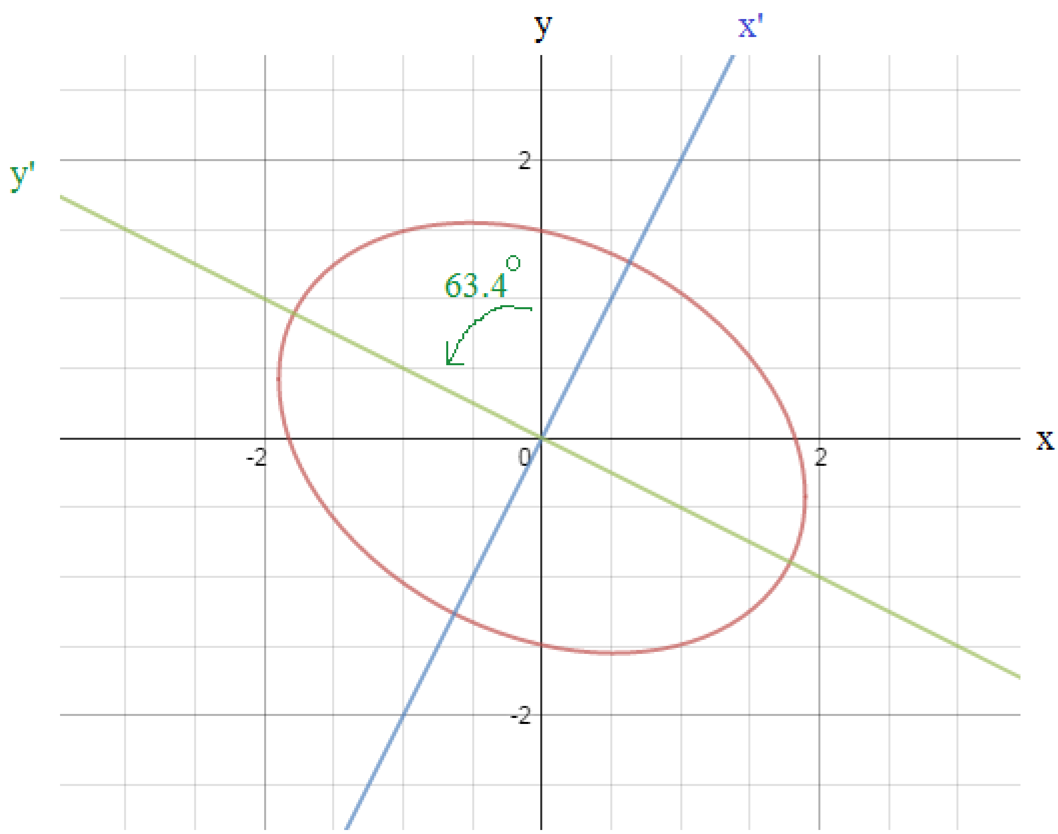
Rotation of Axes: Conics

Formulas, Examples, and practice test (with solutions)



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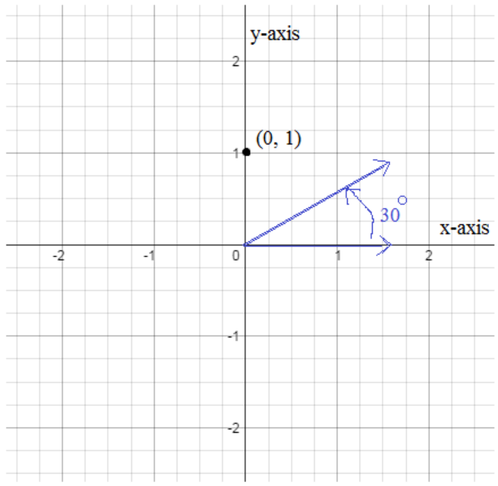
Rotation of Axes

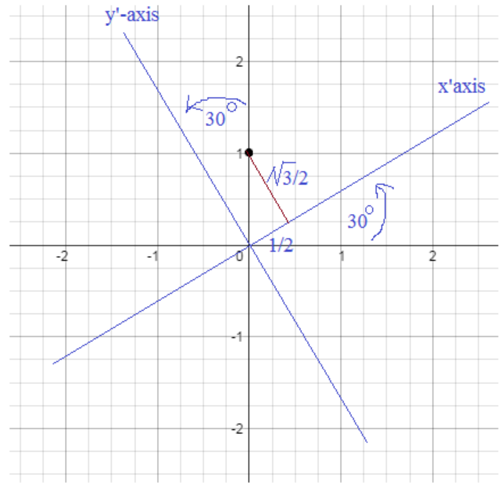
Determine the x'y' coordinates of a given point if the coordinate axes are rotated through a given angle.

Example: (0, 1) 30°

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$





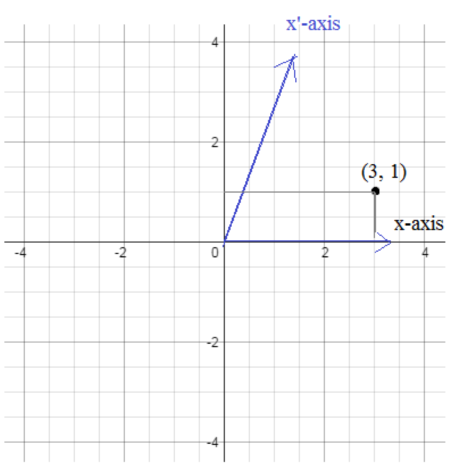
$x' = 0\cos(30) + 1\sin(30)$ $\quad x' = 1/2$

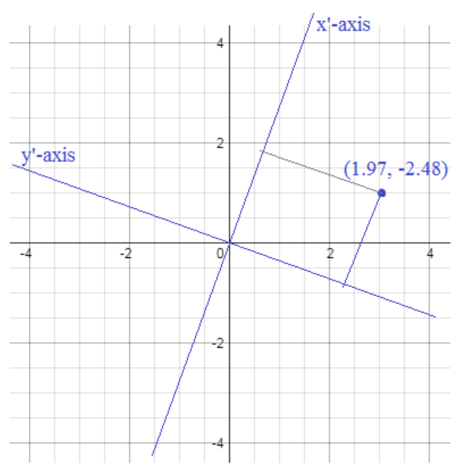
$y' = -0\sin(30) + 1\cos(30)$ $\quad y' = \sqrt{3}/2$

The coordinates of the point *related to the xy-axes* (0, 1)

The coordinates of the point *related to the rotated x'y'-axis* $(1/2, \sqrt{3}/2)$

Example: (3, 1) 70°





$x' = 3\cos(70) + 1\sin(70)$ $\quad x' = 1.97$

$y' = -3\sin(70) + 1(\cos 70)$ $\quad y' = -2.48$

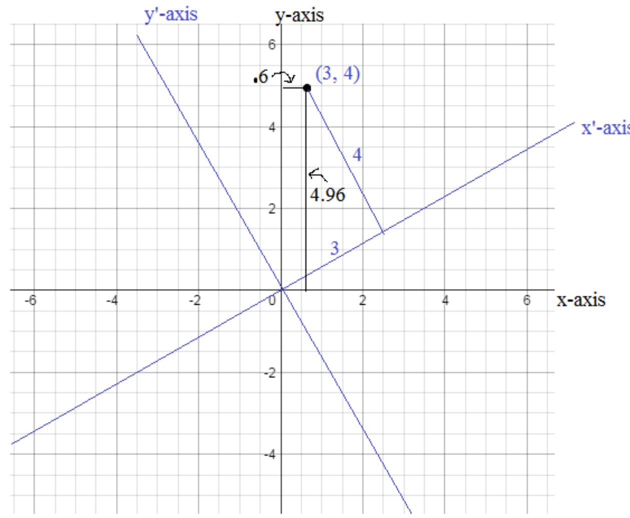
The coordinates of the point *related to the xy-axes* (3, 1)

The coordinates of the point *related to the rotated x'y'-axis* (1.97, −2.48)

Rotation of Axes

Determine the original xy-coordinates from a given point in a rotated x'y'-coordinate axes.

Example: (3, 4) inside a 30 degree rotated xy-axes



$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

$x = 3\cos(30) - 4\sin(30) \quad \frac{3\sqrt{3}}{2} - 2 \quad = .60$

$y = 3\sin(30) + 4\cos(30) \quad \frac{3}{2} + 2\sqrt{3} \quad = 4.96$

Application/Example: Show that xy = 4 is a conic rotated though an angle of 45 degrees.

$x = x'\cos(45) - y'\sin(45) \qquad x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'$

$y = x'\sin(45) + y'\cos(45) \qquad y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$

$x = \frac{\sqrt{2}}{2}(x' - y')$

$y = \frac{\sqrt{2}}{2}(x' + y')$

Then, substitute:

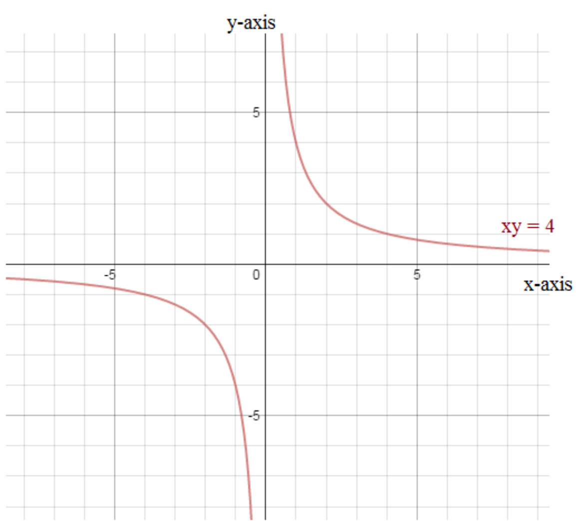
$xy = 4$

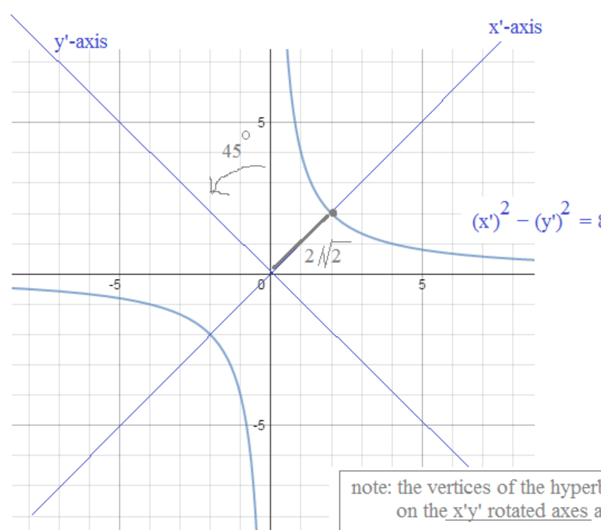
$\frac{\sqrt{2}}{2}(x' - y') \cdot \frac{\sqrt{2}}{2}(x' + y') = 4$

$\frac{2}{4}(x' - y') \cdot (x' + y') = 4$

$(x'^2 - y'^2) = 8$

Hyperbola!

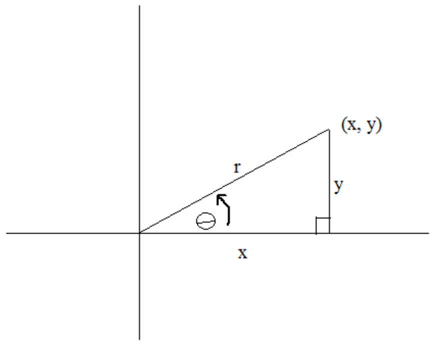




note: the vertices of the hyperbola on the x'y' rotated axes are

$(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$

Where does the rotation formula come from?

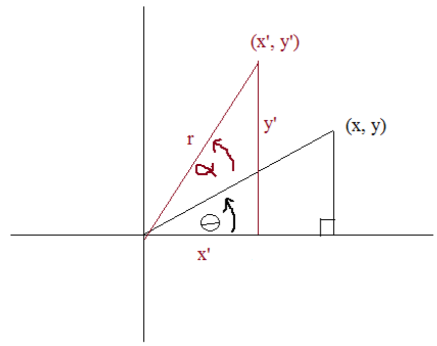


$\cos\theta = \dfrac{x}{r}$ $\qquad x = r\cos\theta$

$\sin\theta = \dfrac{y}{r}$ $\qquad y = r\sin\theta$

Suppose we want to rotate the point counter-clockwise α degrees around the origin.

(This is the same as rotating the x and y-axes clockwise)



What is $(\theta + \alpha)$?

θ is the original angle

α is the rotated angle (counterclockwise)

$x' = r\cos(\theta + \alpha)$

$y' = r\sin(\theta + \alpha)$

Using trigonometry addition identities

$x' = r[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$

$y' = r[\sin\theta\cos\alpha + \cos\theta\sin\alpha]$

remember, $r = \dfrac{x}{\cos\theta}$ and $r = \dfrac{y}{\sin\theta}$

Using substitution...

$$x' = \frac{x}{\cancel{\cos\theta}}\cancel{\cos\theta}\cos\alpha - \frac{y}{\cancel{\sin\theta}}\cancel{\sin\theta}\sin\alpha$$

$$y' = \frac{y}{\cancel{\sin\theta}}\cancel{\sin\theta}\cos\alpha + \frac{x}{\cancel{\cos\theta}}\cancel{\cos\theta}\sin\alpha$$

$x' = x\cos\alpha - y\sin\alpha$

$y' = y\cos\alpha + x\sin\alpha$

$x' = x\cos\alpha - y\sin\alpha$

$y' = x\sin\alpha + y\cos\alpha$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

General Form: $A^2 + Bxy + C^2 + Dx + Ey + F = 0$

$$B^2 - 4AC < 0 \Rightarrow A'C' > 0 \Rightarrow A' \text{ and } C' \text{ are the same sign} \Rightarrow \text{is an } \underline{\text{ellipse}} \;;$$

$$B^2 - 4AC > 0 \Rightarrow A'C' < 0 \Rightarrow A' \text{ and } C' \text{ are of different sign} \Rightarrow \text{is a } \underline{\text{hyperbola}} \;;$$

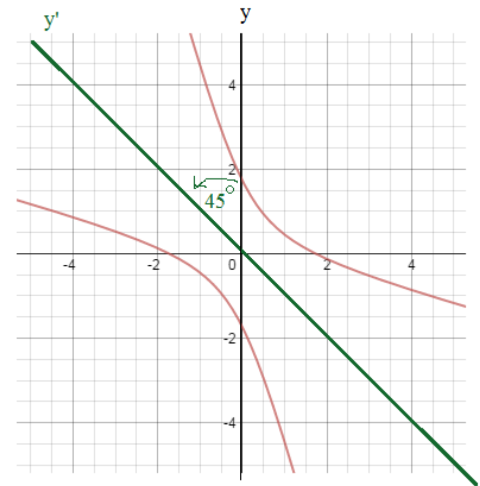
$$B^2 - 4AC = 0 \Rightarrow A'C' = 0 \Rightarrow A' \text{ or } C' \text{ is zero} \Rightarrow \text{is a } \underline{\text{parabola}} \;.$$

Example: $x^2 + 4xy + y^2 - 3 = 0$

What type of conic is it?

It appears to be a circle, because the A and C terms are the same..
But, there is a B term...

$\boxed{B^2 - 4AC = 12 > 0 \quad \text{therefore, it is a hyperbola!}}$



Rotate the axes so that the new expression contains no "xy" term.

$$\cot(2\theta) = \frac{A - C}{B}$$

$$\cot(2\theta) = \frac{1 - 1}{4} = 0$$

$$2\theta = 90^\circ$$

$$\theta = 45^\circ$$

Convert the x and y coordinates into x' and y' terms...

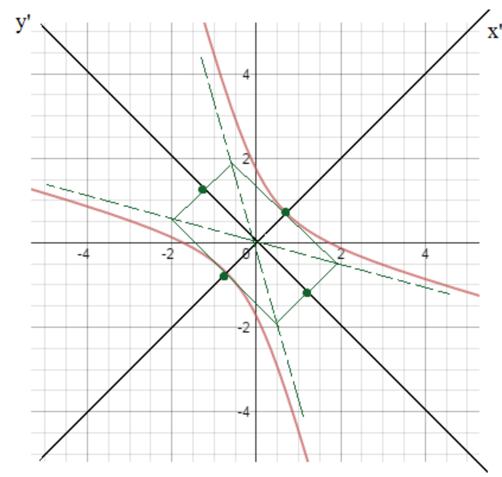
$$\boxed{\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}}$$

$$x = x'\cos(45) - y'\sin(45)$$

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'$$

$$y = x'\sin(45) + y'\cos(45)$$

$$y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$



Substitute and simplify...

$$x^2 + 4xy + y^2 - 3 = 0$$

$$\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 + 4\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + \left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 3$$

$$\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 + 4\left(\frac{1}{2}x'^2 - \frac{1}{2}y'^2\right) + \frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2 = 3$$

Note: the x'y' term disappears
because there is no rotation!

$$3x'^2 - y'^2 = 3$$

$$\frac{x'}{1} - \frac{y'}{3} = 1$$

center: (0, 0)

vertex: (1, 0) and (-1, 0) on the x'y'-coordinate plane..

foci: (2, 0) and (-2, 0) on the x'y'-coordiante plane..

asymptotes: $y' = \sqrt{3}\,x'$ and $y' = -\sqrt{3}\,x'$

Example: Given $17x^2 + 6xy + 9y^2 = 72$

Find the angle of rotation for the axes that will align this conic (and eliminate the xy-term).

Define the sine and cosine of this angle.

Then, find the equation of the conic relative to the rotated axes.

First, what is the conic? $B^2 - 4AC = (6)^2 - 4(17)(9) \Longrightarrow$ less than 0; therefore it's an ELLIPSE

Now, to find the angle of rotation....

$a = 1$

$2(9-17)b + 6(1-b^2) = 0$

solve for b...

$-16b + 6 - 6b^2 = 0$

$3b^2 + 8b - 3 = 0$

$(3b-1)(b+3) = 0$

$b = \frac{1}{3}$ or -3

Note: one rotates clockwise; the other rotates counterclockwise... (Either way will eliminate the xy-term)

For $Ax^2 + Bxy + Cy^2 + D = 0$

the angle of rotation is

$\sin\theta = \frac{b}{c}$ $\quad\cos\theta = \frac{a}{c}$

found from: $a = 1$

$2(C-A)b + B(1-b^2) = 0$

$a^2 + b^2 = c^2$

We'll use $b = -3$

$a^2 + b^2 = c^2$ $\quad (1)^2 + (-3)^2 = c^2$

$c = \sqrt{10}$

$\sin\theta = \frac{-3}{\sqrt{10}}$

$\cos\theta = \frac{1}{\sqrt{10}}$

rotation of axes

$x = \cos\theta\, x' - \sin\theta\, y'$

$y = \sin\theta\, x' + \cos\theta\, y'$

$x = \frac{1}{\sqrt{10}}x' - \frac{-3}{\sqrt{10}}y'$

$y = \frac{-3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'$

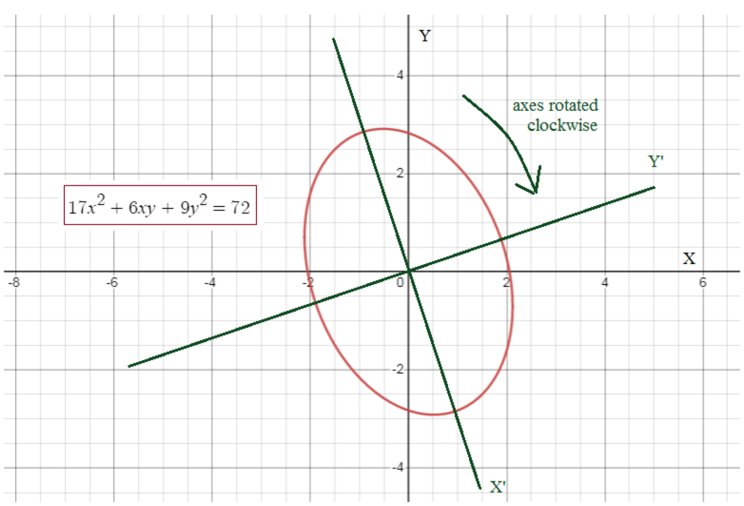
Substitute into the original equation $17x^2 + 6xy + 9y^2 = 72$

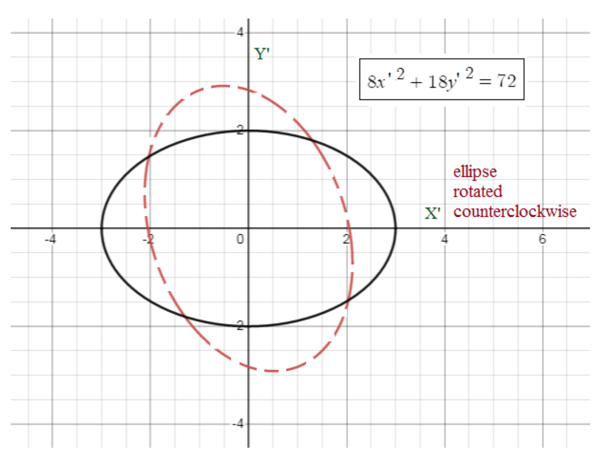
$$17\left(\frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y'\right)^2 + 6\left(\frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y'\right)\left(\frac{-3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'\right) + 9\left(\frac{-3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'\right)^2 = 72$$

$$17\left(\frac{1}{10}x'^2 + \frac{6}{10}x'y' + \frac{9}{10}y'^2\right) + 6\left(\frac{-3}{10}x'^2 - \frac{8}{10}x'y' + \frac{3}{10}y'^2\right) + 9\left(\frac{9}{10}x'^2 - \frac{6}{10}x'y' + \frac{1}{10}y'^2\right) = 72$$

$$\frac{80}{10}x'^2 + 0x'y' + \frac{180}{10}y'^2 = 72 \Longrightarrow \boxed{8x'^2 + 18y'^2 = 72 \quad\text{or}\quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1}$$

As expected, the x'y' term is zero





Example: Identify the following rotated conic.
Then, rotate the axes to eliminate the xy term.

$$x^2 + 4xy - 2y^2 - 6 = 0$$

To identify the conic, find $B^2 - 4AC$

$$4^2 - 4(1)(-2) = \boxed{24 > 0} \Longrightarrow \text{hyperbola}$$

To find the angle of rotation, we'll use

$$\boxed{\operatorname{Tan}(2\theta) = \frac{B}{A - C}}$$

$$\operatorname{Tan}(2\theta) = \frac{4}{1 - (-2)}$$

$$\frac{2\operatorname{Tan}\theta}{1 - \operatorname{Tan}^2\theta} = \frac{4}{3}$$

$$4(1 - \operatorname{Tan}^2\theta) = 6\operatorname{Tan}\theta$$

$$4\operatorname{Tan}^2\theta + 6\operatorname{Tan}\theta - 4 = 0$$

$$2\operatorname{Tan}^2\theta + 3\operatorname{Tan}\theta - 2 = 0$$

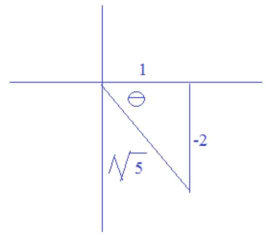
$$(2\operatorname{Tan}\theta - 1)(\operatorname{Tan}\theta + 2) = 0$$

$\operatorname{Tan}\theta = 1/2$

or

$\boxed{\operatorname{Tan}\theta = -2}$

We can use either..
(One rotates clockwise, and one rotates counterclockwise..
But, either will eliminate the xy term)



$$\boxed{\begin{aligned} \operatorname{Cos}\theta &= \frac{1}{\sqrt{5}} \\ \operatorname{Sin}\theta &= \frac{-2}{\sqrt{5}} \end{aligned}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \operatorname{Cos}\theta & -\operatorname{Sin}\theta \\ \operatorname{Sin}\theta & \operatorname{Cos}\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Rotating the axes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \Longrightarrow \begin{aligned} x &= \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y' \\ y &= \frac{-2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' \end{aligned}$$

$$x^2 + 4xy - 2y^2 - 6 = 0$$

Substitute:

$$\left(\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right)^2 + 4\left(\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'\right)\left(\frac{-2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right) - 2\left(\frac{-2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right)^2 - 6 = 0$$

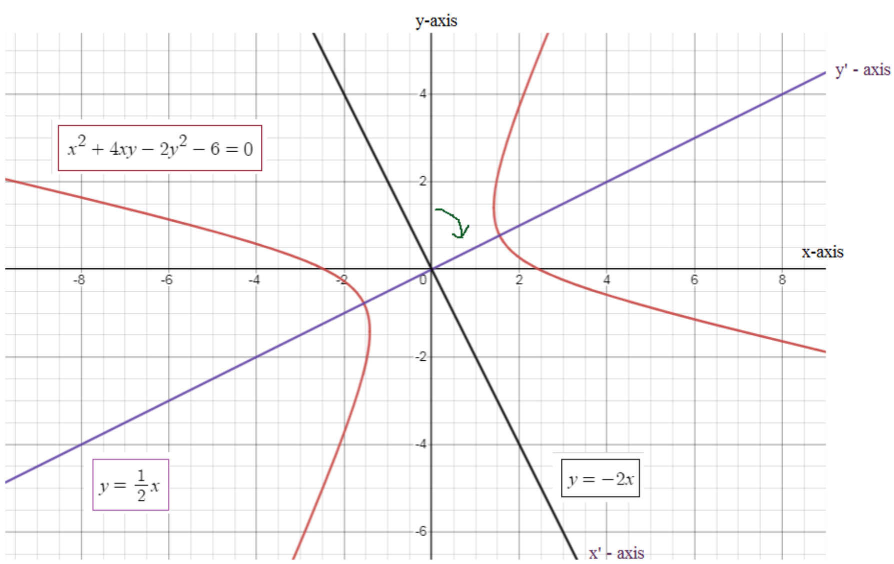
Expand:

$$\frac{1}{5}x'^2 + \frac{4}{5}x'y' + \frac{4}{5}y'^2 \quad - \frac{8}{5}x'^2 + \frac{4}{5}x'y' - \frac{16}{5}x'y' + \frac{8}{5}y'^2 \quad - \frac{8}{5}x'^2 + \frac{8}{5}x'y' - \frac{2}{5}y'^2 - 6 = 0$$

Collect terms:

$$\frac{-15}{5}x'^2 + 0x'y' + \frac{10}{5}y'^2 - 6 = 0 \Longrightarrow -3x'^2 + 2y'^2 = 6 \quad \text{or} \quad \boxed{\frac{y'^2}{3} - \frac{x'^2}{2} = 1}$$

Note: the x'y' term is 0



For the conic $16x^2 + 24xy + 9y^2 + 105x + 110y + 225 = 0$,

a) Identify the conic

b) Find the angle of rotation that aligns the axes with this conic

c) Sketch a graph

For $Ax^2 + Bxy + Cy^2 + D = 0$

the angle of rotation is

$\sin\theta = \frac{b}{c}$ found from: $a = 1$

$\cos\theta = \frac{a}{c}$ $2(C - A)b + B(1 - b^2) = 0$

$a^2 + b^2 = c^2$

a) $B^2 - 4AC = (24)^2 - 4(16)(9) = 0$

$\Longrightarrow$ PARABOLA

b) $A = 16$
$B = 24$
$C = 9$

$a = 1$

$2(9 - 16)b + 24(1 - b^2) = 0$

$-14b + 24 - 24b^2 = 0$

$12b^2 + 7b - 12 = 0$

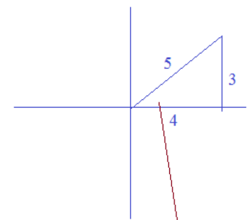
$(4b - 3)(3b + 4) = 0$

$b = 3/4$ or $-4/3$

rotation of axes

$x = \cos\theta\, x' - \sin\theta\, y'$

$y = \sin\theta\, x' + \cos\theta\, y'$



37 degree angle

for convenience, we'll select the postive value 3/4

$a^2 + b^2 = c^2$

$1 + 9/16 = c^2$

$c = 5/4$

$\cos\theta = \frac{4}{5}$

$\sin\theta = \frac{3}{5}$

$x = \frac{4}{5}x' - \frac{3}{5}y'$

$y = \frac{3}{5}x' + \frac{4}{5}y'$

$16x^2 + 24xy + 9y^2 + 105x + 110y + 225 = 0$,

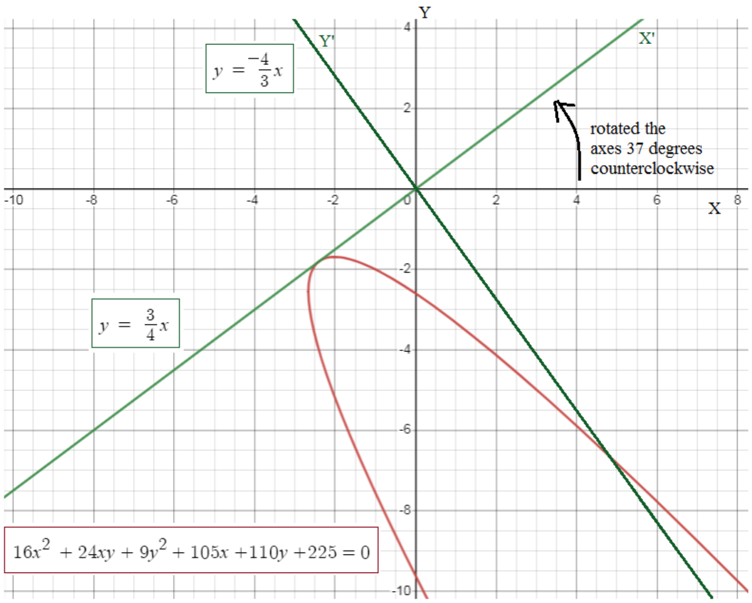
$$16\left(\frac{4}{5}x' - \frac{3}{5}y'\right)^2 + 24\left(\frac{4}{5}x' - \frac{3}{5}y'\right)\left(\frac{3}{5}x' + \frac{4}{5}y'\right) + 9\left(\frac{3}{5}x' + \frac{4}{5}y'\right)^2 + 105\left(\frac{4}{5}x' - \frac{3}{5}y'\right) + 110\left(\frac{3}{5}x' + \frac{4}{5}y'\right) + 225 = 0$$

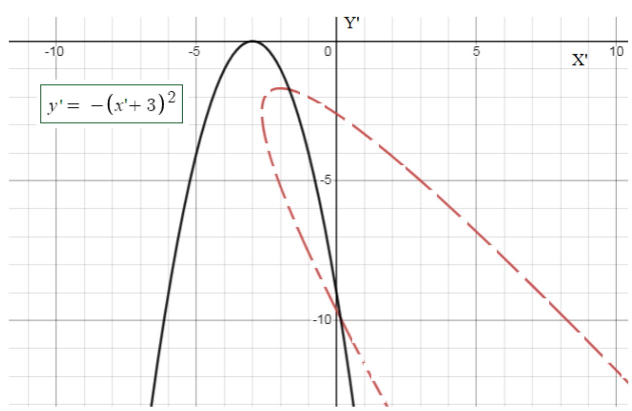
simplifies to $\boxed{25x'^2 + 150x' + 25y' + 225 = 0}$ $x'^2 + 6x' + y' + 9 = 0$

$y' = -1(x'^2 + 6'x + 9)$

or

$\boxed{y' = -(x + 3)^2}$





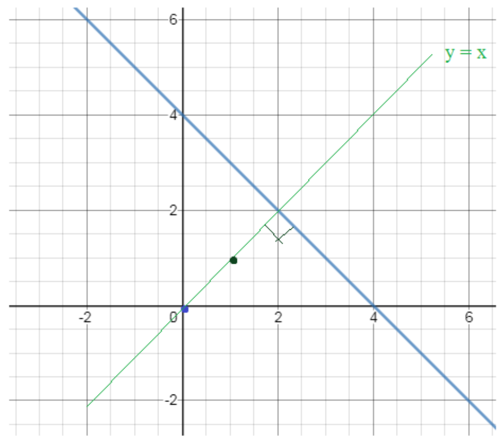
Example: Find the equation of the parabola with focus (0, 0) and directrix $x + y = 4$

First, let's find the vertex....

We know the vertex is equidistant from the focus and directrix....

focus: (0, 0)

directrix: $y = -x + 4$



Using geometry, we can determine the vertex...

Since directrix slope is -1, we know the slope of a perpendicular line is 1...

and, since the perpendicular line goes through (0, 0), we have $y = x$

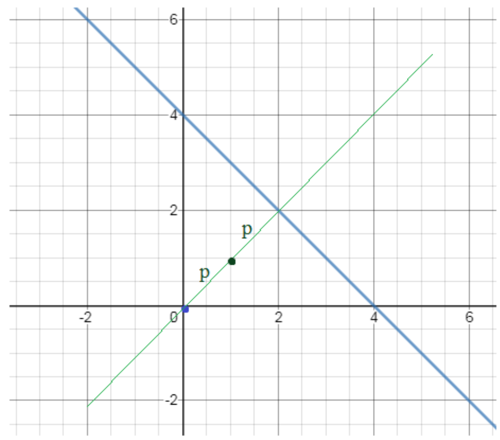
then, solving system of equations, we know the intersection of

$y = x$ and $y = -x + 4$ is (2, 2)

The intersection is (2, 2).... therefore, the midpoint between the focus (0, 0) and the intersection (2, 2) is (1, 1)

We now know the vertex is (1, 1)....

and, we see that the directrix has been rotated 45 degrees....



***definition of parabola: any point on the parabola is equidistant to the focus and directrix!

$x + y = 4$ -----> $x + y - 4 = 0$ A = 1, B = 1, C = -4

Distance from a point to a line: (directrix) $\dfrac{Ax + By + C}{\sqrt{A^2 + B^2}}$ -----> $\dfrac{x + y + (-4)}{\sqrt{1^2 + 1^2}}$

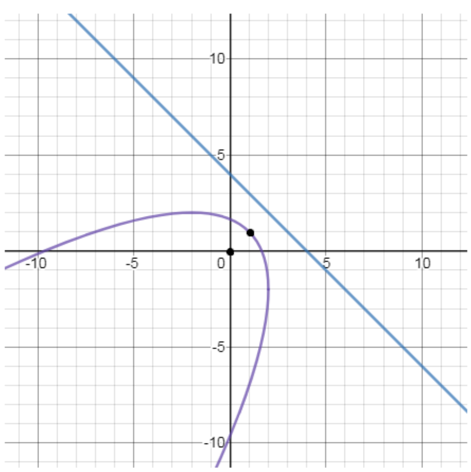
Distance from a point to the focus: $\sqrt{(x - 0)^2 + (y - 0)^2}$

focus: (0, 0) $= \sqrt{x^2 + y^2}$

Let's remember the definition of a parabola....

distance p from vertex to directrix (or focus) is

$p = \sqrt{(1 - 2)^2 + (1 - 2)^2} = \sqrt{2}$



Distance from any point to the directrix

Distance from any point to the focus

$$\frac{x + y + (-4)}{\sqrt{1^2 + 1^2}} = \sqrt{x^2 + y^2}$$

cross multiply....

$$\sqrt{2x^2 + 2y^2} = x + y - 4$$

square both sides...

$$2x^2 + 2y^2 = (x + y - 4)^2$$

$$2x^2 + 2y^2 = x^2 + xy + (-4x) + xy + y^2 + (-4y) - 4x - 4y + 16$$

collect like terms...

$$x^2 + y^2 = 2xy - 8x - 8y + 16$$

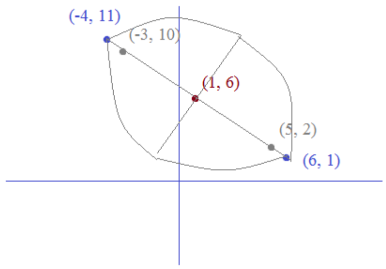
$$\boxed{x^2 - 2xy + y^2 + 8x + 8y - 16 = 0}$$

Example: Given: Vertices (-4, 11) and (6, 1)

Foci (-3, 10) and (5, 2)

Find the equation of the ellipse...

Step 1: make a sketch and find properties of ellipse...



The midpoint of the vertices is the center (1, 6)

$$\frac{(x-1)^2}{\quad} + \frac{(y-6)^2}{\quad} = 1$$

distance from center to each focus

$$c = \sqrt{(-3-1)^2 + (10-6)^2} = \sqrt{32}$$

$$c^2 = a^2 + b^2$$

$$a^2 = 50$$

$$c^2 = 32$$

$$\Rightarrow \quad b^2 = 18$$

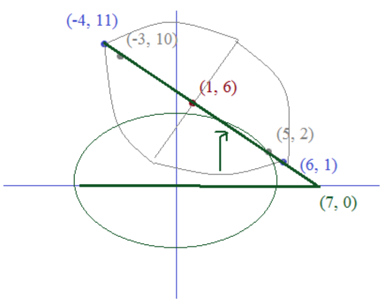
and, the distance from the center to each vertex is

$$a = \sqrt{(-4-1)^2 + (11-6)^2} = \sqrt{50}$$

$$\frac{(x-1)^2}{50} + \frac{(y-6)^2}{b^2} = 1$$

$$\frac{(x-1)^2}{50} + \frac{(y-6)^2}{18} = 1$$

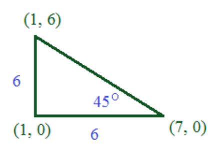
Step 2: find the angle of rotation...



The slope of the major axis is -1...

The x-axis is horizontal...

This ellipse is rotated 45 degrees *clockwise*



Step 3: shift and rotate the ellipse...

rotate 45 degrees clockwise — after shifting center to origin — translate < 1, 6 >

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(-45) & -\sin(-45) \\ \sin(-45) & \cos(-45) \end{bmatrix} \begin{bmatrix} x-1 \\ y-6 \end{bmatrix} + \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

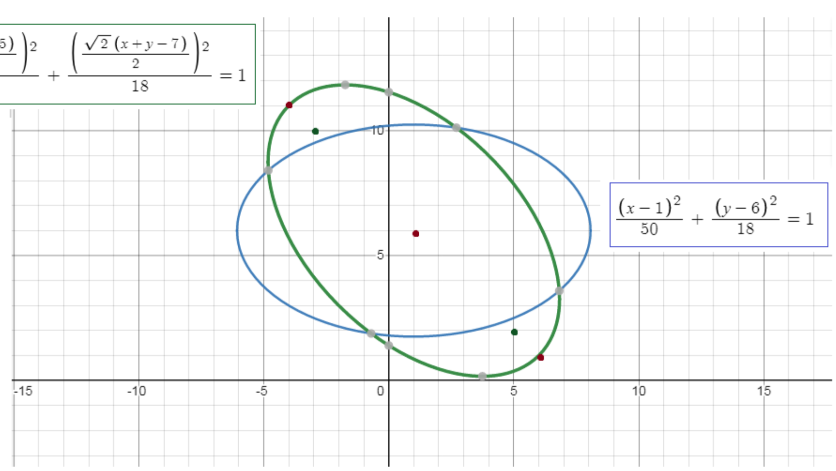
$$\begin{bmatrix} x'-1 \\ y'-6 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x-1 \\ y-6 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^{-1} \begin{bmatrix} x'-1 \\ y'-6 \end{bmatrix} = \begin{bmatrix} x-1 \\ y-6 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{2}\,(x'-y'+5)}{2} \\ -\frac{\sqrt{2}\,(x'+y'-7)}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Step 4: Substitute for x and y...

$$\frac{(x-1)^2}{50} + \frac{(y-6)^2}{18} = 1 \Rightarrow \frac{\left(\frac{\sqrt{2}\,(x'-y'+5)}{2} + 1 - 1\right)^2}{50} + \frac{\left(-\frac{\sqrt{2}\,(x'+y'-7)}{2} + 6 - 6\right)^2}{18} = 1$$

$$\frac{\left(\frac{(x-y+5)}{2}\right)^2}{50} + \frac{\left(\frac{\sqrt{2}\,(x+y-7)}{2}\right)^2}{18} = 1$$

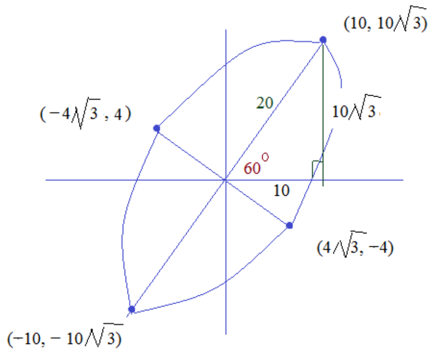


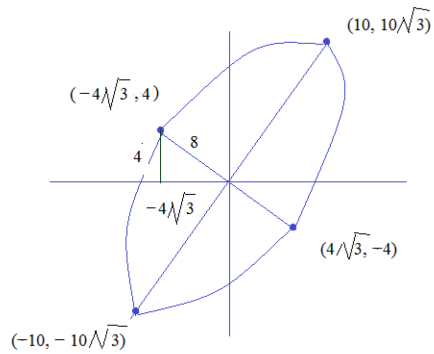
$$\frac{(x-1)^2}{50} + \frac{(y-6)^2}{18} = 1$$

Example: Find the equation of the ellipse with vertices $(10, 10\sqrt{3})$ and $(-10, -10\sqrt{3})$

and co-vertices $(4\sqrt{3}, -4)$ and $(-4\sqrt{3}, 4)$

Step 1: Draw a diagram and find the ellipse's properties





The midpoint of the vertices and co-vertices is the center (0, 0)

The "a" value (or, semi-major axis) is 20

The "b" value (or, semi-minor axis) is 8

$$\Rightarrow \quad \frac{x^2}{400} + \frac{y^2}{64} = 1$$

Step 2: Identify the rotation and write the transformation

The 'original' ellipse above is rotated 60 degrees counterclockwise...

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow$$

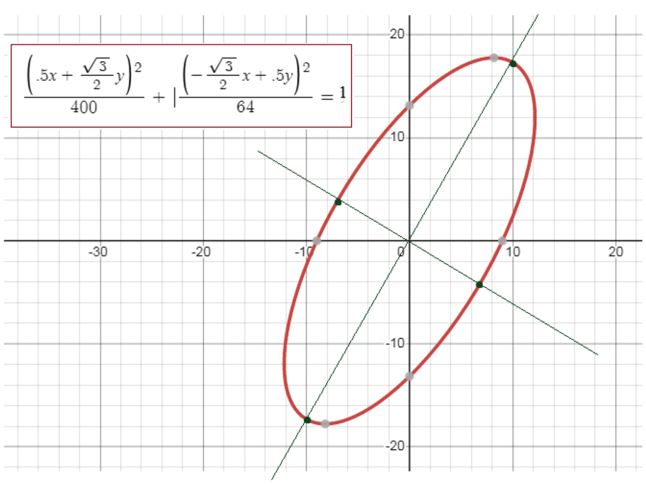
Find x and y....

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(60) & \sin(60) \\ -\sin(60) & \cos(60) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$$

$$y = \frac{-\sqrt{3}}{2}x' + \frac{1}{2}y'$$

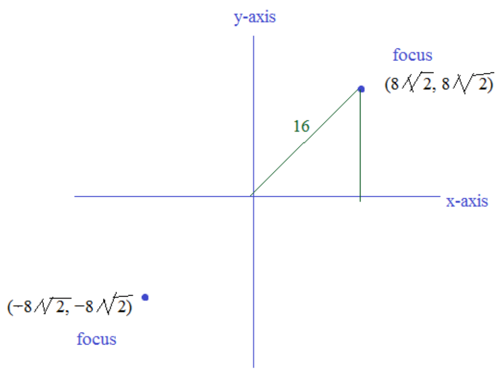


Step 3: Substitute into the 'original' ellipse...

$$\frac{x^2}{400} + \frac{y^2}{64} = 1 \quad \Rightarrow \quad \boxed{\frac{\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2}{400} + \frac{\left(\frac{-\sqrt{3}}{2}x' + \frac{1}{2}y'\right)^2}{64} = 1}$$

Example: A hyperbola whose foci are $(8\sqrt{2}, 8\sqrt{2})$ and $(-8\sqrt{2}, -8\sqrt{2})$ has asymptotes on the x-axis and y-axis.

Find the equation of the hyperbola.



The midpoint of the hyperbola is the center (0, 0)

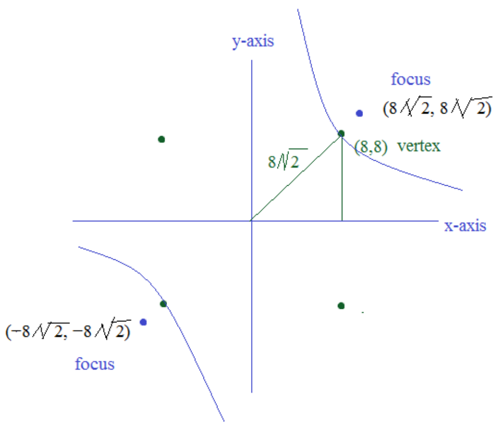
Distance from (0, 0) to each focus is the c value 16

Since the x-axis and y-axis are the asymptotes, they are perpendicular and imply that a = b

$$a^2 + b^2 = c^2$$

$$a^2 + b^2 = 16^2 \qquad \text{and} \quad a = b$$

$$a^2 = b^2 = 128 \qquad a = b = 8\sqrt{2}$$



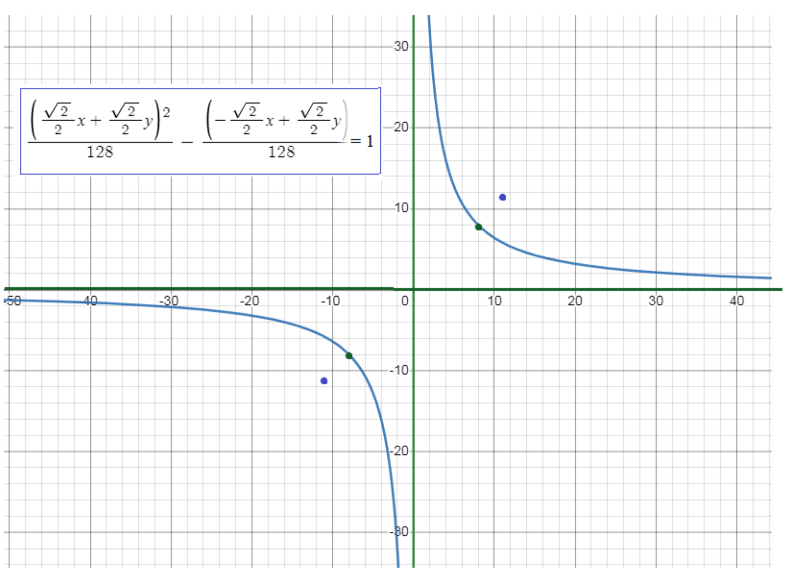
$$\frac{x^2}{128} - \frac{y^2}{128} = 1$$

This is the equation of the hyperbola using a, b, and c values...

Now, we will reorient and rotate it 45 degrees counterclockwise!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \cos(45) & \sin(45) \\ -\sin(45) & \cos(45) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\frac{x^2}{128} - \frac{y^2}{128} = 1 \qquad x = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

$$y = -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

Then, substitute into the hyperbola numbers..

$$\boxed{\frac{\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2}{128} - \frac{\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2}{128} = 1}$$

Using Matrices for rotation of axes

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Clockwise rotation of θ degrees

To find x and y, we use the inverse matrix!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

inverse

1

$$\boxed{\frac{1}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}}$$

Example. For the following ellipse $3(x-4)^2 + (y+2)^2 = 27$

Rotation of axes using matrices

Find the equation of the ellipse after it is rotated 45 degrees counterclockwise

a) around the origin

b) around the center of the ellipse

a) rotation 45 degrees around the origin...

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Multiply each side by the inverse of the rotation matrix...

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

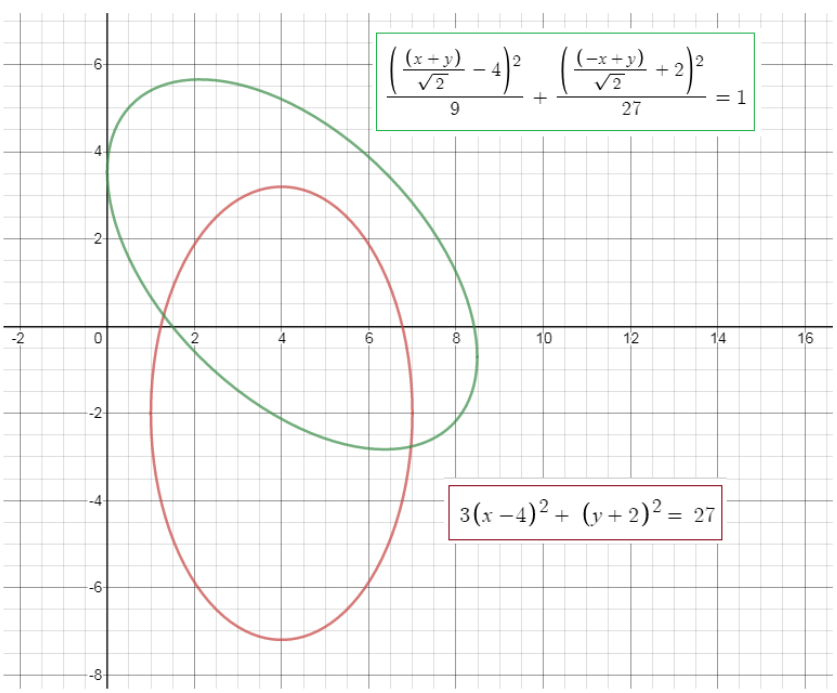
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

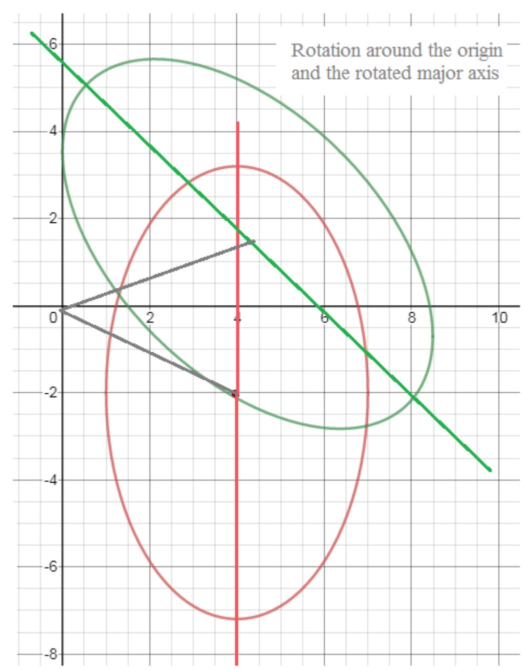
$$x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \Longrightarrow \frac{x' + y'}{\sqrt{2}}$$

$$y = \frac{-1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \Longrightarrow \frac{-x' + y'}{\sqrt{2}}$$

Substitute into original equation...

$$3(x-4)^2 + (y+2)^2 = 27 \Longrightarrow \frac{\left(\frac{x' + y'}{\sqrt{2}} - 4\right)^2}{9} + \frac{\left(\frac{-x' + y'}{\sqrt{2}} + 2\right)^2}{27} = 1$$





Example (continued): For the following ellipse $3(x-4)^2 + (y+2)^2 = 27$

Rotation of axes using matrices

Find the equation of the ellipse after it is rotated 45 degrees counterclockwise

b) around the center of the ellipse

b) rotating around the center of the ellipse....

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x-4 \\ y+2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

(2) rotation 45 degrees

(1) moving each point to the origin

(3) reapplying the shift back to the rotated point

Using matrix algebra, we'll solve for x and y...

$$\begin{bmatrix} x' - 4 \\ y' + 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x-4 \\ y+2 \end{bmatrix}$$

Multiply each side by the inverse of the rotation matrix...

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' - 4 \\ y' + 2 \end{bmatrix} = \begin{bmatrix} x-4 \\ y+2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{x' - 4}{\sqrt{2}} + \frac{y' + 2}{\sqrt{2}} \\ \frac{-x' + 4}{\sqrt{2}} + \frac{y' + 2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} x-4 \\ y+2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{x' - 4}{\sqrt{2}} + \frac{y' + 2}{\sqrt{2}} + 4 \\ \frac{-x' + 4}{\sqrt{2}} + \frac{y' + 2}{\sqrt{2}} - 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Finding the inverse of the rotation matrix...

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} =$$

$$\Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

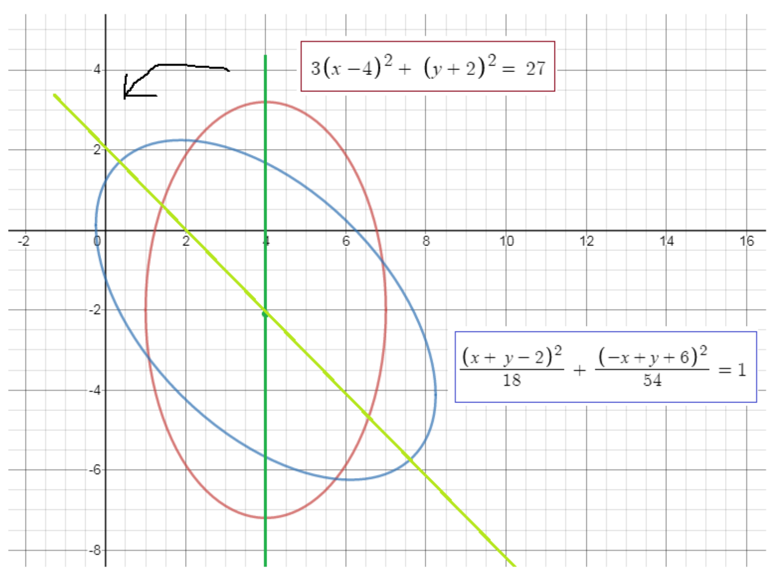
Then, plug into original equation....

$$3(x-4)^2 + (y+2)^2 = 27$$

$$3\left(\frac{x' - 4}{\sqrt{2}} + \frac{y' + 2}{\sqrt{2}} + 4 - 4\right)^2 + \left(\frac{-x' + 4}{\sqrt{2}} + \frac{y' + 2}{\sqrt{2}} - 2 + 2\right)^2 = 27$$

$$3\left(\frac{x' + y' - 2}{\sqrt{2}}\right)^2 + \left(\frac{-x' + y' + 6}{\sqrt{2}}\right)^2 = 27$$

$$\frac{(x' + y' - 2)^2}{18} + \frac{(-x' + y' + 6)^2}{54} = 1$$



Invariant Circle

Show that the equation $x^2 + y^2 = 49$ is invariant under any rotation.

Intuitively, we know this equation is invariant, because it's a circle centered at the origin.
So, any rotation, and it remains a circle centered at the origin...

Let's prove it algebraically...

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

is any rotation of angle θ (rotation could be clockwise or counter-clockwise)

So, we substitute $x = x'\cos\theta + y'\sin\theta$, $y = -x'\sin\theta + y'\cos\theta$ into the original equation....

$x^2 + y^2 = 49$

$$(x'\cos\theta + y'\sin\theta)^2 + (-x'\sin\theta + y'\cos\theta)^2 = 49$$

$$x'^2\cos^2\theta + \cancel{2x'y'\cos\theta\sin\theta} + y'^2\sin^2\theta + x'^2\sin^2\theta - \cancel{2x'y'\cos\theta\sin\theta} + y'^2\cos^2\theta = 49$$

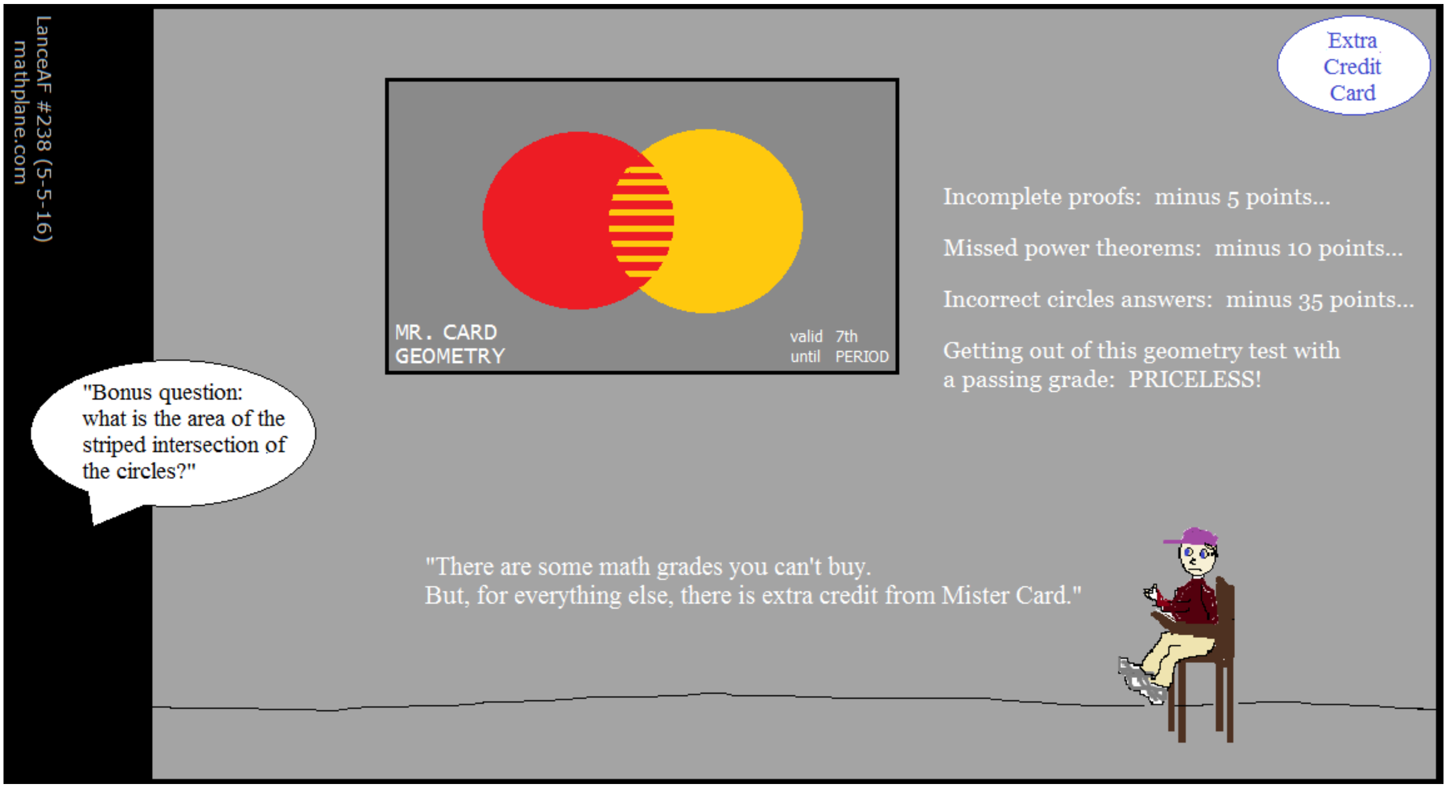
cancel, rearrange, and factor...

$$x'^2\cos^2\theta + x'^2\sin^2\theta + y'^2\sin^2\theta + y'^2\cos^2\theta = 49$$

$$x'^2(\cos^2\theta + \sin^2\theta) + y'^2(\cos^2\theta + \sin^2\theta) = 49$$

trigonometry identity..

$$x'^2 + y'^2 = 49$$



Practice Quiz-→

Rotation of Conics Exercise

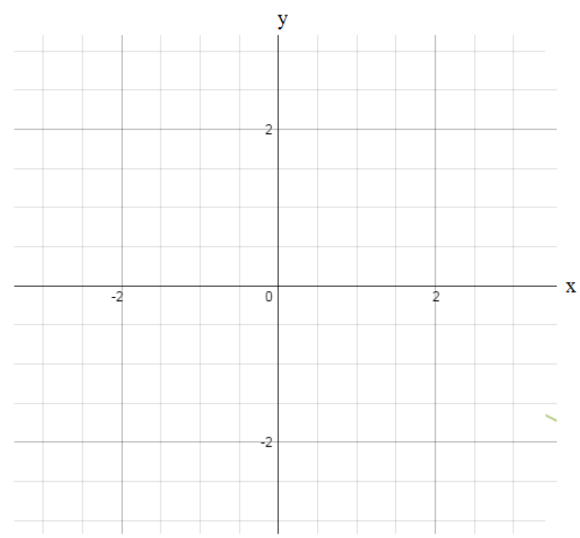
In the following general equations,

a) Identify the conic

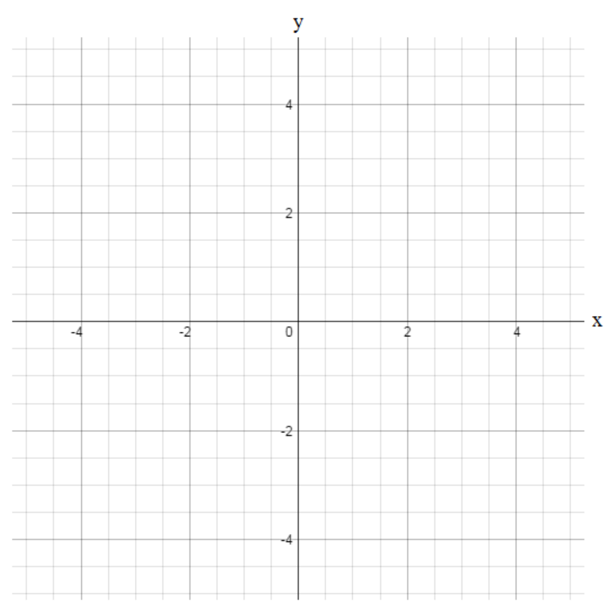
b) Rotate the axes, and write the new expression containing no 'xy' term

c) Graph

1) $6x^2 + 4xy + 9y^2 - 20 = 0$

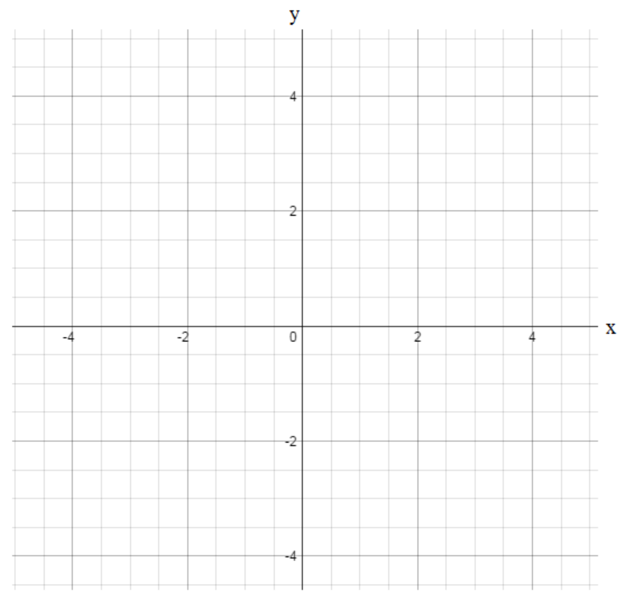


2) $4x^2 - 12xy + 9y^2 + 12x + 8y = 0$

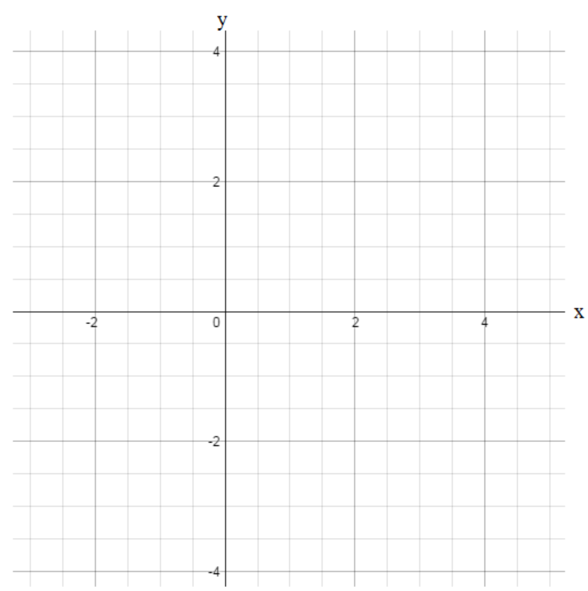


Rotation of Conics Exercise

3) $2x^2 - 8xy + 2y^2 - 6 = 0$

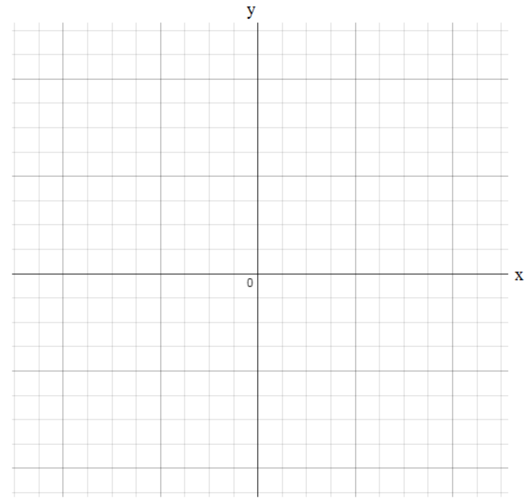


4) $4x^2 - 6xy + 4y^2 - 6y - 2 = 0$

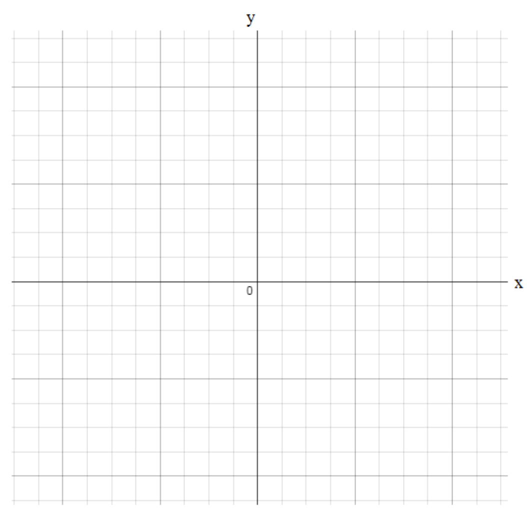


Rotation of Conics Exercise

5) $7x^2 + 6xy - y^2 - 32 = 0$

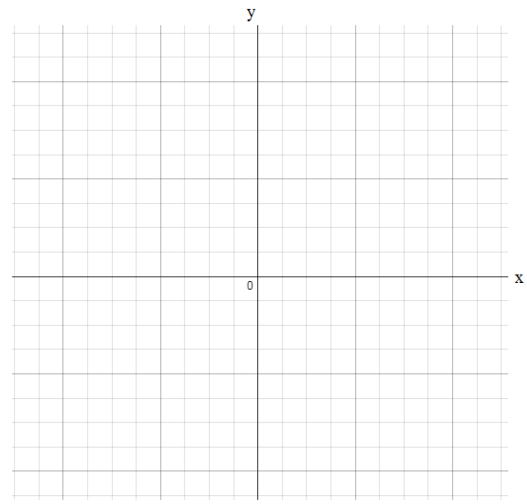


6) $4x^2 + 12xy + 9y^2 + 8\sqrt{13}\,x + 12\sqrt{13}\,y - 65 = 0$

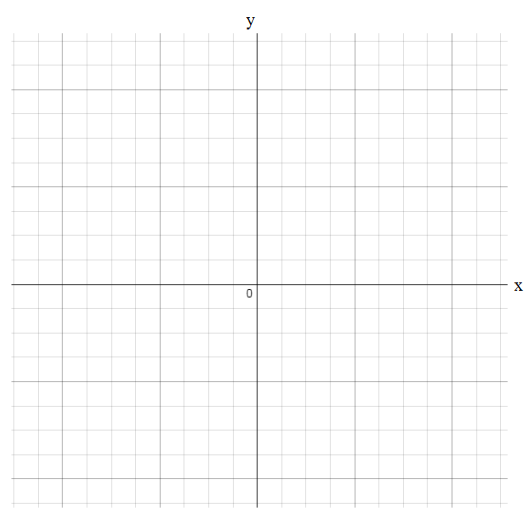


Rotation of Conics Exercise

7) $6x^2 + 4xy + 9y^2 + 27y = 30$



8) $16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0$



In the following general equations,

a) Identify the conic

b) Rotate the axes, and write the new expression containing no 'xy' term

c) Graph

Rotation of Conics Exercise

SOLUTIONS

1) $6x^2 + 4xy + 9y^2 - 20 = 0$

a) $B^2 - 4AC$

$(4)^2 - 4(6)(9) = -200 < 0$

Since less than zero, it's a $\boxed{\text{rotated ellipse...}}$

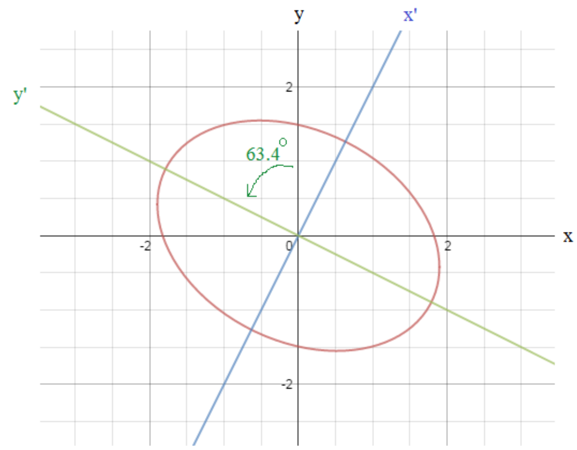
b) $\cot(2\theta) = \dfrac{A - C}{B}$

$\cot(2\theta) = \dfrac{6 - 9}{4} = -3/4$

$\text{arccot}(-3/4) = 2\theta$

$126.87 = 2\theta$

$\boxed{\theta \approx 63.4^\circ}$



$\tan(63.4) = 2$ (slope of x'-axis)

then, $-1/2$ (slope of y'-axis)

c) $x = x'\cos(63.4) - y'\sin(63.4)$

$x = .45x' - .89y'$

$y = x'\sin(63.4) + y'\cos(63.4)$

$y = .89x' + .45y'$

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

then, substitute:

$6x^2 + 4xy + 9y^2 - 20 = 0 \implies 6(.45x' - .89y')^2 + 4(.45x' - .89y')(.89x' + .45y') + 9(.89x' + .45y')^2 = 20$

$6(.20x'^2 - .8x'y' + .79y'^2) + 4(.40x'^2 - .79x'y' + .20x'y' - .40y'^2) + 9(.79x'^2 + .8x'y' + .20y'^2) = 20$

$9.91x'^2 + 0x'y' + 4.94y'^2 = 20$ $\boxed{\dfrac{x'^2}{2} + \dfrac{y'^2}{4} = 1}$ center: (0, 0)

minor semi-axis: 1.4
major semi-axis: 2

2) $4x^2 - 12xy + 9y^2 + 12x + 8y = 0$

a) $B^2 - 4AC$

$(-12)^2 - 4(4)(9) = 0$

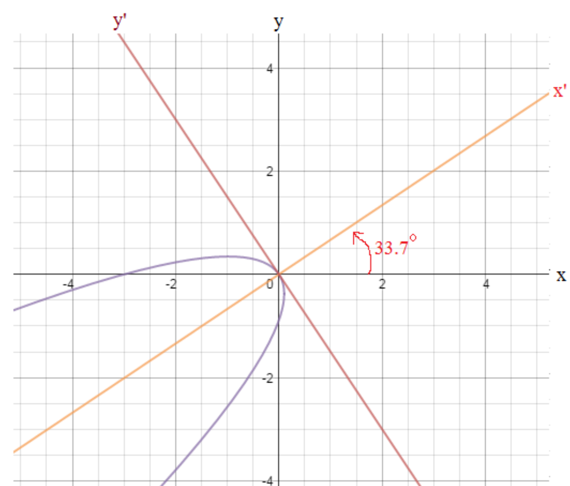
Since it equals 0, it's a $\boxed{\text{rotated parabola...}}$

b) $\cot(2\theta) = \dfrac{A - C}{B}$

$\cot(2\theta) = \dfrac{4 - 9}{-12} = 5/12$

$2\theta = 67.38$

$\boxed{\theta \approx 33.7^\circ}$



$\tan(33.7) = .67$ (slope of x'-axis)

then, -1.5 (slope of y'-axis)

perpendicular

c) $x = x'\cos(33.7) - y'\sin(33.7)$

$x = .83x' - .55y'$

$y = x'\sin(33.7) + y'\cos(33.7)$

$y = .55x' + .83y'$

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

then, substitute..

$4(.83x' - .55y')^2 - 12(.83x' - .55y')(.55x' + .83y') + 9(.55x' + .83y')^2 + 12(.83x' - .55y') + 8(.55x' + .83y') = 0$

$4(.69x'^2 - .91x'y' + .30y'^2) - 12(.46x'^2 + .39x'y' - .46y'^2) + 9(.30x'^2 + .91x'y' + .69y'^2) + 9.96x' - 6.6y' + 4.4x' + 6.64y' = 0$

$0x'^2 + 0x'y' + 12.9y'^2 + 14.35x' + 0y' = 0 \implies 14.35x' = -12.9y'^2$ $\boxed{x' = -.9(y')^2}$

vertex: (0, 0) Opens to the left...

Rotation of Conics Exercise

SOLUTIONS

3) $2x^2 - 8xy + 2y^2 - 6 = 0$

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

a) $B^2 - 4AC$

$(-8)^2 - 4(2)(2) = 48 > 0$

Since it is greater than 0, it's a rotated hyperbola ...

b) $\cot(2\theta) = \dfrac{A - C}{B}$

$\cot(2\theta) = \dfrac{2-2}{-8} = 0$

$2\theta = 90^\circ$

$\boxed{\theta = 45^\circ}$

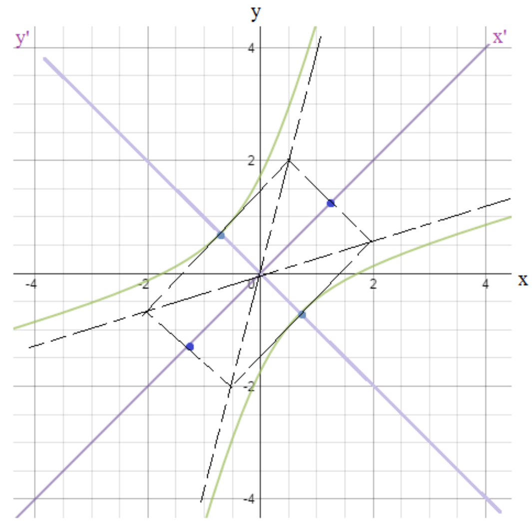
c) $x = x'\cos(45) - y'\sin(45)$

$x = \dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'$

$y = x'\sin(45) + y'\cos(45)$

$y = \dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'$

then, substitute..



$$2\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 - 8\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 2\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 6$$

$$2\left(\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2\right) - 8\left(\frac{1}{2}x'^2 - \frac{1}{2}y'^2\right) + 2\left(\frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2\right) = 6$$

$$-2x'^2 + 0x'y' + 6y'^2 = 6$$

$$\boxed{\frac{y'^2}{1} - \frac{x'^2}{3} = 1}$$

'vertical hyperbola' center: (0, 0)

$\tan(45) = 1$ so, $y = (1)x$ becomes the x'axis

and, $y = (-1)x$ becomes the y'axis

vertex (on the x'y'- coordinate plane): (0, 1) (0, -1)

co-vertex (on the x'y'-coordinate plane): $(\sqrt{3}, 0)$ $(-\sqrt{3}, 0)$

4) $4x^2 - 6xy + 4y^2 - 6y - 2 = 0$

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

a) $B^2 - 4AC$

$(-6)^2 - 4(4)(4) = -28 < 0$

rotated AND shifted ellipse

b) $\tan(2\theta) = \dfrac{B}{A - C}$

$\tan(2\theta) = \dfrac{-6}{4 - 4}$ undefined

$2\theta = 90^\circ$

$\boxed{\theta = 45^\circ}$

c) $x = x'\cos(45) - y'\sin(45)$

$x = \dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'$

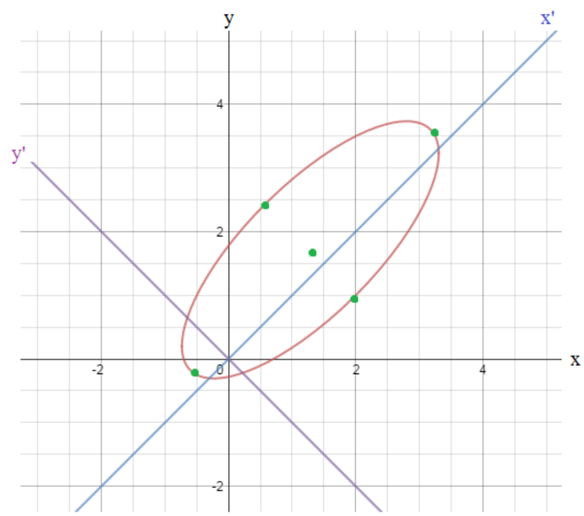
$x = \dfrac{\sqrt{2}}{2}(x' - y')$

$y = x'\sin(45) + y'\cos(45)$

$y = \dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'$

$y = \dfrac{\sqrt{2}}{2}(x' + y')$

then, substitute..



$$2(x' - y')^2 - 3(x'^2 - y'^2) + 2(x' + y')^2 - 3\sqrt{2}\,(x' + y') = 2$$

$$x'^2 + 0x'y' + 7y'^2 - 3\sqrt{2}\,x' - 3\sqrt{2}\,y' = 2$$

(complete the square)

$$x'^2 - 3\sqrt{2}\,x' + \frac{9}{2} + 7\left(y'^2 - \frac{3\sqrt{2}}{7}y' + \frac{18}{196}\right) = 2 + \frac{9}{2} + \frac{18}{28}$$

$$\left(x' - \frac{3}{\sqrt{2}}\right)^2 + 7\left(y' - \frac{3}{\sqrt{98}}\right)^2 = \frac{50}{7} \qquad (x' - 2.12)^2 + 7(y' - .30)^2 = 7.14$$

center: (2.12, .30) on the x'y'-coordinate plane

vertices: (-.55, .30) and (4.79, .30)

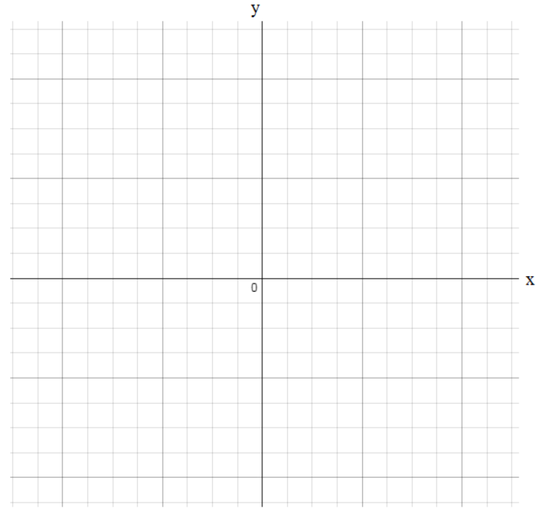
co-vertices: (2.12, 1.30) and (2.12, -.70)

(approximate values)

$$\frac{(x' - 2.12)^2}{7.14} + \frac{(y' - .30)^2}{1.02} = 1$$

SOLUTIONS

5) $7x^2 + 6xy - y^2 - 32 = 0$



$B^2 - 4AC = 36 - 4(7)(-1) > 0$ HYPERBOLA

To find the angle of rotation...

$\boxed{a = 1}$

$\boxed{2(C - A)b + B(1 - b^2) = 0}$

$2(-1 - 7)b + 6(1 - b^2) = 0$

$-6b^2 - 16b + 6 = 0$

$3b^2 + 8b - 3 = 0$

$(3b - 1)(b + 3) = 0$

$\boxed{b = 1/3 \text{ or } -3}$

$a^2 + b^2 = c^2$

If we use $b = -3$

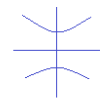
$a = 1 \quad b = -3 \quad \text{and} \quad c = \sqrt{10}$

$\cos\theta = \dfrac{1}{\sqrt{10}}$

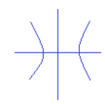
$\sin\theta = \dfrac{-3}{\sqrt{10}}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \Rightarrow \begin{aligned} x &= \frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y' \\ y &= \frac{-3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y' \end{aligned}$$

Note: If we use $b = -3$, we get



If we use $b = 1/3$, we get



Then, substitution $\quad 7x^2 + 6xy - y^2 - 32 = 0$

$$7\left(\frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y'\right)^2 + 6\left(\frac{1}{\sqrt{10}}x' + \frac{3}{\sqrt{10}}y'\right)\left(\frac{-3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'\right) - \left(\frac{-3}{\sqrt{10}}x' + \frac{1}{\sqrt{10}}y'\right)^2 = 32$$

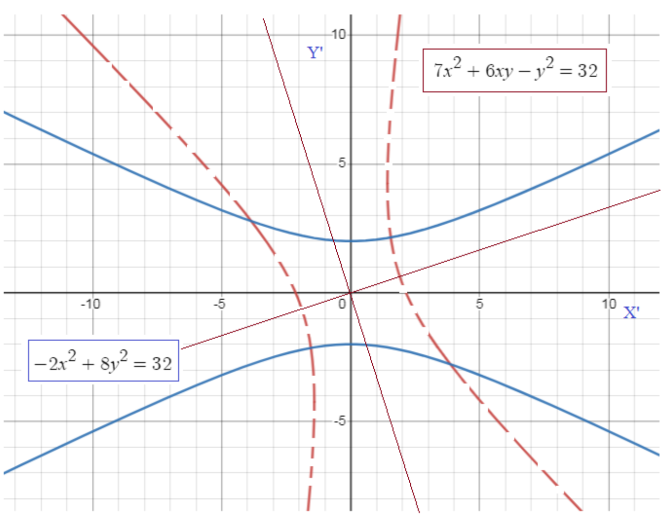
$$\boxed{-2x'^2 + 8y'^2 - 32 = 0 \quad \text{or} \quad \frac{y'^2}{4} - \frac{x'^2}{16} = 1}$$

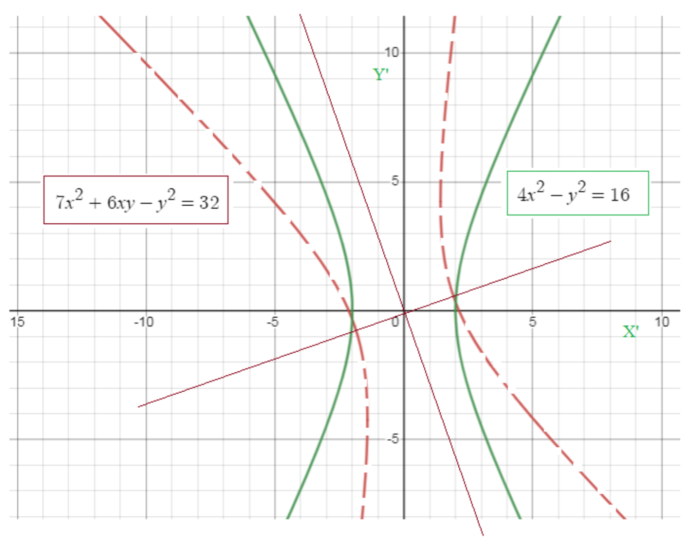
If rotated the other direction:

$4x'^2 - y'^2 - 16 = 0$

or

$\dfrac{x'^2}{4} - \dfrac{y'^2}{16} = 1$





SOLUTIONS

6) $4x^2 + 12xy + 9y^2 + 8\sqrt{13}\,x + 12\sqrt{13}\,y - 65 = 0$

$B^2 - 4AC = 12^2 - 4(4)(9) = 0$ PARABOLA

$a = 1$ $\quad 2(C - A)b + B(1 - b^2) = 0$

$2(9 - 4)b + 12(1 - b^2) = 0$

$10b + 12 - 12b^2 = 0$

$6b^2 - 5b - 6 = 0$

$(3b + 2)(2b - 3) = 0$

$b = -2/3$ or $3/2$

Using $b = 3/2$,

$a^2 + b^2 = c^2$

$1 + 9/4 = c^2$

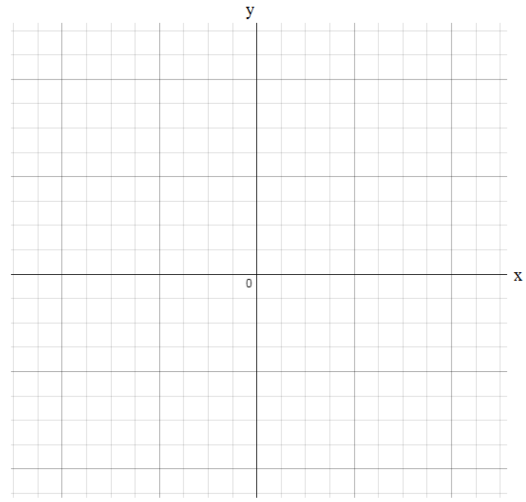
$c = \dfrac{\sqrt{13}}{2}$

$\sin\theta = \dfrac{b}{c} = \dfrac{3}{\sqrt{13}}$

$\cos\theta = \dfrac{a}{c} = \dfrac{2}{\sqrt{13}}$

$x = \dfrac{2}{\sqrt{13}}x' + \dfrac{-3}{\sqrt{13}}y'$

$y = \dfrac{3}{\sqrt{13}}x' + \dfrac{2}{\sqrt{13}}y'$



$4x^2 + 12xy + 9y^2 + 8\sqrt{13}\,x + 12\sqrt{13}\,y - 65 = 0$

$$4\left(\frac{2}{\sqrt{13}}x' + \frac{-3}{\sqrt{13}}y'\right)^2 + 12\left(\frac{2}{\sqrt{13}}x' + \frac{-3}{\sqrt{13}}y'\right)\left(\frac{3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y'\right) + 9\left(\frac{3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y'\right)^2 + 8\sqrt{13}\left(\frac{2}{\sqrt{13}}x' + \frac{-3}{\sqrt{13}}y'\right) + 12\sqrt{13}\left(\frac{3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y'\right) = 65$$

(Simplified with calculator) $\Longrightarrow$ $13x'^2 + 52x' = 65$

(divide by 13)

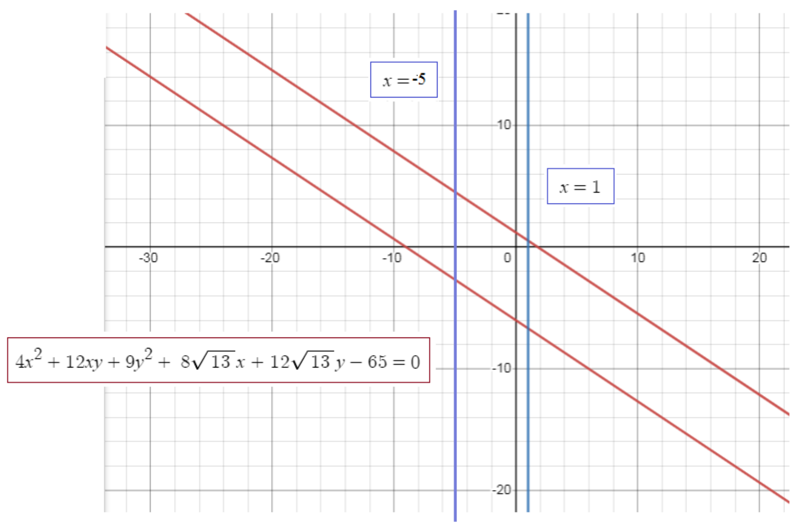
$x'^2 + 4x' = 5$

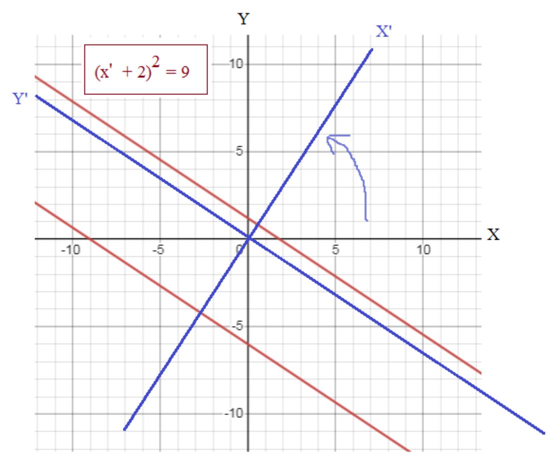
(complete the square)

$x'^2 + 4x' + 4 = 5 + 4$

$\boxed{(x' + 2)^2 = 9}$

$\boxed{\text{vertical lines: } x' = 1 \text{ and } x' = -5}$





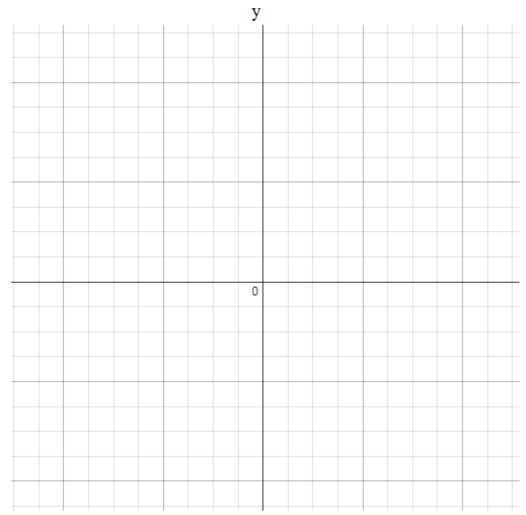
NOTE: If we had used $b = -2/3$, the axes would have been rotated the other direction, and the result would have been $(y' + 2)^2 = 9$

SOLUTIONS

7) $6x^2 + 4xy + 9y^2 + 27y = 30$

To identify the conic, find $B^2 - 4AC$

$(4)^2 - 4(6)(9) < 0 \implies$ ELLIPSE



To find the angle of rotation, we'll use

$$\boxed{\tan(2\theta) = \frac{B}{A - C}}$$

$$\tan(2\theta) = \frac{4}{6 - 9}$$

$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{4}{-3}$$

$$-6\tan\theta = 4 - 4\tan^2\theta$$

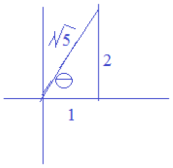
$$2\tan^2\theta - 3\tan\theta - 2 = 0$$

$$(2\tan\theta + 1)(\tan\theta - 2) = 0$$

$$\boxed{\tan\theta = -1/2 \qquad \tan\theta = 2}$$

We can pick either -1/2 or 2..

If we choose 2,



$$\sin\theta = \frac{2}{\sqrt{5}} \qquad \cos\theta = \frac{1}{\sqrt{5}}$$

$$\boxed{x = \frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' \qquad y = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'}$$

$6x^2 + 4xy + 9y^2 + 27y = 30$

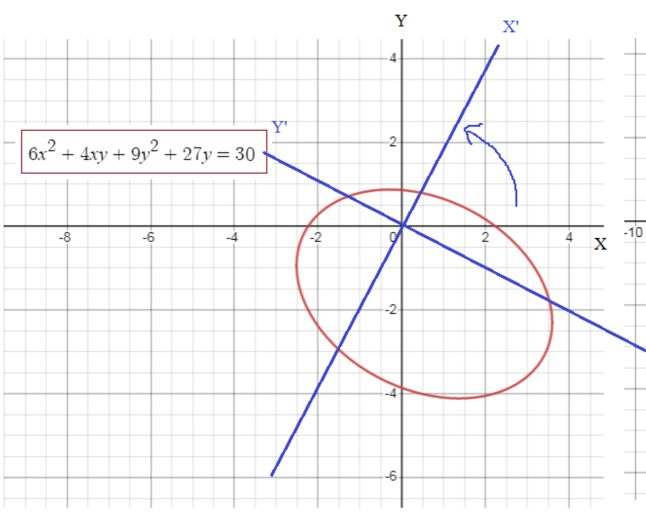
$$6\left(\frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y'\right)^2 + 4\left(\frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y'\right)\left(\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right) + 9\left(\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right)^2 + 27\left(\frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'\right) = 30$$

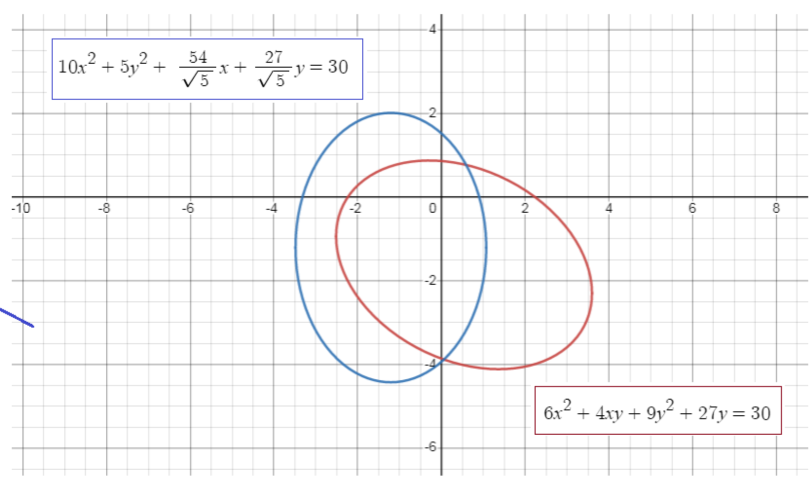
$$\frac{6}{5}x'^2 \boxed{-\frac{24}{5}x'y'} + \frac{24}{5}y'^2 + \frac{8}{5}x'^2 \boxed{+\frac{4}{5}x'y'} \boxed{-\frac{16}{5}x'y'} - \frac{8}{5}y'^2 + \frac{36}{5}x'^2 \boxed{+\frac{36}{5}x'y'} + \frac{9}{5}y'^2 + \frac{54}{\sqrt{5}}x' + \frac{27}{\sqrt{5}}y' = 30$$

Note: the x'y' cancels to zero... (eliminating the rotation)

$$\frac{6}{5}x'^2 + \frac{24}{5}y'^2 + \frac{8}{5}x'^2 - \frac{8}{5}y'^2 + \frac{36}{5}x'^2 + \frac{9}{5}y'^2 + \frac{54}{\sqrt{5}}x' + \frac{27}{\sqrt{5}}y' = 30$$

$$\frac{50}{5}x'^2 + \frac{25}{5}y'^2 + \frac{54}{\sqrt{5}}x' + \frac{27}{\sqrt{5}}y' = 30 \implies \boxed{10x'^2 + 5y'^2 + \frac{54}{\sqrt{5}}x' + \frac{27}{\sqrt{5}}y' = 30}$$





Rotation of Conics Exercise

SOLUTIONS

8) $16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0$

To identify the conic, find $B^2 - 4AC$

$(24)^2 - 4(16)(9) = 0$ PARABOLA

To find the angle of rotation, we'll use

$$\boxed{\operatorname{Tan}(2\theta) = \frac{B}{A - C}}$$

$$\operatorname{Tan}(2\theta) = \frac{-24}{16 - 9}$$

$$\frac{2\operatorname{Tan}\theta}{1 - \operatorname{Tan}^2\theta} = \frac{-24}{7}$$

$$14\operatorname{Tan}\theta = -24 + 24\operatorname{Tan}^2\theta$$

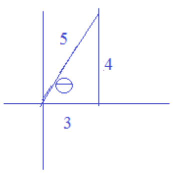
$$12\operatorname{Tan}^2\theta - 7\operatorname{Tan}\theta - 12 = 0$$

$$(3\operatorname{Tan}\theta - 4)(4\operatorname{Tan}\theta + 3) = 0$$

$$\boxed{\operatorname{Tan}\theta = 4/3 \quad \operatorname{Tan}\theta = -3/4}$$

We can select either 4/3 or -3/4 to rotate and remove the xy term..

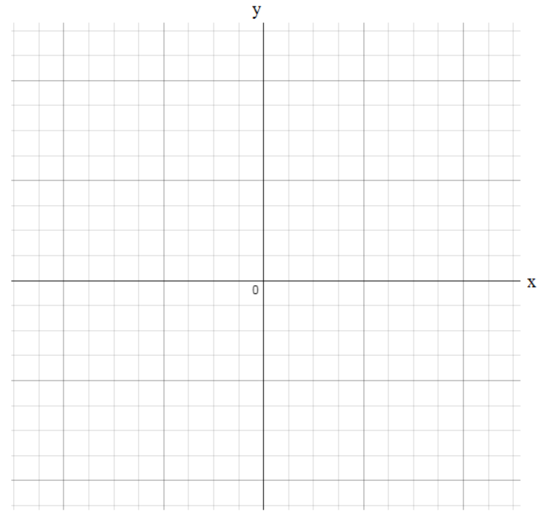
If we choose 4/3,



$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

$$\boxed{\begin{aligned} x &= \frac{3}{5}x' - \frac{4}{5}y' \\ y &= \frac{4}{5}x' + \frac{3}{5}y' \end{aligned}}$$



$16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0$

$$16\left(\frac{3}{5}x' - \frac{4}{5}y'\right)^2 - 24\left(\frac{3}{5}x' - \frac{4}{5}y'\right)\left(\frac{4}{5}x' + \frac{3}{5}y'\right) + 9\left(\frac{4}{5}x' + \frac{3}{5}y'\right)^2 + 110\left(\frac{3}{5}x' - \frac{4}{5}y'\right) - 20\left(\frac{4}{5}x' + \frac{3}{5}y'\right) + 100 = 0$$

$$\frac{144}{25}x'^2 \boxed{-\frac{384}{25}x'y'} + \frac{256}{25}y'^2 - \frac{288}{25}x'^2 \boxed{+\frac{168}{25}x'y'} + \frac{288}{25}y'^2 + \frac{144}{25}x'^2 \boxed{+\frac{216}{25}x'y'} + \frac{81}{25}y'^2 + 66x' - 88y' - 16x' - 12y' + 100 = 0$$

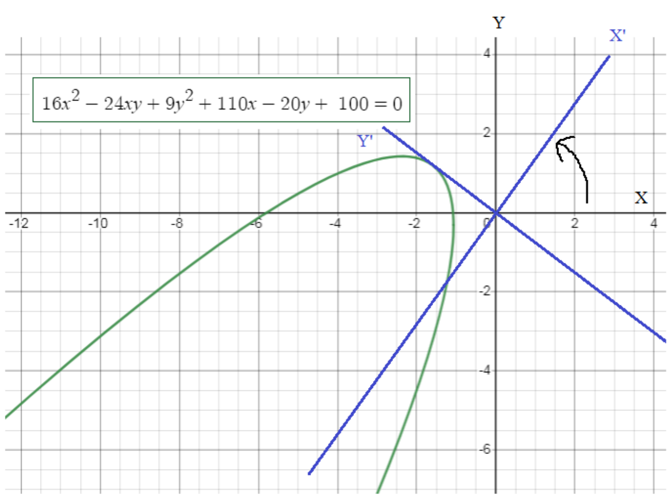
The x'y' cancel to zero, eliminating the rotation on fhe x'y' coordinate plane..

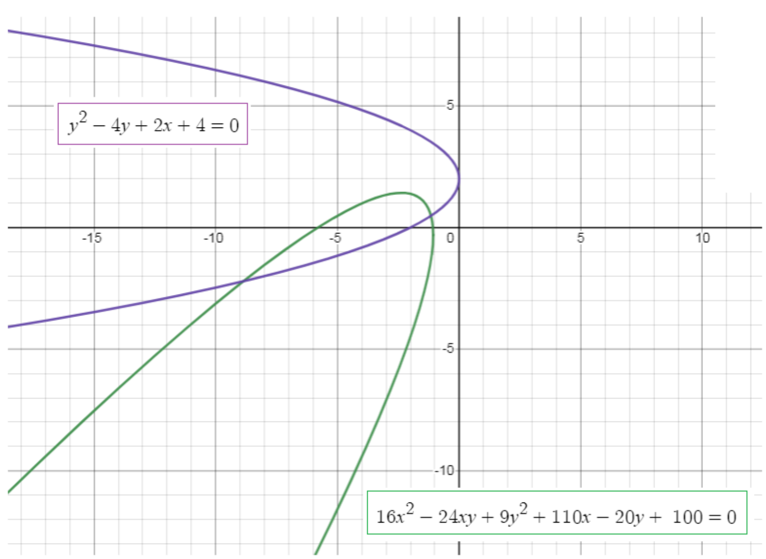
$$\boxed{\frac{144}{25}x'^2} + \frac{256}{25}y'^2 \boxed{-\frac{288}{25}x'^2} + \frac{288}{25}y'^2 \boxed{+\frac{144}{25}x'^2} + \frac{81}{25}y'^2 + 66x' - 88y' - 16x' - 12y' + 100 = 0$$

The x'^2 terms cancel, leaving us with a parabola...

$$\frac{256}{25}y'^2 + \frac{288}{25}y'^2 + \frac{81}{25}y'^2 + 66x' - 88y' - 16x' - 12y' + 100 = 0$$

$$25y'^2 - 100y' + 50x' + 100 = 0 \Rightarrow \boxed{y'^2 - 4y' + 2x' + 4 = 0 \quad \text{or} \quad (y' - 2)^2 = -2x'}$$

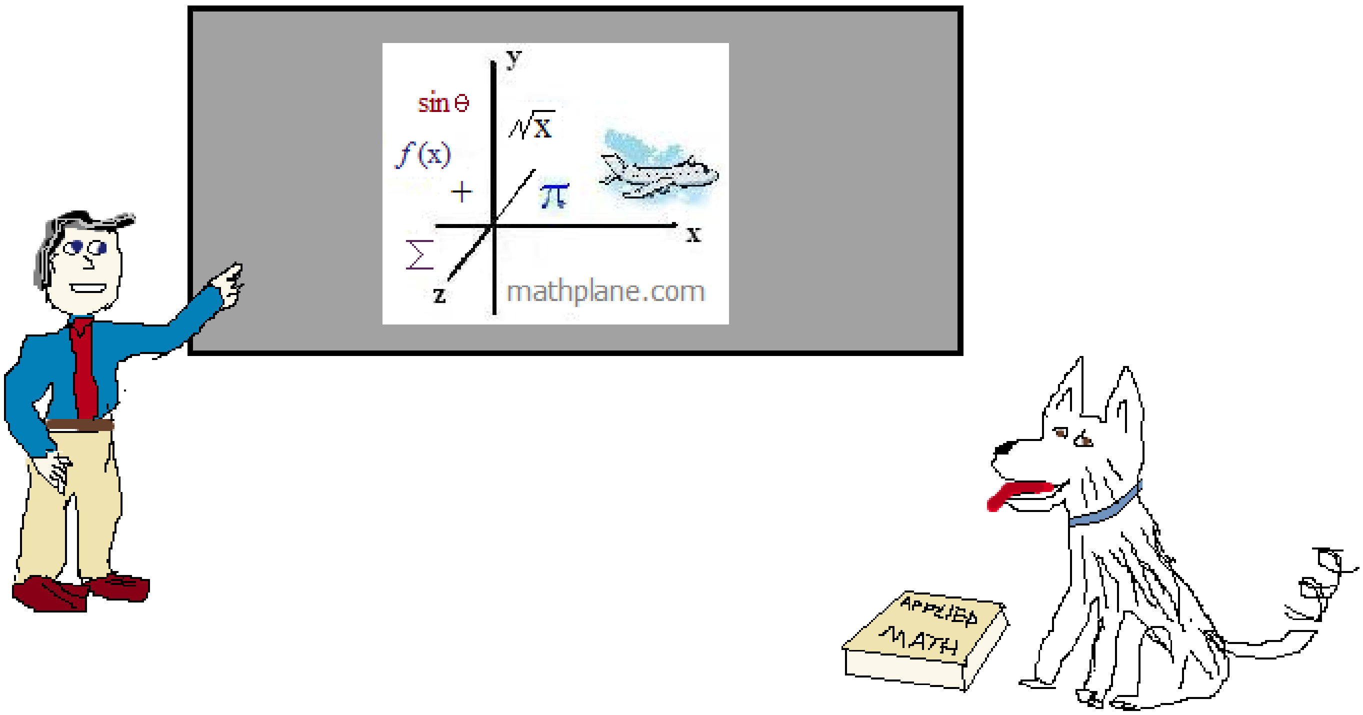




Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know

Cheers



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And, Mathplane.ORG for mobile and tablets