
(using a suggested method listed at the right)

SOLUTIONS

Series Convergence and Divergence

Suggested tests:

a) p-series
b) geometric series
c) comparison
d) nth root
e) integral
f) telescoping
g) alternate series
h) ratio

5) $\sum_{n=0}^{\infty} 8\left(\frac{-2}{5}\right)^n$

Using the geometric series...

since the $|r| = \frac{2}{5}$ which is < 1,

the series **converges..**

$\frac{8}{1-(-2/5)} = \boxed{\frac{40}{7}}$

$8 - (16/5) + 32/25 - (64/125) + 144/625 \ldots$

6) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

Using the integral test...

$\lim_{n\to\infty} \frac{n}{(n^2+1)^2} = 0$

so, the series can converge OR diverge...
to find out, we'll use the integral test...

$\lim_{b\to\infty} \int_1^b \frac{x}{(x^2+1)^2}\,dx$

$\lim_{b\to\infty} \frac{1}{2}\int_1^b 2x\,(x^2+1)^{-2}\,dx$

$\lim_{b\to\infty} \frac{1}{2}\frac{(x^2+1)^{-1}}{-1} = \lim_{b\to\infty} \frac{-1}{2(x^2+1)} = 0$

Since the improper integral goes to 0, this series **converges...**

7) $\sum_{n=1}^{\infty} \frac{(n+1)!}{8^n}$

Using the ratio test...

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+2)!}{8^{n+1}}}{\frac{(n+1)!}{8^n}} = \lim_{n\to\infty} \frac{(n+2)!}{8^{n+1}} \cdot \frac{8^n}{(n+1)!}$

$= \lim_{n\to\infty} \frac{n+2}{8} = \infty$

Since the limit > 1, this series **diverges...**

8) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Is $0 < a_{n+1} < a_n$?

$0 < \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$

this is true for all $n \geq 1$

Series does **converge...**

Determine if the following series converge or diverge.

Series Convergence and Divergence

SOLUTIONS

9) $\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$ Using the limit comparison test:

Use $\frac{1}{n^3}$ (a p-series that converges)

$$\frac{\frac{2n^2 - 1}{3n^5 + 2n + 1}}{\frac{1}{n^3}} = \frac{2n^5 - n^3}{3n^5 + 2n + 1} \qquad \lim_{n \to \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} = \frac{2}{3}$$

Since limit exists, the two sequences either
BOTH diverge OR
BOTH converge...

Since $\frac{1}{n^3}$ converges, this series converges...

10) $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \ldots$

This series is $\sum_{n=1}^{\infty} \frac{1}{(n)200} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{200} \Rightarrow \frac{1}{200} \sum_{n=1}^{\infty} \frac{1}{n}$

harmonic series (i.e. p-series where p = 1)
therefore, it diverges...

11) $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} + \ldots$

This series is $\sum_{n=1}^{\infty} \frac{1}{(200 + n^2)}$

Using Comparison Test: $\sum_{n=1}^{k} \frac{1}{(200 + n^2)} < \sum_{n=1}^{k} \frac{1}{n^2}$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series), we know the other series must converge as well

12) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ Can verify using two tests...

Integral Test

$$\int \frac{1}{n^{3/2}} = -2n^{-1/2}$$

$$= \frac{-2}{\sqrt{n}} \Big|_1^{\infty}$$

$$= 0 + 2$$

series converges...

p-test

$\frac{1}{n^{3/2}}$ p = 3/2

since 3/2 > 1

series converges...

the sequence does converge...

$\frac{1}{1}, \frac{1}{2\sqrt{2}}, \frac{1}{3\sqrt{3}}, \Rightarrow$ decreasing ✓

SOLUTIONS

1) Find the MacLaurin Series of the 5th order for the function $f(x) = \sin(2x)$

$f(x) = \sin(2x)$ $\quad f(0) = 0$

$f'(x) = 2\cos(2x)$ $\quad f'(0) = 2$

$f''(x) = -4\sin(2x)$ $\quad f''(0) = 0$

$f'''(x) = -8\cos(2x)$ $\quad f'''(0) = -8$

$f^{(4)}(x) = 16\sin(2x)$ $\quad f^{(4)}(0) = 0$

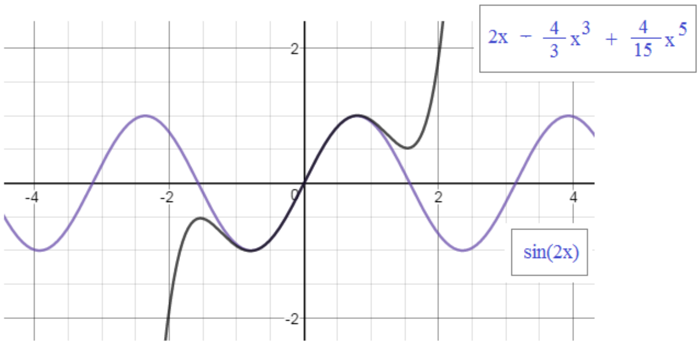
$f^{(5)}(x) = 32\cos(2x)$ $\quad f^{(5)}(0) = 32$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n$$

Since a Maclaurin series is around x = 0, we'll let a = 0

$$f(x) \dashrightarrow \frac{0}{0!}(x)^0 + \frac{2}{1!}(x)^1 + \frac{0}{2!}(x)^2 + \frac{-8}{3!}(x)^3 + \frac{0}{4!}(x)^4 + \frac{32}{5!}(x)^5$$

$$2x - \frac{4}{3}x^3 + \frac{4}{15}x^5$$



2) Find the polynomial of order 4 at 0 for $f(x) = e^{-x}$
Use this to approximate $e^{(.5)}$

$f(x) = e^{-x}$ $\quad f(0) = 1$

$f'(x) = -e^{-x}$ $\quad f'(0) = -1$

$f''(x) = e^{-x}$ $\quad f''(0) = 1$

$f'''(x) = -e^{-x}$ $\quad f'''(0) = -1$

$f^{4}(x) = e^{-x}$ $\quad f^{4}(0) = 1$

$$e^{-x} = 1 + (-1)\frac{x}{1!} + (1)\frac{x^2}{2!} + (-1)\frac{x^3}{3!} + (1)\frac{x^4}{4!} + \ldots$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

To approximate $e^{(.5)}$ we'll let x = -1/2

$$f(-1/2) = e^{.5}$$

$$f(-1/2) = 1 - (-1/2) + \frac{(-1/2)^2}{2} - \frac{(-1/2)^3}{6} + \frac{(-1/2)^4}{24}$$

$e^{.5} = 1.64872$ (approx)

1 + 1/2 + 1/8 + 1/48 + 1/384

= 1.64844

3) What is the coefficient of $(x-2)^3$ in the Taylor Series generated by ln(x) @ x = 2

$f(x) = \ln(x)$ $\quad f(2) = \ln(2)$

$f'(x) = \frac{1}{x}$ $\quad f'(2) = 1/2$

$f''(x) = \frac{-1}{x^2}$ $\quad f''(2) = -1/4$

$f'''(x) = \frac{2}{x^3}$ $\quad f'''(2) = 2/8$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n$$

$$\ln(2)(x-2) + \frac{1/2}{1!}(x-2) + \frac{-1/4}{2!}(x-2)^2 + \frac{2/8}{3!}(x-2)^3$$

coefficient is 1/24

SOLUTIONS

4) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ Does the series converge or diverge?

Try the ratio test...

$$\lim_{n\to\infty} \frac{\dfrac{(n+4)!}{3!\,(n+1)!\,3^{n+1}}}{\dfrac{(n+3)!}{3!\,n!\,3^n}} = \lim_{n\to\infty} \frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}} \bullet \frac{3!\,n!\,3^n}{(n+3)!}$$

$$\lim_{n\to\infty} \frac{(n+4)!}{\cancel{3!}\,(n+1)!\,3^{n+1}} \bullet \frac{\cancel{3!}\,n!\,3^n}{(n+3)!}$$

$$\lim_{n\to\infty} \frac{(n+4)}{(n+1)\cdot 3} = \frac{1}{3} \lim_{n\to\infty} \frac{(n+4)}{(n+1)} = \frac{1}{3} \bullet 1$$

Since the limit < 1, the series CONVERGES

5) $1 + \dfrac{1}{\sqrt[5]{2}^{\,2}} + \dfrac{1}{\sqrt[5]{3}^{\,2}} + \dfrac{1}{\sqrt[5]{4}^{\,2}} + \ldots$ Does the series converge or diverge?

rewrite.... $\dfrac{1}{1^{2/5}} + \dfrac{1}{2^{2/5}} + \dfrac{1}{3^{2/5}} + \dfrac{1}{4^{2/5}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/5}}$

This is a p-series where $p = 2/5$

Since $p = 2/5 < 1$, this series DIVERGES

6) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \frac{1}{7^n} =$

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n$$

(geometric series) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n \Rightarrow \frac{\frac{1}{7}}{1 - 1/7} = \frac{\frac{1}{7}}{\frac{6}{7}}$

so, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n = \frac{1}{6}$

using partial fractions..

$$\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{(n+3)}$$

$$\frac{3}{n(n+3)} = \frac{A(n+3)}{n(n+3)} + \frac{B(n)}{n(n+3)}$$

$3 = An + 3A + Bn$

$3A = 3$

and $n(A + B) = 0n$

$A = 1$

$B = -1$

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{-1}{n+3}$$

using "telescoping"...

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

$$- \frac{1}{4} - \frac{1}{5} - \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

The sum is

$$\frac{1}{6} + \frac{11}{6} = 2$$

7) Find the intervals of convergence.

SOLUTIONS

a) $\sum_{n=0}^{\infty} 3\left(\frac{x}{4}\right)^n$ This is a geometric series, so $\left|\frac{x}{4}\right| < 1$ $\boxed{-4 < x < 4}$

b) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ Using the nth root

$$\lim_{n\to\infty} \sqrt[n]{\frac{n(x+3)^n}{5^n}}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n} \cdot \sqrt[n]{(x+3)^n}}{\sqrt[n]{5^n}} = \frac{1 \cdot (x+3)}{5}$$

$$-1 < \frac{(x+3)}{5} < 1$$

$$\boxed{-8 < x < 2}$$

8) $\sum_{n=1}^{\infty} \frac{2}{n^2+n} =$

First, we'll confirm the sequence converges to 0...

there is a pattern: 1, 1/3, 1/6, 1/10, 1/15....

the limit sequence goes to zero... ✓

"Telescoping"

Now, we'll see if the series converges...

Use partial fractions $\frac{2}{n^2+n} = \frac{A}{n} + \frac{B}{n+1}$

$2 = A(n+1) + B(n)$ ⇨ if n = 0, then A = 2; if n = -1, then B = -2

$$\frac{2}{n^2+n} = \frac{2}{n} + \frac{-2}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{2}{n^2+n} = 2 - 1 + 1 - 2/3 + 2/3 - 2/4 + 2/4 \; ..... = \boxed{2}$$

9) Find the sum....

$$1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots + (-1)^n \frac{1}{4^n} + \ldots$$

Using "telescoping", we'll split the positives and negatives...

Two geometric series...

$$1 + \frac{1}{16} + \frac{1}{256} + \ldots + = \sum_{n=0}^{\infty} \frac{1}{16^n}$$

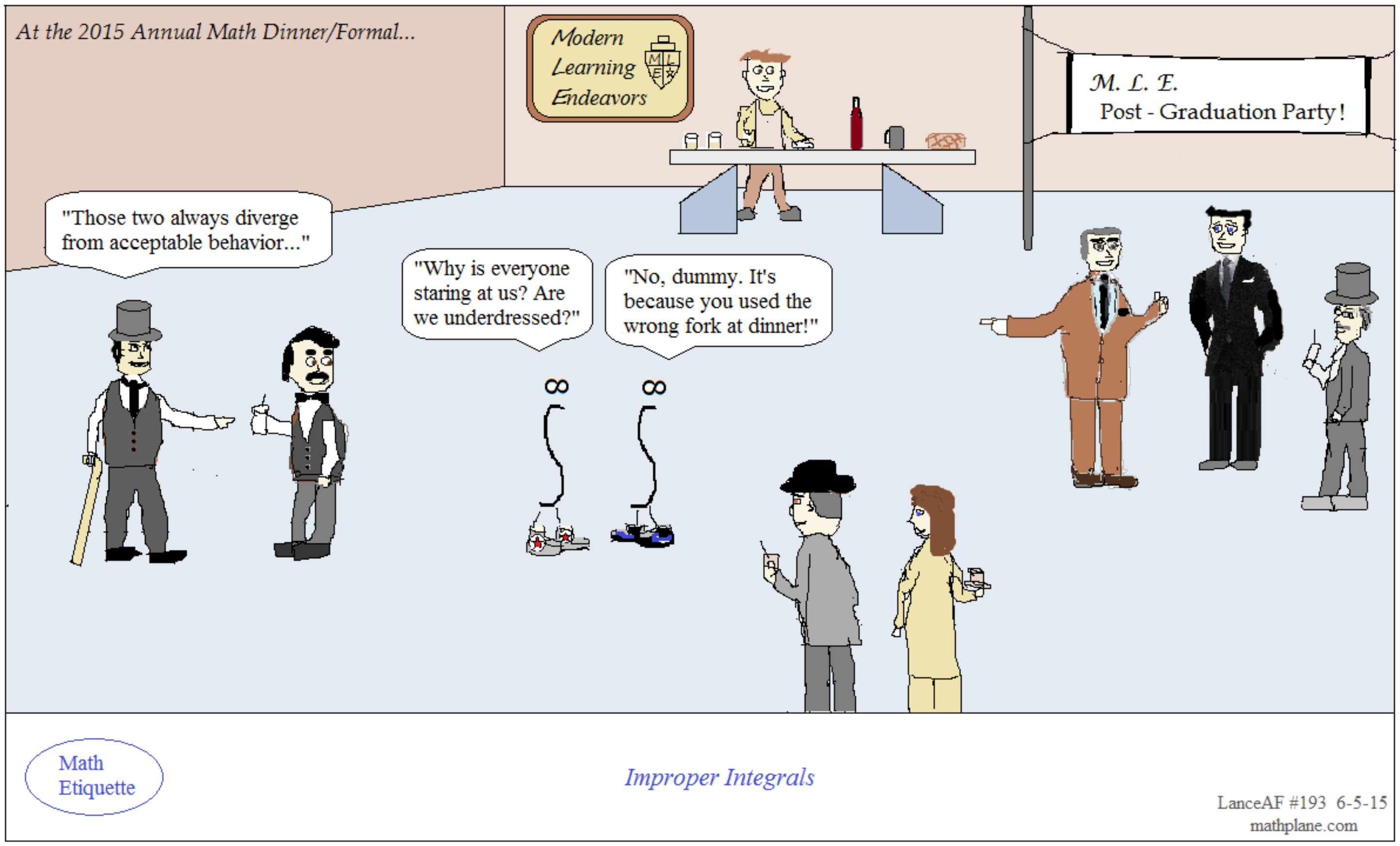
$$\frac{1}{1 - \frac{1}{16}} = 16/15$$

$$\boxed{\frac{12}{15}}$$

$$-\frac{1}{4} - \frac{1}{64} - \frac{1}{1024} - \ldots = -\frac{1}{4}\left(1 + \frac{1}{16} + \frac{1}{256} + \ldots +\right) = -\frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{16^n}$$

$$-\frac{1}{4} \cdot 16/15 = -4/15$$

Improper Integrals



Examples-→

Improper Integrals

Definition: A definite integral where the integrand has a discontinuity between the bounds of integration.
(or, the upper/lower bound is $+/- \infty$)

An improper integral can be evaluated using limits!

if the limit exists (and is finite), it *converges*

if the limit does not exist (or, is infinite), it *diverges*

Example:

$$\int_1^{\infty} \frac{1}{x^{1.1}}\,dx$$

Step 1: If possible, sketch a graph

We're looking for the area under the curve.
(Since it goes on forever, we are looking for the value of convergence it approaches.)

Step 2: Evaluate the integral, substituting limits

$$\int_1^{\infty} x^{-1.1}\,dx = \left.\frac{x^{-.1}}{-.1}\right|_1^{\infty} = \lim_{b \to \infty} \frac{1}{-.1b^{.1}} - \frac{1}{-.1(1)^{.1}}$$

("bottom heavy", so it goes to 0)

Step 3: Find the limits

$$= 0 - -10 = 10$$

Example:

$$\int_0^{\ln 4} x^{-2} e^{\frac{1}{x}}\,dx$$

$$-1\int_0^{\ln 4} -1\, x^{-2} e^{\frac{1}{x}}\,dx = -1 \cdot \left. e^{\frac{1}{x}} \right|_0^{\ln 4} = -e^{\frac{1}{\ln 4}} - -e^{\frac{1}{0}}$$

Since 1/0 is undefined, this integral *diverges*

Since the derivative of $\frac{1}{x}$ is $-x^{-2}$, we insert a -1

$$= \infty$$

Comparison Test: Determining Convergence/Divergence

"When it's difficult to evaluate an integral, try a similar equation."

Example: Does $\int_1^{\infty} \frac{dx}{1+e^x}$ converge or diverge?

$\frac{1}{1+e^x}$ is difficult to integrate...

However, $\frac{1}{e^x}$ is much easier....

$$\frac{1}{e^x} > \frac{1}{1+e^x}$$

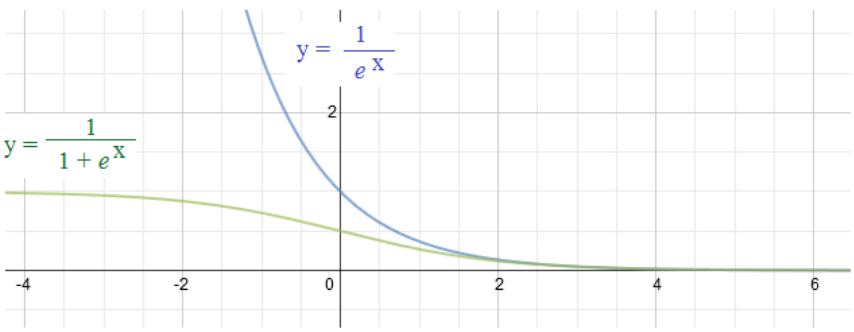
Since the larger value (greater area) converges, the lesser value must converge, too...

$$\int_1^{\infty} \frac{1}{e^x}\,dx = \int_1^{\infty} e^{-x}\,dx = -\int_1^{\infty} -e^{-x}\,dx$$

$$= -e^{-x}\Big|_1^{\infty} = \lim_{b\to\infty} -e^{-x}\Big|_1^{b}$$

$$\lim_{b\to\infty} -e^{-b} - -e^{-1}$$

$$0 + \frac{1}{e} = \frac{1}{e}$$



Example: Does $\int_{\pi}^{\infty} \frac{2+\cos\theta}{\theta}\,d\theta$ converge or diverge?

Again, this integral is difficult to find.
But,

$\frac{2}{\theta}$ is similar and much easier.

$$\frac{2+\cos\theta}{\theta} > \frac{2}{\theta}$$

Since the <u>smaller</u> value diverges, the larger value must diverge, too.

$$\int_{\pi}^{\infty} \frac{2}{\theta}\,d\theta = 2\int_{\pi}^{\infty} \frac{1}{\theta}\,d\theta =$$

$$2\ln\theta\Big|_{\pi}^{\infty} = \ln\theta^2\Big|_{\pi}^{\infty} = \infty - \ln(\pi)^2$$

$$= \infty$$

Determining Convergence/Divergence: *Comparison Test*

Example: Does $\displaystyle\int_1^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx$ converge or diverge?

First, let's rewrite the equation: $\dfrac{1}{e^{x}\sqrt{x}}$

Then, to test for convergence, let's pick a function that is greater...

$$\frac{1}{\sqrt{x}} > \frac{1}{e^{x}\sqrt{x}} \quad \text{for all } x \geq 1$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\int_1^{b} (x)^{\frac{-1}{2}}\,dx \longrightarrow \lim_{b\to\infty} 2x^{\frac{1}{2}}\Big|_1^{b} = \infty - 2$$

DIVERGES

Since the 'larger' equation diverges,
the comparison test is inconclusive....

Now, let's test another function....

$$\frac{1}{e^{x}} > \frac{1}{e^{x}\sqrt{x}} \quad \text{for all } x \geq 1$$

$$\int_1^{\infty} \frac{1}{e^{x}}\,dx = \lim_{b\to\infty}\int_1^{b} e^{-x}\,dx \longrightarrow \lim_{b\to\infty} -e^{-x}\Big|_1^{b} = \lim_{b\to\infty} \frac{-1}{e^{x}}\Big|_1^{b} = 0 + \frac{1}{e}$$

CONVERGES

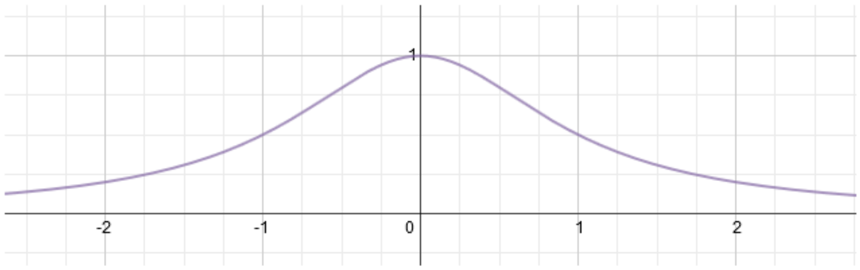
Since the 'larger' equation converges,
the integral must converge, too!

Using Inverse Trigonometry Function

Improper Integral

What is the area under the curve $y = \dfrac{1}{x^2 + 1}$ in Quadrant I ?

Step 1: If possible, sketch the graph



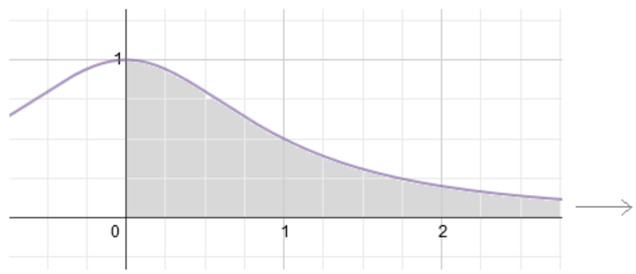
The curve *approaches* 0 in both directions.

Step 2: Determine boundaries of integrand (ends of the integral)

We're looking for the area in quadrant I.
(under the curve and above the x-axis)

Since the curve never gets to the x-axis,
the boundaries of the integral will be

$x = 0$ and ∞



Step 3: Evaluate integral

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_0^{b} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \tan^{-1}(x)\Big|_0^b = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

$\tan(\frac{\pi}{2})$ is undefined

$\tan(0) = 0$

Evaluate $\displaystyle\int_1^{\infty} \frac{\tan^{-1}(t)}{1+t^2}\,dt$

$$\int_1^{\infty} \tan^{-1}(t)\,\frac{1}{1+t^2}\,dt = \lim_{b\to\infty} \frac{(\tan^{-1}(t))^2}{2}\Big|_1^b = \lim_{b\to\infty} \frac{(\tan^{-1}(b))^2}{2} - \frac{(\tan^{-1}(1))^2}{2}$$

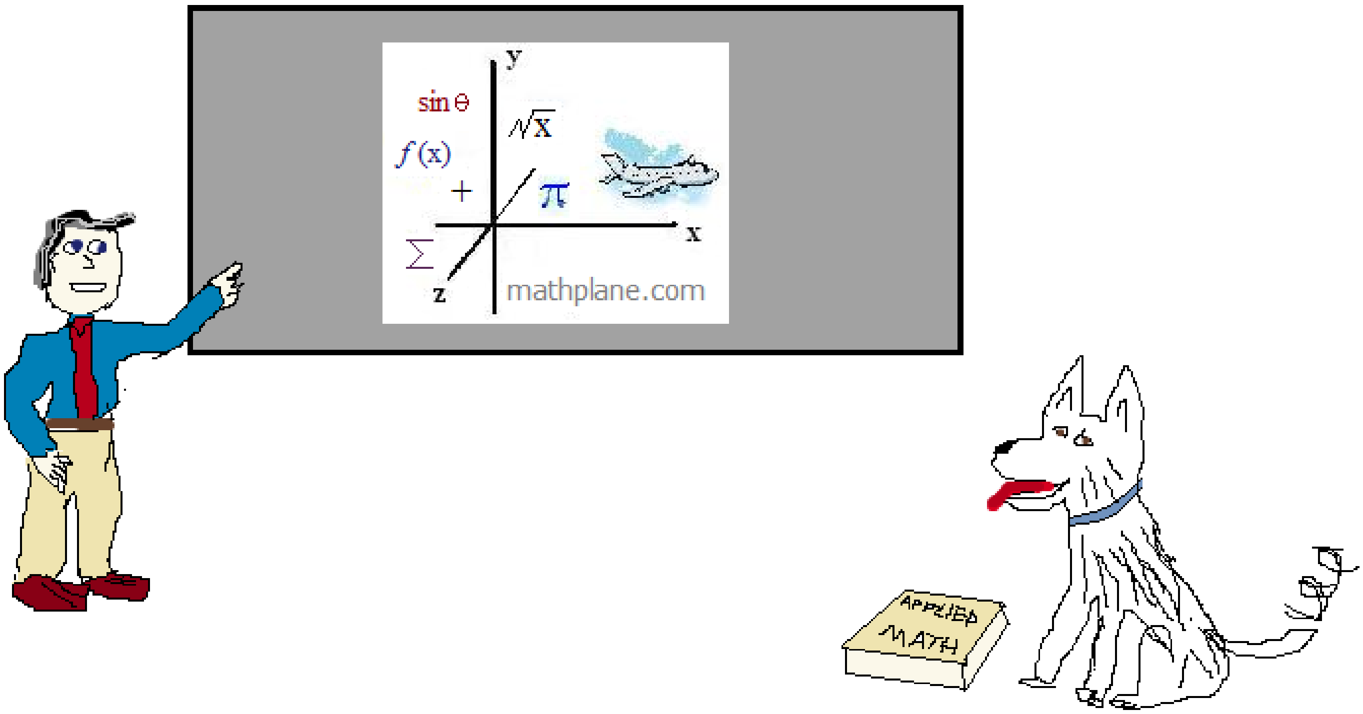
$$\frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{\left(\frac{\pi}{4}\right)^2}{2} = \frac{\pi^2}{8} - \frac{\pi^2}{32}$$

$$\boxed{\frac{3\pi^2}{32} \approx .925}$$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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Saturday morning, December 13th, 2014
Holiday Photos
"This is lame..."
"Hurry up. It's almost time!"
"Set up the camera! Make sure you get the clock and calendar..."
9:08:51
Season's Greetings!
9:10:11
Today is 12-13-14
Twelve hours later, the Kodak family did try one more pose...
(The evening photo wasn't much better....)
LanceAF #168 (12-12-14)
mathplane.com