

# Geometry: Surface Area and Volume (Advanced)

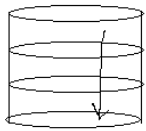
Notes, Examples, and Practice Questions (with Solutions)

Quick review of formulas

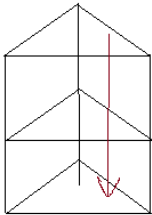
Volume

2 Bases  
Prisms  
"0 points"

(area of base) x (height)



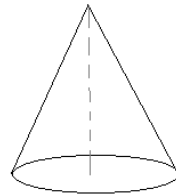
Cylinder Volume:  
area of the circle x  
the depth of the figure..



Triangular Prism Volume:  
area of the triangle x  
the depth of the figure

1 Base  
Pyramids/Cones  
"1 point"

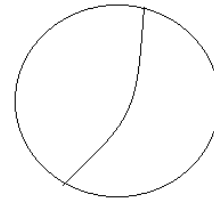
$\frac{1}{3}$  (area of base) x (height)



height is  
inside...

0 Bases  
Sphere  
"0 points, 0 edges"

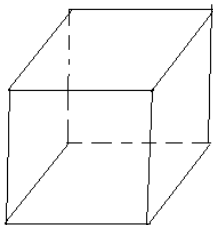
$\frac{4}{3} \pi$  (radius)<sup>3</sup>



Surface Area

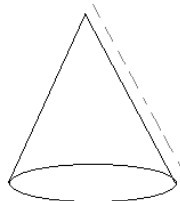
2 Bases  
Prisms  
"0 points"

$2 \times$  (area of base) + (base perimeter)(height)  
area of bases      lateral area



1 Base  
Pyramids/Cones  
"1 point"

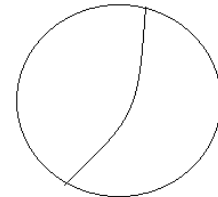
$1 \times$  (area of base) +  $\frac{1}{2}$  (base perimeter)(slant height)  
area of base      lateral area



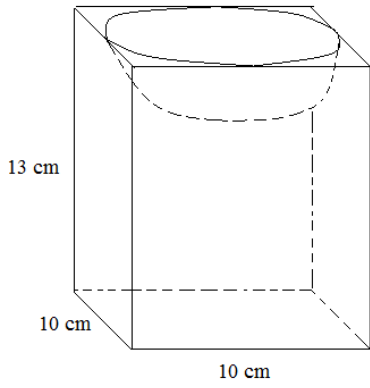
slant height is  
outside

0 Bases  
Sphere  
"0 points, 0 edges"

$4 \pi$  (radius)<sup>2</sup>



Example:



A hemisphere is "scooped out of a prism".  
What is the surface area and volume?

Surface area:

$$\text{Base} = 100 \text{ cm}^2$$

$$\text{Lateral Area} = 4 \times (10 \text{ cm} \times 13 \text{ cm}) = 520 \text{ cm}^2$$

$$\text{Top: } 100 - 25\pi$$

$$\text{Scooped out hemisphere: } \frac{4\pi \cdot 25}{2}$$

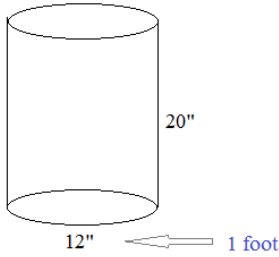
Volume: Prism =  $10 \times 10 \times 13 = 1300 \text{ cm}^3$

$$\text{Hemisphere} = \frac{1}{2} \cdot \frac{4}{3} \pi (\text{radius})^3$$

$$= \frac{2}{3} (125) \pi \text{ cm}^3$$

$$\text{TOTAL} = 1300 \text{ cm}^3 - \frac{2}{3} (125) \pi \text{ cm}^3$$

Example: The density of a metal is  $160 \text{ lbs/ft}^3$   
What is the weight of the cylinder?



1.67' method 1: convert inches to feet...

$$\pi (.5)^2 (1.67) = 1.31$$

$$209.75$$

method 2: convert feet to inches

$$160 \text{ lbs/ft}^3$$

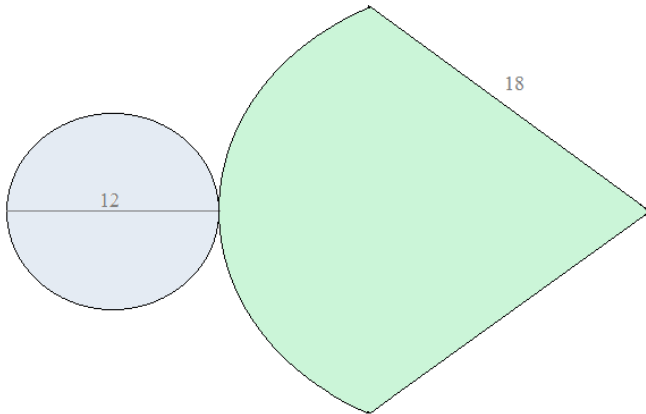
$$160 \text{ lbs}/(12 \times 12 \times 12)$$

$$.0926 \text{ lbs/in}^3$$

$$\pi (6'')^2 (20'') = 2260.8$$

$$209.35$$

Example: Find the area of the shapes.



The blue circle is straightforward:

the area is  $36\pi$

Then, to find the green area, we'll consider the reconstructed shape: a cone

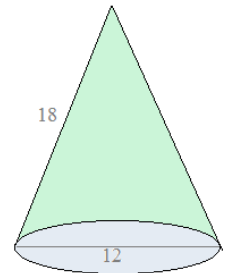
Surface area of the cone:

$$36\pi + 108\pi = 144\pi$$

Base      Lateral Area

The green area is the lateral area..

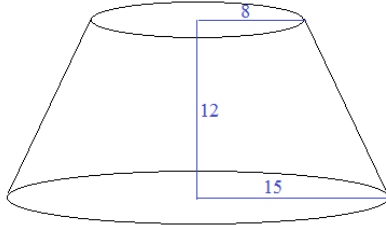
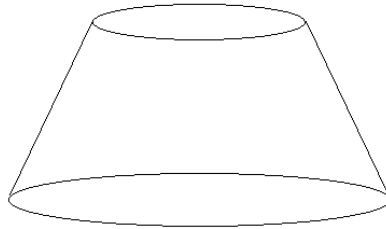
$$108\pi$$



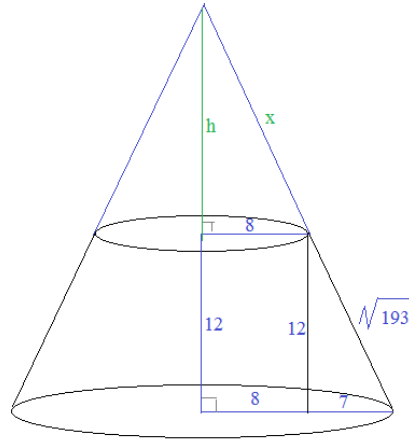
Example: Find the volume and surface area of the frustum...

Frustum: Volume and Surface Area

radius of top: 8  
 radius of bottom: 15  
 height: 12



HINT: extend the frustum top upward to create a cone...  
 Then, find ratio of top cone to entire cone...  
 then, frustum = entire cone - top cone..



ratio of top cone to entire cone

$$\frac{h}{h + 12} = \frac{8}{15}$$

$$15h = 8h + 96$$

$$7h = 96$$

$$h = \frac{96}{7}$$

Pythagorean Theorem for top cone slant height

$$64 + \frac{96^2}{7^2} = (\text{slant height})^2$$

$$x = \frac{8\sqrt{193}}{7}$$

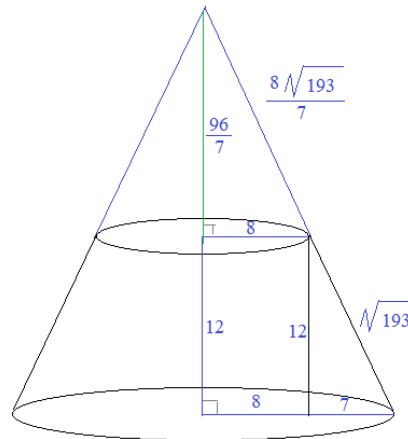
Volume of top cone:

$$\frac{1}{3} (64\pi) \left(\frac{96}{7}\right) = \frac{2048\pi}{7} \text{ approx. } 919$$

Volume of entire cone:  $919/6059 \approx 512/3375$

$$\frac{1}{3} (225\pi) \left(\frac{180}{7}\right) = \frac{13500\pi}{7} \text{ approx. } 6059$$

Volume of frustum: 5140 (approx.)



Note ratio of top cone to entire cone:  
 8:15  
 therefore, area will be 64:225  
 and volume will be 512:3375

Lateral area of top cone:

$$\frac{1}{2} (16\pi)(15.9) = 399.6$$

base circumference      slant height

$$399.6/1404.3 \approx 64/225$$

Lateral area of entire cone:

$$\frac{1}{2} (30\pi)(29.8) = 1404.3$$

Lateral area of frustum: 1004.7

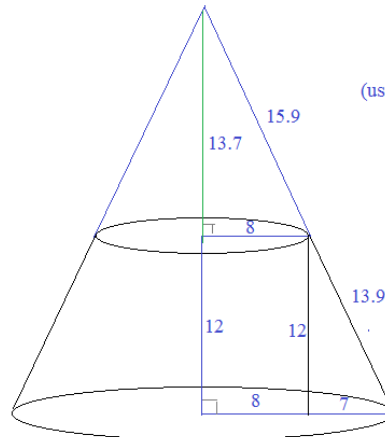
+

Area of top circle:  $64\pi$

+

Area of bottom circle:  $225\pi$

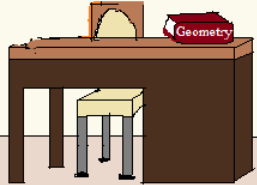
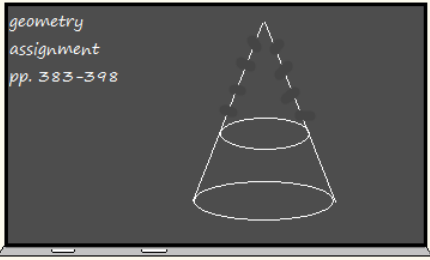
Surface area =  $1004.7 + 289\pi$



(using decimal approximations)

Frustum  
Frustration

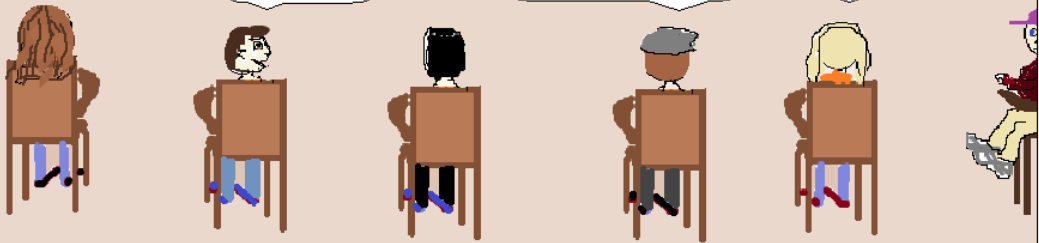
"Can anyone tell us how to find the volume of a frustum?"



"Frust-what?"

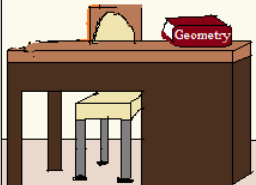
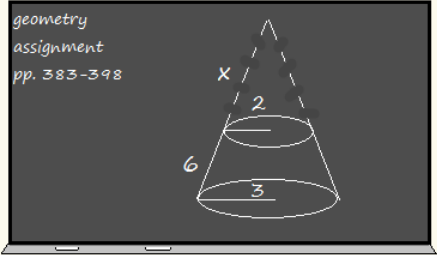
"Why did he cut off the top of that birthday hat?"

"I don't know."

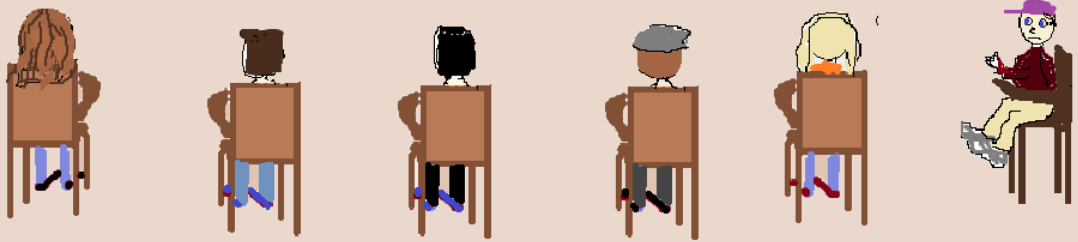


"... volume of 'large' cone minus the volume of the 'top' cone...."  
"... in this example, how do we find the dimensions and volumes of the cones?  
... Someone?... Anyone?"

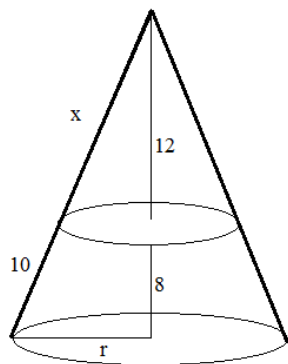
'Doesn't anyone ever read the book?!?!?'



"Since it's an upside down ice cream cone, why don't they call it a 'frost-mmm'?"



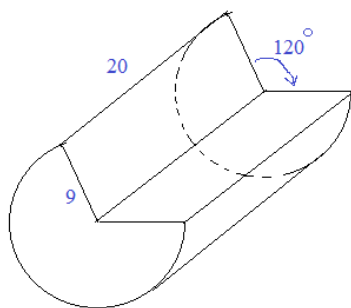
1) Find the volume and surface area of the entire cone (with radius  $r$ ).



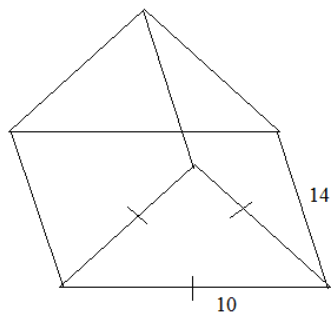
2) A 20" log with a radius of 9" has a uniform part carved out of it. (See diagram)

What is the volume of the carved log?

If you wanted to paint the log, what is the surface area that will be painted?

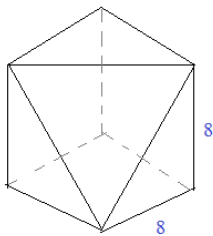


3) What is the volume of the triangle prism? And, surface area?

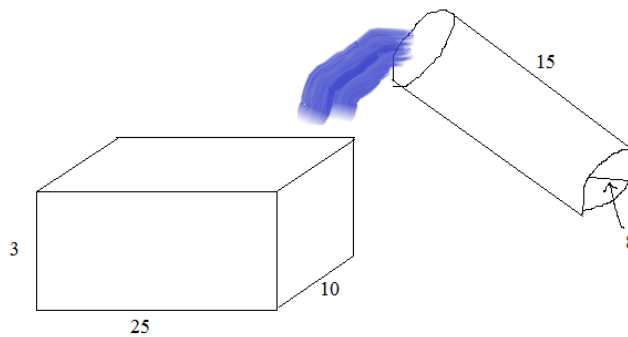


- 4) A cone has dimensions in which the ratio of the radius to height is 3:4. If the surface area is  $384\pi$ , what is the length of the height?

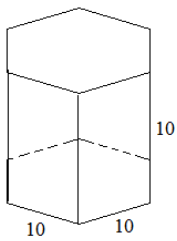
- 5) A cube with edges of length 8 cm has been sliced at 3 vertices as shown in the figure. What is the surface area?



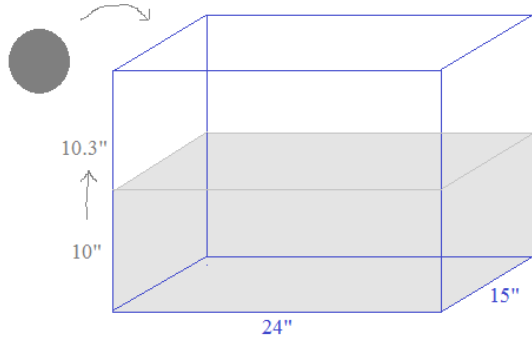
- 6) A cylinder with these dimensions is full of water. If it is poured into the open rectangular prism, will it overflow?



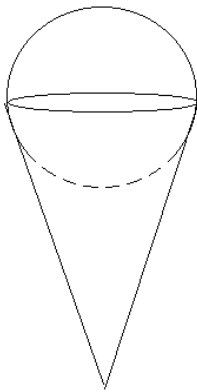
- 7) Find the volume and surface area of the hexagonal prism:



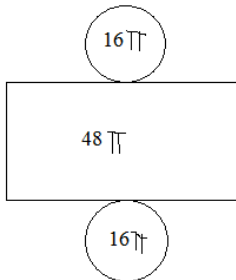
- 8) A rectangular tank has length 24 inches, height 20 inches, and depth 15 inches. After filling half the tank with water, a solid steel ball is placed in the tank, raising the water level from 10 inches to 10.3 inches. What is the diameter of the solid ball?



- 9) An ice cream cone is 7" deep and 3" across. If a round scoop of ice cream that is 3" wide sits on the cone, would the ice cream overflow after it melts?

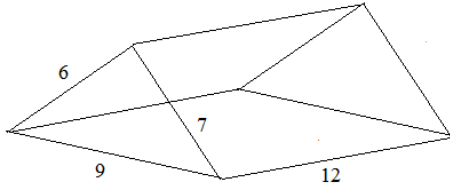


- 10) The areas of each shape within the "flattened cylinder" are shown in the diagram. What is the volume of the constructed cylinder?



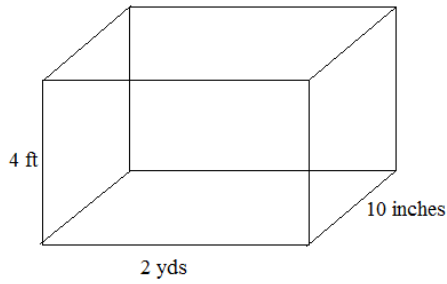


11) Find the volume and surface area of the triangular prism:

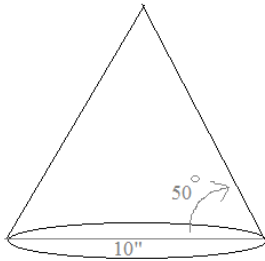


12) Find the surface area:

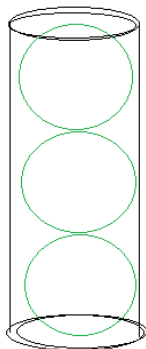
(diagram not drawn to scale!)



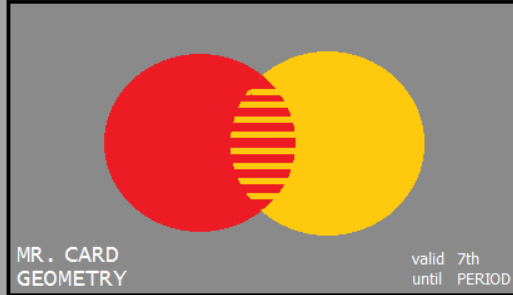
13) The diameter of a cone's base is 10 inches.  
The slant height makes a 50 degree angle with the diameter of the base.  
What is the volume and surface area of the cone?



14) Three tennis balls come in a cannister.  
Is there more space or more tennis balls occupying the cylinder container?  
What is the ratio?



Extra  
Credit  
Card



Incomplete proofs: minus 5 points...

Missed power theorems: minus 10 points...

Incorrect circles answers: minus 35 points...

Getting out of this geometry test with  
a passing grade: PRICELESS!

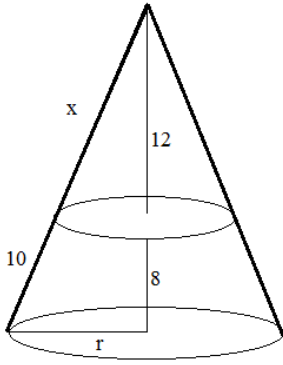
"Bonus question:  
what is the area of the  
striped intersection of  
the circles?"

"There are some math grades you can't buy.  
But, for everything else, there is extra credit from Mister Card."



Solutions-→

1) Find the volume and surface area of the entire cone (with radius r).



similar triangles

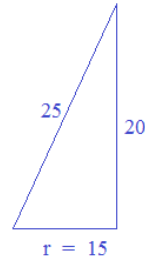
$$\frac{x}{(x+10)} = \frac{12}{20}$$

$$20x = 12x + 120$$

$$x = 15$$

SOLUTIONS

Pythagorean Theorem



Surface Area and Volume (Advanced)

$$\text{Volume} = \frac{1}{3} (\text{Base})(\text{Height})$$

$$= \frac{1}{3} (225\pi)(20)$$

$$= 1500\pi$$

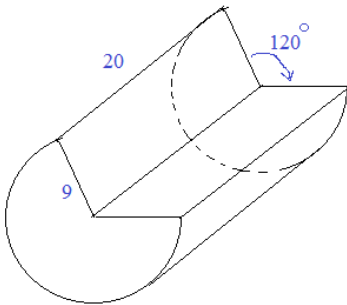
Surface Area = Base + Lateral Area

$$225\pi + \frac{1}{2} (30\pi)(25)$$

$$= 600\pi$$

2) A 20" log with a radius of 9" has a uniform part carved out of it. (See diagram)

What is the volume of the carved log?  
If you wanted to paint the log, what is the surface area that will be painted?



Volume of entire log (cylinder)

$$\frac{240}{360} (81\pi)(20)$$

$$= 1080\pi$$

Surface Area:

Top and bottom:

$$\frac{240}{360} (81\pi) = 54\pi \times 2$$

"Curved Lateral Area"

$$\frac{240}{360} (20)(18\pi)$$

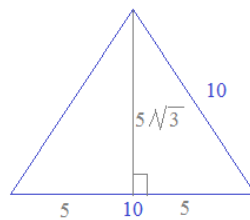
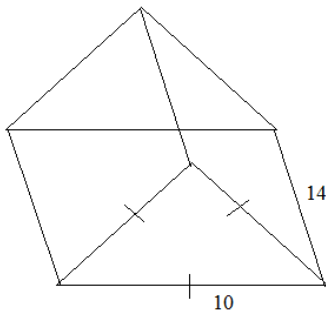
$$240\pi$$

"rectangles"

$$(9)(20) = 180 \times 2$$

$$\begin{array}{r} 108\pi \\ 240\pi \\ 360 \\ \hline 360 + 348\pi \end{array}$$

3) What is the volume of the triangle prism? And, surface area?



Area of Base triangle is  $25\sqrt{3}$

$$\text{so, volume is } 14 \times 25\sqrt{3} = 350\sqrt{3} \text{ units}^3$$

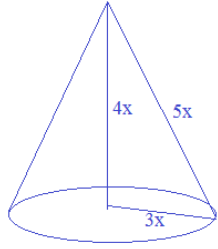
Surface Area: Bases  $\rightarrow 50\sqrt{3}$

$$3 \text{ sides } \rightarrow 3 \times 140 = 420$$

$$420 + 50\sqrt{3} \text{ units}^2$$

4) A cone has dimensions in which the ratio of the radius to height is 3:4. If the surface area is  $384\pi$ , what is the length of the height?

SOLUTIONS



$$384\pi = \pi(3x)^2 + \frac{1}{2}\pi(6x)(5x)$$

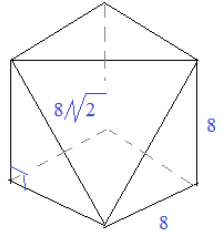
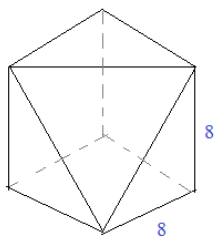
base                      lateral area

$$384\pi = 9x^2\pi + 15x^2\pi$$

$$x = 16$$

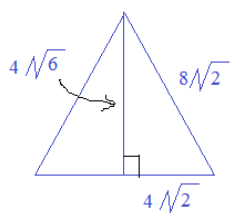
the height is 64

5) A cube with edges of length 8 cm has been sliced at 3 vertices as shown in the figure. What is the surface area?



Area of bottom and backs..  
(3 full sides)  
 $64 \times 3 \dots$   
Area of top and fronts...  
(3 half sides)  
 $32 \times 3 \dots$

Area of front sliced equilateral triangle..



$288 + 32\sqrt{3}$

$16\sqrt{12} = 32\sqrt{3}$

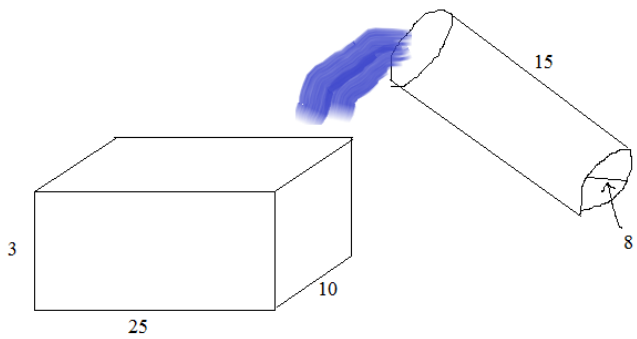
6) A cylinder with these dimensions is full of water. If it is poured into the open rectangular prism, will it overflow?

Volume of prism:

$3 \times 25 \times 10 = 750$

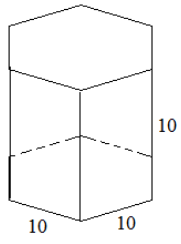
Volume of cylinder:

$16\pi \times 15 = 753.6$  Approx.



YES, it will overflow!

7) Find the volume and surface area of the hexagonal prism:



surface area: 2 hexagon bases + 6 rectangle sides

volume: hexagon base x (height)

height = 10

base area =  $\frac{1}{2}$  (apothem)(perimeter)

$= \frac{1}{2} (5\sqrt{3})(60) = 150\sqrt{3}$

Volume:  $1500\sqrt{3}$   
Surface Area:  $600 + 300\sqrt{3}$

SOLUTIONS

- 8) A rectangular tank has length 24 inches, height 20 inches, and depth 15 inches. After filling half the tank with water, a solid steel ball is placed in the tank, raising the water level from 10 inches to 10.3 inches. What is the diameter of the solid ball?

Step 1: Find volume of water in tank

$$24 \times 10 \times 15 = 3600 \text{ cubic inches}$$

(The actual height of tank is irrelevant. The water level height is what matters)

Step 2: Find volume of water AND solid ball in tank

$$24 \times 10.3 \times 15 = 3708 \text{ cubic inches}$$

Step 3: Find dimensions of solid ball...

$$3708 - 3600 = 108 \text{ cubic inches is the volume of the ball...}$$

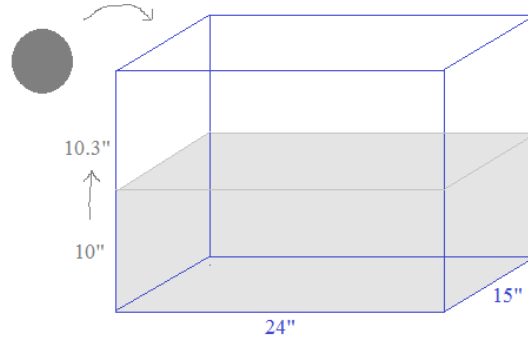
$$\text{Volume of sphere} = \frac{4}{3} \pi (\text{radius})^3$$

$$108 = \frac{4}{3} \pi (r)^3$$

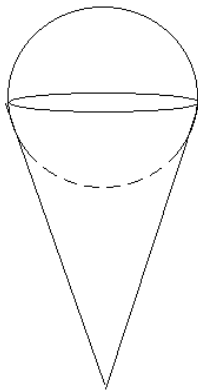
$$81 = \pi (r)^3$$

$$25.78 = r^3$$

$$\text{radius} = 2.95 \text{ so, diameter} = 5.9 \text{ (approximately)}$$



- 9) An ice cream cone is 7" deep and 3" across. If a round scoop of ice cream that is 3" wide sits on the cone, would the ice cream overflow after it melts?



When the ice cream melts, the volume of the scoop will fill the cone...

So, is the volume of the scoop greater than the volume of the cone?

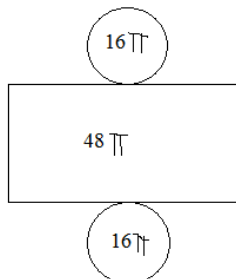
$$\text{Volume of sphere} = \frac{4}{3} \pi (\text{radius})^3 = \frac{4}{3} \pi \left(\frac{3}{2} \text{ inch}\right)^3 = 14.137$$

$$\text{Volume of cone} = \frac{1}{3} (\text{base area})(\text{height})$$

$$\frac{1}{3} \pi (\text{radius})^2 (\text{height}) = \frac{1}{3} \pi \left(\frac{3}{2} \text{ inch}\right)^2 (7 \text{ inches}) = 16.49$$

The ice cream would melt into the cone WITHOUT overflowing!

- 10) The areas of each shape within the "flattened cylinder" are shown in the diagram. What is the volume of the constructed cylinder?



radius of each circle: 4

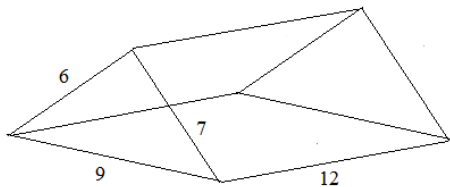
diameter of each circle: 8

circumference:  $8\pi$

therefore, height is 6

so, the volume is  $96\pi$

11) Find the volume and surface area of the triangular prism:



**SOLUTIONS**

Since it is a prism, the volume is the (area of the base) x (depth)

$$\text{area of base (triangle)} = \sqrt{(11)(11-6)(11-7)(11-9)} = \sqrt{440}$$

$$\text{Volume} = \sqrt{440} \cdot (12) = 251.7 \text{ cubic units}$$

$$\text{Surface Area} = 2(\text{base area}) + \text{lateral area}$$

$$= 2 \cdot \sqrt{440} + (22)(12)$$

$$= 306 \text{ square units}$$

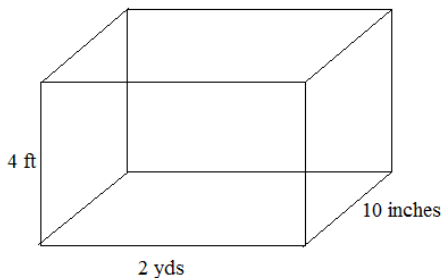
Surface Area and Volume (Advanced)

Using Heron's formula where semiperimeter is 1/2(perimeter) of triangle = 11

$$\text{LA} = \text{perimeter}(\text{height})$$

12) Find the surface area:

(diagram not drawn to scale!)



Step 1: Convert to common units...  
For ease, we'll use inches...

length x width x height

$$72'' \times 10'' \times 48''$$

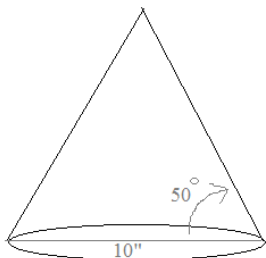
Step 2: Find area of each side...

$$\begin{aligned} \text{"left and right"} & 480 \text{ sq inches} \times 2 = 960 \\ \text{"top and bottom"} & 720 \text{ sq inches} \times 2 = 1440 \\ \text{"front and back"} & 3456 \text{ sq inches} \times 2 = 6912 \end{aligned}$$

Step 3: Add them up

$$\begin{aligned} & 9312 \text{ square inches} \\ & 64.667 \text{ square feet} \\ & 7.18 \text{ square yards} \end{aligned}$$

13) The diameter of a cone's base is 10 inches.  
The slant height makes a 50 degree angle with the diameter of the base.  
What is the volume and surface area of the cone?



$$\cos(50^\circ) = \frac{5}{\text{slant height}}$$

$$\text{slant height } l = 7.78$$

$$\tan(50^\circ) = \frac{\text{height}}{5}$$

$$\text{height } h = 5.96$$

$$\text{Volume} = \frac{1}{3} (25\pi) (5.96)$$

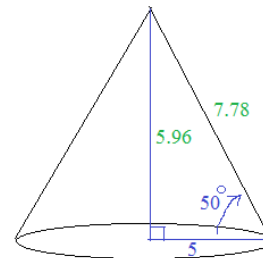
$$= 156 \text{ cubic inches}$$

$$\text{Lateral Area} = \frac{1}{2} (10\pi) (7.78)$$

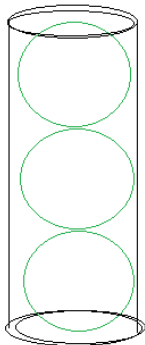
$$122.2 \text{ square inches}$$

$$\text{Surface Area} = 25\pi + 122.2$$

$$= 200.7 \text{ square inches}$$



14) Three tennis balls come in a canister.  
Is there more space or more tennis balls occupying the cylinder container?  
What is the ratio?



Assume the radius of a tennis ball is 2  
diameter is 4

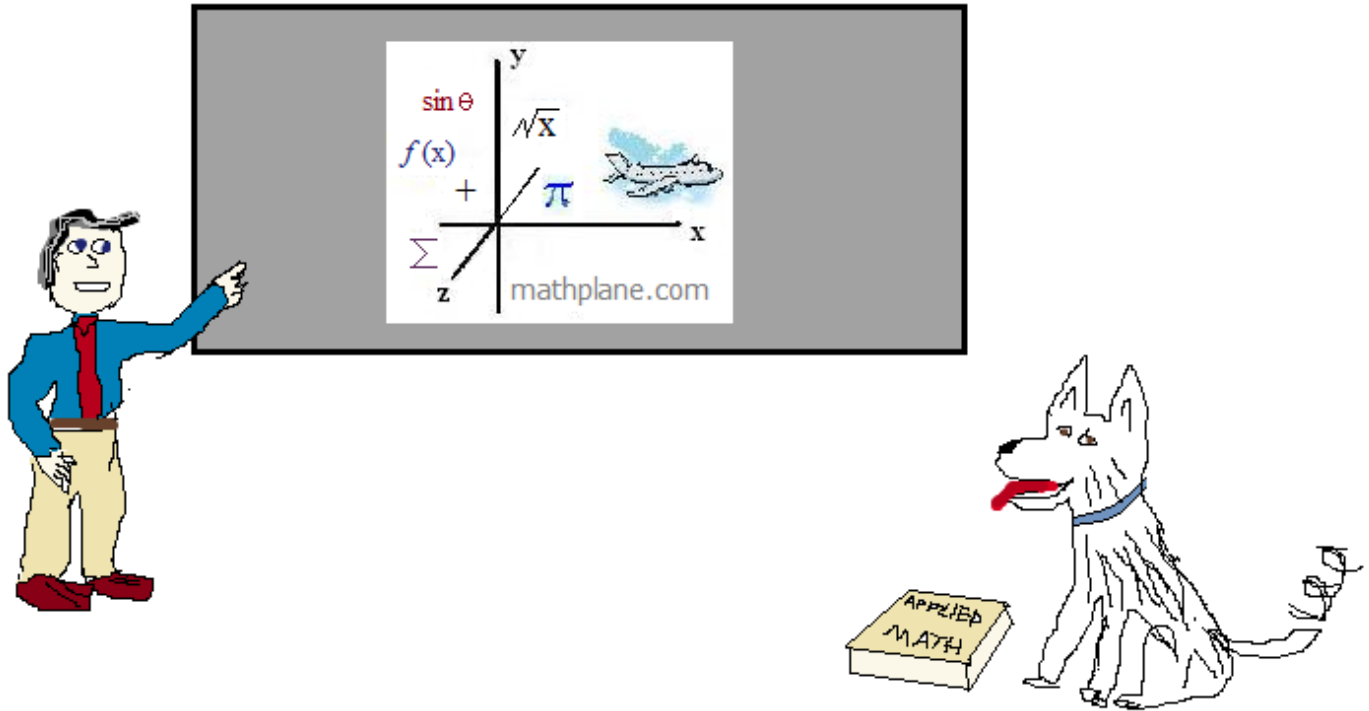
height of cylinder is 12...

$$\text{Volume of cylinder: } 48\pi$$

$$\text{Volume of one tennis ball: } \frac{4}{3}\pi (\text{radius})^3 = \frac{4}{3}\pi (8) = \frac{32}{3}\pi$$

$$3 \text{ tennis balls: } 32\pi \quad \text{remaining space: } 16\pi$$

$$\text{tennis balls occupy more space at a ratio of } 2:1$$



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