More Integrals (for BC Calculus)

Examples and Practice (with Solutions)

Topics include partial fractions, integration by parts, quotient rule, product rule, area, volume, and more…

Example: $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx$

Method 1: Power Rule

$$\int \sin x \cdot (\cos x)^{-3}\,dx$$

U-Substitution: $U = \cos x$

$$dU = -\sin x\,dx \qquad dx = \frac{1}{-\sin x}\,dU$$

$$\int \sin x\, U^{-3}\,\frac{1}{-\sin x}\,dU$$

$$\int U^{-3}\,(-1)\,dU = \frac{(-1)U^{-2}}{-2}$$

$$-\int -\sin x \cdot (\cos x)^{-3}\,dx = \frac{-(\cos x)^{-2}}{-2} \Rightarrow \boxed{\frac{1}{2\cos^2 x} + C}$$

Method 2: Apply Trig Identity

$$\int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}\,dx$$

$$\int \tan x \cdot \sec^2 x\,dx$$

$$\boxed{\frac{\tan^2 x}{2} + C}$$

These are equivalent....

$$\frac{1}{2}\bullet\frac{\sin x}{\cos x}\cdot\frac{\sin x}{\cos x}$$

$$\frac{\sin^2 x}{2\cos^2 x}$$

$$\frac{1-\cos^2 x}{2\cos^2 x}$$

$$\frac{1}{2\cos^2 x} - \frac{1}{2} + C$$

Example: $$\int \frac{\frac{1}{2\sqrt{x}}(4x^5) - (20x^4)\sqrt{x}}{16x^{10}}\,dx$$

$$\int \frac{\overset{f'}{\frac{1}{2\sqrt{x}}}\overset{g}{(4x^5)} - \overset{g'}{(20x^4)}\overset{f}{\sqrt{x}}}{16x^{10}}\,dx$$

looks like
the quotient rule

$$\boxed{\frac{\sqrt{x}}{4x^5} + C}$$

Perfect square!

Example: $\displaystyle\int \frac{1}{x^2 + 4x + 5}\,dx$

cannot factor the denominator, so
we can't use partial fraction...

$$\int \frac{1}{x^2 + 4x + 4 + 1}\,dx$$

Rearrange/separate the denominator...

$$\int \frac{1}{(x+2)^2 + 1}\,dx$$

Inverse Tangent!!

$$\boxed{\tan^{-1}(x+2) + C}$$

Example: $\int x\cos(5x)\ dx$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

First try....

$u = \cos(5x)$	$du = -5\sin(5x)$
$dv = x$	$v = \dfrac{x^2}{2}$

$$\int \underbrace{x}_{dv}\underbrace{\cos(5x)}_{u}\ dx = \underbrace{\frac{x^2}{2}}_{v}\underbrace{\cos(5x)}_{u} - \int \underbrace{-5\sin(5x)}_{du}\ \underbrace{\frac{x^2}{2}}_{v}\ dx$$

Still have a problem with the integral...
Let's try the other way...

Second try...

$u = x$	$du = 1$
$dv = \cos(5x)$	$v = \dfrac{1}{5}\sin(5x)$

$$\int \underbrace{x}_{u}\underbrace{\cos(5x)}_{dv}\ dx = \underbrace{x}_{u}\ \underbrace{\frac{1}{5}\sin(5x)}_{v} - \int \underbrace{\frac{1}{5}\sin(5x)}_{v}\ \underbrace{(1)}_{du}\ dx$$

$$= x\,\frac{1}{5}\sin(5x) - \left(-\frac{1}{25}\cos(5x)\right)$$

$$= \boxed{\frac{x\sin(5x)}{5} + \frac{\cos(5x)}{25} + C}$$

Example: $\int \sec^2 x\tan x\ dx$

Approach 1:

$$\int \sec x(\sec x\tan x)\ dx$$

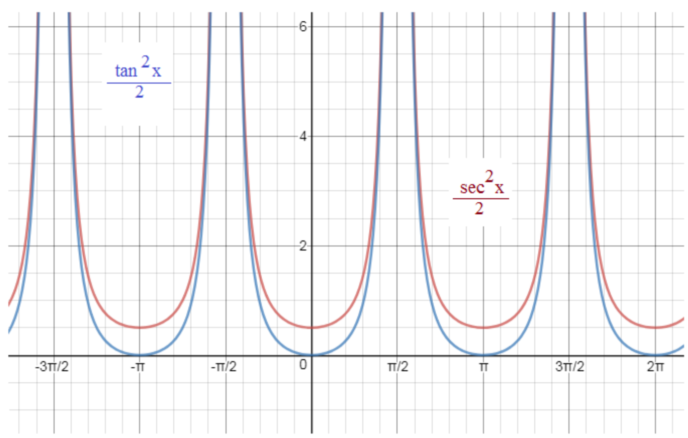
$$\boxed{\frac{\sec^2 x}{2} + C}$$

OR

Approach 2:

$$\int \tan x(\sec^2 x)\ dx$$

$$\boxed{\frac{\tan^2 x}{2} + C}$$



Example: $\int_0^2 \dfrac{x^2 - 2}{x + 1}\ dx$

Applying U-Substitution...

Let $U = x + 1 \implies x = U - 1$

$1\ dU = 1\ dx$

since we're changing the x's to U's, we must change the boundaries...

x boundaries are 0 to 2

$U = x + 1$

so, U boundaries are 1 to 3

$$\int \frac{(U-1)^2 - 2}{U}\ dU \qquad \int_1^3 \frac{U^2 - 2U - 1}{U}\ dU \qquad \int_1^3 U - 2 - \frac{1}{U}\ dU = \left.\frac{U^2}{2} - 2U - \ln U\right|_1^3$$

$$\frac{9}{2} - 6 - \ln(3) - \left(\frac{1}{2} - 2 - \ln(1)\right)$$

$$\boxed{-\ln(3)}$$

Example: $\displaystyle\int \frac{5x - 4}{2x^2 + x - 1}\,dx$ Use Partial Fractions

Decompose the expression...

$$\frac{5x - 4}{2x^2 + x - 1} = \frac{5x - 4}{(2x - 1)(x + 1)} = \frac{A}{(2x - 1)} + \frac{B}{(x + 1)}$$

$$5x - 4 = Ax + A + 2Bx - B$$

$$5x = (A + 2B)x$$

$$-4 = A - B$$

$$5 = A + 2B$$

$$-9 = -3B$$

$$B = 3 \quad A = -1$$

$$\int \frac{5x - 4}{2x^2 + x - 1}\,dx = \int \frac{-1}{(2x - 1)} + \frac{3}{(x + 1)}\,dx$$

$$\frac{-1}{2}\int \frac{2}{(2x - 1)}\,dx + 3\int \frac{1}{(x + 1)}\,dx$$

$$\boxed{\frac{-1}{2}\ln|2x - 1| + 3\ln|x + 1| + C}$$

or, the equivalent:

$$\frac{\ln\left(\dfrac{(x + 1)^6}{|2x - 1|}\right)}{2}$$

Example: $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$ rewrite $\displaystyle\int \frac{\cos x \cdot \cos^2 x}{\sqrt{\sin x}}\,dx$

Using U-substitution

Let $U = \sqrt{\sin x}$

$U^2 = \sin x$

$U^4 = \sin^2 x$

$1 - U^4 = \cos^2 x$

$2U\,du = \cos x\,dx$

apply U-substitution $\displaystyle\int \frac{\cos x\,(1 - U^4)}{U}\,dx$

$$\int \frac{\cos x\,(1 - U^4)}{U}\,\frac{2U\,du}{\cos x}$$

integrate

$$\int \frac{\cancel{\cos x}\,(1 - U^4)}{\cancel{U}}\,\frac{2\cancel{U}\,du}{\cancel{\cos x}} = \int 2\,(1 - U^4)\,du$$

substitute back

$$= 2U - \frac{2U^5}{5} + C$$

$$= \boxed{2\sqrt{\sin x} - \frac{2}{5}\sqrt{\sin^5 x} + C}$$

Example: $\displaystyle\int_2^8 \frac{f(x)g'(x) - g(x)f'(x)}{(f(x))^2}\,dx$ quotient rule!

so, the integral is $\displaystyle\left.\frac{g(x)}{f(x)}\right|_2^8 = \frac{g(8)}{f(8)} - \frac{g(2)}{f(2)} = \frac{-2}{7} - \frac{10}{-2} = \boxed{\frac{33}{7}}$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	2	10	-3
8	7	-1	-2	5

Example: $\displaystyle\int_1^4 x\,f''(x)\,dx$

$f(1) = 2 \qquad f'(1) = 5$

$f(4) = 7 \qquad f'(4) = 3$

apply integration by parts

$u = x \qquad du = 1$

$dv = f''(x) \qquad v = f'(x)$

$$= \underset{u\ \ v}{x\,f'(x)} - \int_1^4 \underset{v\ \ du}{f'(x)(1)}\,dx$$

$$\left. x\,f'(x) - f(x)\right|_1^4 = 4\,f'(4) - f(4) - [\,1\,f'(1) - f(1)\,]$$

$$4(3) - 7 - [5 - 2]$$

$$= \boxed{2}$$

Example: $\int (\ln x)^2 \, dx$

$$\int (\ln x)^2 (1) \, dx$$

$u = (\ln x)^2$ $\quad du = 2(\ln x) \cdot \frac{1}{x}$

$dv = 1$ $\quad v = x$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

$$\underbrace{\int (\ln x)^2 (1)\,dx}_{u \quad dv} = \underbrace{x\,(\ln x)^2}_{v \quad u} - \int \underbrace{x \cdot 2(\ln x) \cdot \frac{1}{x}}_{v \quad du}$$

$$x\,(\ln x)^2 - \int 2(\ln x)$$

We'll apply integration by parts again!

$u = (\ln x)$ $\quad du = \frac{1}{x}$

$dv = 2$ $\quad v = 2x$

$$\underbrace{2x(\ln x)}_{v \quad u} - \int \underbrace{2x \cdot \frac{1}{x}}_{v \quad du}$$

Collect all the final terms....

$$\boxed{x\,(\ln x)^2 - 2x(\ln x) + 2x + C}$$

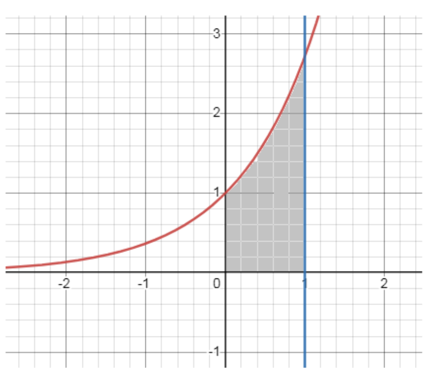
Example: Find the area bounded by $y = e^x$

x-axis, y-axis, and $x = 1$

Then, determine the volume of the solid found by rotating the area around $x = 3$...

a) Area

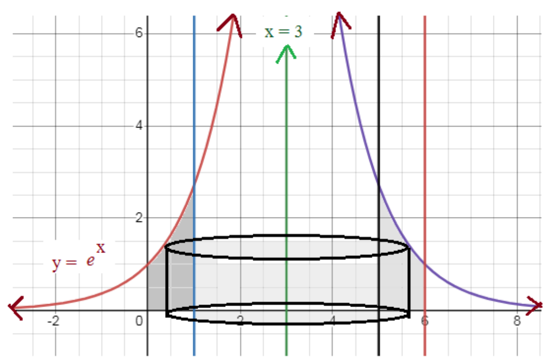
We want to sketch and establish the boundaries...



$$\int_0^1 e^x \, dx = e^x \Big|_0^1 = \boxed{e - 1}$$

b) Volume

We'll use shell method... Then, integration by parts...



The cross section of one cylinder ("shell")..

$$\text{Volume} = \int 2\pi \,(\text{radius})(\text{height})\, dx$$

$$2\pi \int_0^1 \underbrace{(3 - x)}_{u}\, \underbrace{e^x \, dx}_{dv} = 2\pi \left(\underbrace{(3 - x)}_{u}\, \underbrace{e^x}_{v} - \int \underbrace{(-1)}_{du} \underbrace{e^x}_{v}\, dx \right)$$

$u = (3 - x)$ $\quad dv = e^x \, dx$

$du = (-1)$ $\quad v = e^x$

$$2\pi \left((3 - x)\, e^x - (-e^x) \Big|_0^1 \right) = \boxed{2\pi\,(3e - 4)}$$

Example: $$\int_0^1 \frac{x}{(2x+1)^3}\,dx$$

let $U = 2x+1$ $\quad dU = 2dx$

$U - 1 = 2x$

$x = \dfrac{U-1}{2}$ $\quad dx = \dfrac{1}{2}\,dU$

Using double U-substitution

$$\int_1^3 \frac{\frac{U-1}{2}}{U^3}\,\frac{1}{2}\,dU$$

And, you can apply the boundaries...

$U = 2x+1$ so, $2(0)+1 = 1$

$2(1)+1 = 3$

The U boundaries are 1 and 3....

$$\frac{1}{4}\int_1^3 \frac{U-1}{U^3}\,dU$$

Split the numerator...

$$\frac{1}{4}\int_1^3 \frac{U}{U^3} - \frac{1}{U^3}\,dU \Rightarrow \frac{1}{4}\int_1^3 \frac{1}{U^2} - \frac{1}{U^3}\,dU$$

$$\frac{1}{4}\left(\frac{-1}{U} + \frac{1}{2U^2}\right)\Bigg|_1^3 = \frac{1}{4}\left[\left(\frac{-1}{3}+\frac{1}{18}\right) - \left(\frac{-1}{1}+\frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{-5}{18}\right) + \left(\frac{1}{2}\right)\right] = \boxed{\frac{1}{18}}$$

If we substitute back to x-values, we maintain the original boundaries...

$$\frac{1}{4}\left(\frac{-1}{2x+1} + \frac{1}{2(2x+1)^2}\right)\Bigg|_0^1$$

$$\frac{1}{4}\left[\left(\frac{-1}{3}+\frac{1}{18}\right) - \left(\frac{-1}{1}+\frac{1}{2}\right)\right]$$

$$= \boxed{\frac{1}{18}}$$

Example: $$\int x\sin(x)\cos(x)\,dx$$

We'll apply a trig identity and use integration by parts...

$\sin(2x) = 2\sin x\cos x$

$\dfrac{\sin(2x)}{2} = \sin x\cos x$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\,\underbrace{\frac{\sin(2x)}{2}\,dx}_{dv}$$

$u = x$ $\quad dv = \dfrac{\sin(2x)}{2}\,dx$

$du = 1\,dx$ $\quad v = -\dfrac{1}{4}\cos(2x)$

$$\int x\,\frac{\sin(2x)}{2}\,dx = -\frac{1}{4}x\cos(2x) - \int -\frac{1}{4}\cos(2x)\,dx$$

$$= -\frac{1}{4}x\cos(2x) - -\frac{1}{8}\sin(2x) \Rightarrow \boxed{\frac{1}{4}\left(\frac{\sin(2x)}{2} - x\cos(2x)\right) + C}$$

Example: $\displaystyle\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$

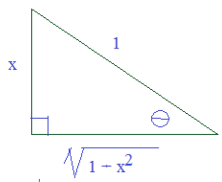
Observation: Judging by the boundary of $\frac{\sqrt{2}}{2}$

and, the square root term, $\sqrt{1-x^2}$

it appears trig substitution would be the technique to try first...

Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$



$x = \sin\theta$

$\sqrt{1-x^2} = \cos\theta$

since $\sin(2x) = 2\sin(x)\cos(x)$,

$\sin 2\theta = 2(x)(\sqrt{1-x^2})$

$\theta = \sin^{-1} x$

$$\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta = \int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$$

$$\int_0^{\frac{\pi}{4}} \frac{1}{2} - \frac{1}{2}\cos 2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin(0)\right) = \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

Boundary substitutions...

$$\int_0^{\frac{\sqrt{2}}{2}} \quad x = \sin\theta \quad \int_0^{\frac{\pi}{4}}$$

Trigonometry identities that are applied....

$\sin^2 x + \cos^2 x = 1$

$1 - \sin^2 x = \cos^2 x$

$\cos(2x) = \cos^2 x - \sin^2 x$

$\Downarrow$

$\cos(2x) = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos(2x)$

$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$

$$\frac{1}{2}\sin^{-1} x - \frac{1}{4}\,2(x)(\sqrt{1-x^2})\Big|_0^{\frac{\sqrt{2}}{2}}$$

$$\frac{1}{2}\sin^{-1}\frac{\sqrt{2}}{2} - \frac{1}{2}\,\frac{\sqrt{2}}{2}\sqrt{\frac{1}{2}} - \frac{1}{2}\sin^{-1} 0 - \frac{1}{4}\,2(0)(\sqrt{1-0^2})$$

$$\frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{4} - 0 - 0 = \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

Example: $\displaystyle\int_0^1 \frac{e^{\arctan(y)}}{1+y^2}\, dy$

Observation: we know the derivative of $\arctan(y) = \frac{1}{1+y^2}$

Using U-substitution

$U = \arctan(y)$

$dU = \frac{1}{1+y^2}\, dy$

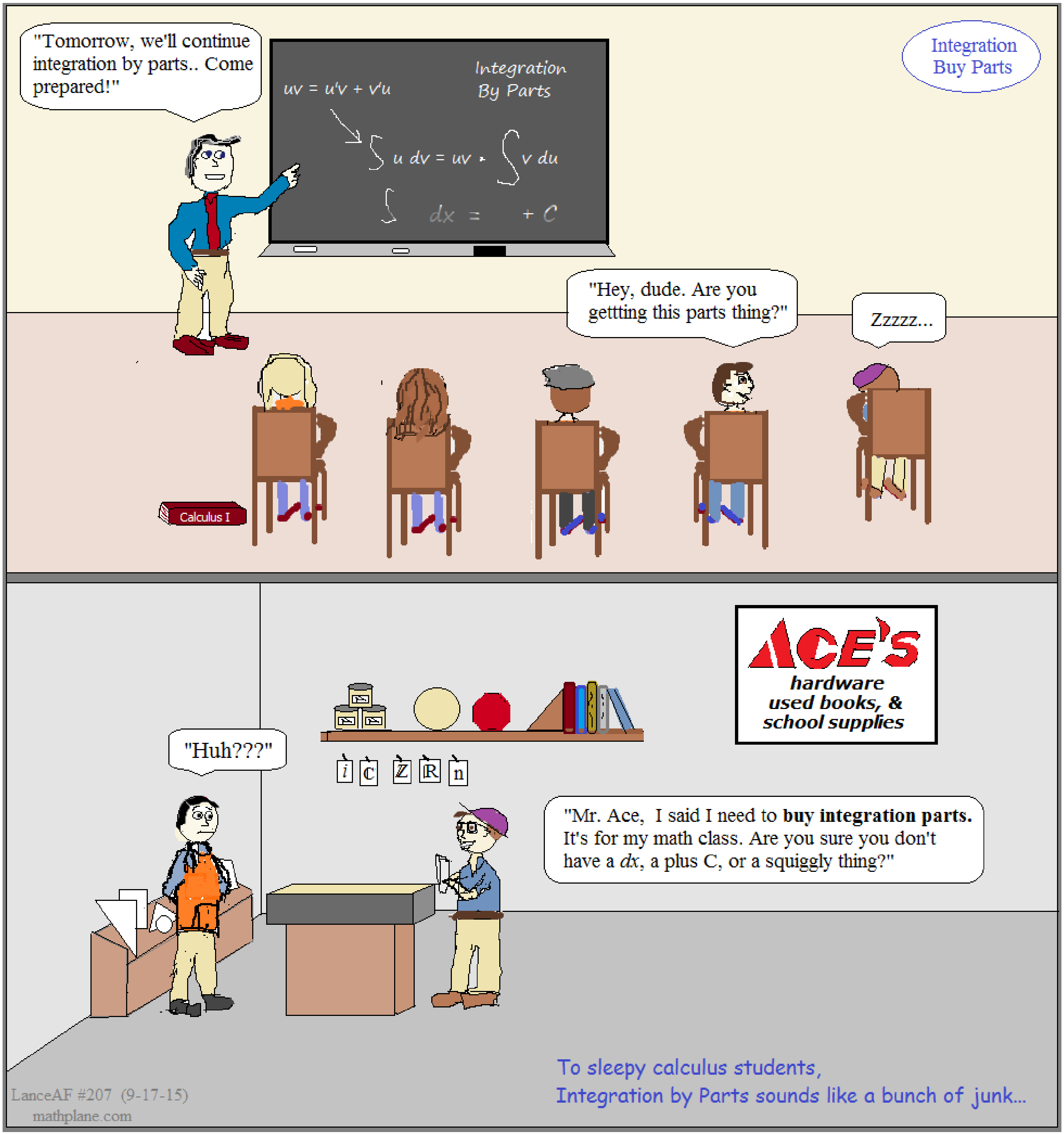
$dy = (1+y^2)\, dU$

$$\int \frac{e^U}{\cancel{1+y^2}} \cancel{(1+y^2)}\, dU$$

$$\int e^U\, dU = e^U \Rightarrow e^{\arctan(y)}$$

$$e^{\arctan(y)}\Big|_0^1 = e^{\arctan(1)} - e^0$$

$$\boxed{e^{\frac{\pi}{4}} - 1}$$



Practice Exercises-→

Evaluate the following indefinite integrals...

Advanced Integrals

1) $$\int \frac{x^3 + 5x + 10}{2x}\, dx$$

2) $$\int \frac{2x\sin(x) - x^2\cos(x)}{\sin^2 x}\, dx$$

3) $$\int 4x^3 e^{-2x} - 2x^4 e^{-2x}\, dx$$

4) $$\int (2x^3 + 7x)\cos(2x)\, dx$$

5) $$\int \cos^2 x\, dx$$

6) $$\int_0^1 \sqrt[4]{x^5} + \sqrt[5]{x^4}\, dx$$

7) $$\int \frac{2 + x^2}{1 + x^2}\, dx$$

8) $$\int \frac{3x^5}{\sqrt{5 + x^3}}\, dx$$

For #9 and #10, find the antiderivatives:

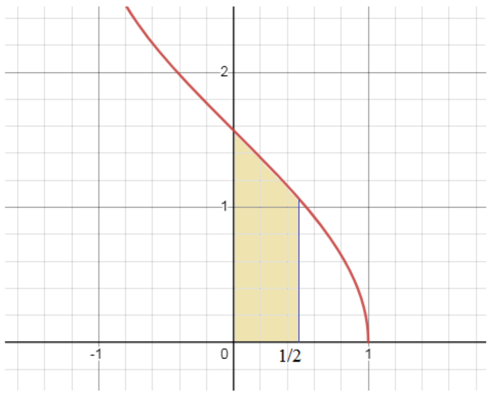
9) $$f(x) = \cos x\sqrt{x+2} + \frac{1}{2\sqrt{x+2}}\sin x$$

10) $$g(x) = 6(\sin\sqrt{3x^4 + 4x})^5 (\cos\sqrt{3x^4 + 4x}) \cdot \left(\frac{1}{2\sqrt{3x^4 + 4x}}\right)(12x^3 + 4)$$

11) $\int_0^{1/2} \cos^{-1} x \, dx$

Integration by Parts

$\int u \, dv = uv - \int v \, du$



12) $\int x^2 \ln x \, dx$

Advanced Integrals

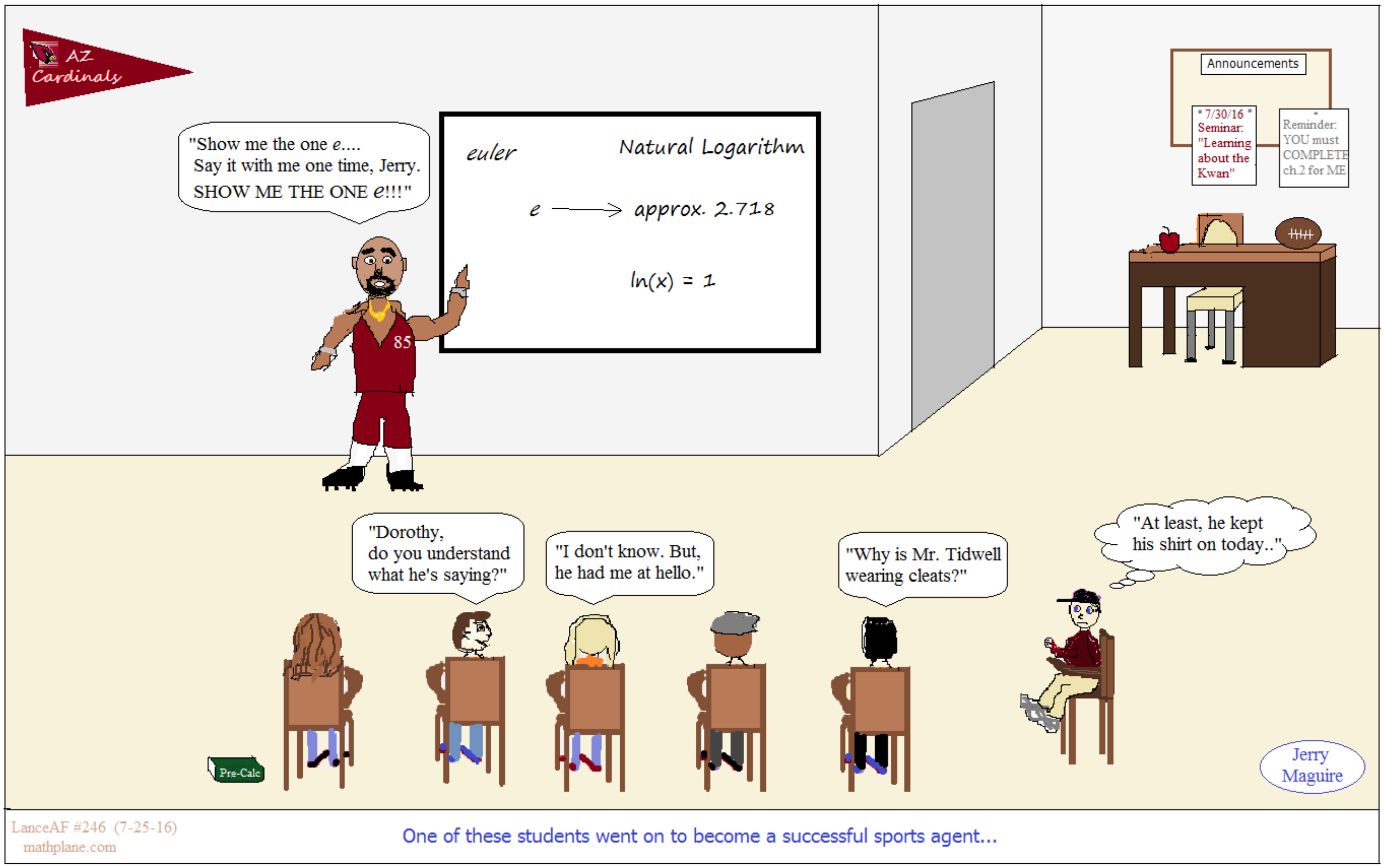
13) $$\int_0^1 \frac{3x - 2}{x + 1}\, dx$$

14) $$\int \frac{5x + 1}{(2x + 1)(x - 1)}\, dx$$

15) $$\int \tan^3 x \sec^4 x\, dx$$

16) $$\int_{0}^{\pi} x\cos^2 x \, dx$$

17) $$\int_{0}^{\frac{\pi}{4}} \sec(x) \, dx$$


AZ Cardinals
"Show me the one e....
Say it with me one time, Jerry.
SHOW ME THE ONE e!!!"
euler
Natural Logarithm
e → approx. 2.718
ln(x) = 1
Announcements
* 7/30/16 * Seminar: "Learning about the Kwan"
Reminder: YOU must COMPLETE ch.2 for ME
85
"Dorothy, do you understand what he's saying?"
"I don't know. But, he had me at hello."
"Why is Mr. Tidwell wearing cleats?"
"At least, he kept his shirt on today.."
Pre-Calc
Jerry Maguire
LanceAF #246 (7-25-16)
mathplane.com
One of these students went on to become a successful sports agent...

SOLUTIONS-→

Evaluate the following indefinite integrals...

Advanced Integrals

SOLUTIONS

1) $\displaystyle\int \frac{x^3 + 5x + 10}{2x}\,dx$

"split the numerator"

$$\int \frac{x^2}{2} + \frac{5}{2} + \frac{5}{x}\,dx$$

$$\boxed{\frac{x^3}{6} + \frac{5}{2}x + 5\ln x + C}$$

2) $\displaystyle\int \frac{2x\sin(x) - x^2\cos(x)}{\sin^2 x}\,dx$

"undo the quotient rule"

$$\frac{\overset{f'(x)}{2x}\,\overset{g(x)}{\sin(x)} - \overset{f(x)}{x^2}\,\overset{g'(x)}{\cos(x)}}{\underset{g(x)^2}{\sin^2 x}}$$

$$\frac{f(x)}{g(x)} \Rightarrow \boxed{\frac{x^2}{\sin(x)} + C}$$

3) $\displaystyle\int 4x^3 e^{-2x} - 2x^4 e^{-2x}\,dx$

"undo the product rule"

$$\overset{f'(x)}{4x^3}\,\overset{g(x)}{e^{-2x}} \;+\; \underbrace{-2}_{g'(x)}\overset{f(x)}{x^4}\,\underbrace{e^{-2x}}_{}$$

$$f(x)\cdot g(x) \Rightarrow \boxed{x^4 e^{-2x} + C}$$

4) $\displaystyle\int (2x^3 + 7x)\cos(2x)\,dx$

"Integration by Parts"
(tabular method)

u		dv
$(2x^3 + 7x)$	+	$\cos(2x)$
$6x^2 + 7$	—	$\frac{1}{2}\sin(2x)$
$12x$	+	$\frac{-1}{4}\cos(2x)$
12	—	$\frac{-1}{8}\sin(2x)$
0		$\frac{1}{16}\cos(2x)$

$$\frac{1}{2}(2x^3 + 7x)\sin(2x) - \frac{-1}{4}(6x^2 + 7)\cos(2x)$$

$$+ \frac{-1}{8}(12x)\sin(2x) - \frac{12}{16}\cos(2x)$$

$$\boxed{\left(\frac{1}{2}(2x^3 + 7x) - \frac{3}{2}x\right)\sin(2x) + \left(\frac{1}{4}(6x^2 + 7) - \frac{3}{4}\right)\cos(2x) + C}$$

5) $\displaystyle\int \cos^2 x\,dx$

Unable to integrate in this form, so we'll use trig identity...

$$\cos(2x) = 2\cos^2 x - 1$$

$$\cos(2x) + 1 = 2\cos^2 x$$

$$\frac{\cos(2x) + 1}{2} = \cos^2 x$$

$$\int \frac{1}{2}\cos(2x) + \frac{1}{2}\,dx$$

$$\int \frac{1}{2}\cos(2x)\,dx + \int \frac{1}{2}\,dx$$

$$\boxed{\frac{1}{4}\sin(2x) + \frac{1}{2}x + C}$$

SOLUTIONS

6) $$\int_0^1 \sqrt[4]{x^5} + \sqrt[5]{x^4}\ dx$$

$$\int_0^1 x^{\frac{5}{4}} + x^{\frac{4}{5}}\ dx \qquad \left. \frac{x^{\frac{9}{4}}}{\frac{9}{4}} + \frac{x^{\frac{9}{5}}}{\frac{9}{5}} \right|_0^1 = \boxed{1}$$

7) $$\int \frac{2+x^2}{1+x^2}\,dx$$

Split the numerator

inverse tangent

integral of 1

$$\int \frac{1+1+x^2}{1+x^2}\,dx \Longrightarrow \int \frac{1}{1+x^2} + \int \frac{1+x^2}{1+x^2}\,dx$$

$$\boxed{\tan^{-1}(x) + x + C}$$

8) $$\int \frac{3x^5}{\sqrt{5+x^3}}\,dx$$

Rewrite the Integral...

$$\int \frac{3x^2\,x^3}{(5+x^3)^{1/2}}\,dx$$

Using "U-substitution"

$$U = 5+x^3$$

$$\frac{dU}{dx} = 3x^2$$

$$3x^2\,dx = dU$$

$$x^3 = U-5$$

Apply U-substitution

$$\int \frac{(U-5)\ dU}{U^{1/2}} \qquad \int U^{1/2}\,dU - \int 5U^{-1/2}\,dU$$

$$\frac{2U^{3/2}}{3} - 2(5)U^{1/2} = \boxed{\frac{2(5+x^3)^{3/2}}{3} - 10(5+x^3)^{1/2} + C}$$

For #9 and #10, find the antiderivatives:

9) $$f(x) = \underset{f'}{\cos x}\,\underset{g}{\sqrt{x+2}} + \underset{g'}{\frac{1}{2\sqrt{x+2}}}\,\underset{f}{\sin x}$$

recognize that these are resultant parts from the product rule!

$$\boxed{F(x) = \sin x\sqrt{x+2} + C}$$

10) $$g(x) = 6\left(\sin\sqrt{3x^4+4x}\,\right)^5 \left(\cos\sqrt{3x^4+4x}\,\right)\left(\frac{1}{2\sqrt{3x^4+4x}}\right)(12x^3+4)$$

recognize that this is the expansion resulting from the chain rule

$$\boxed{G(x) = \sin^6\sqrt{3x^4+4x} + C}$$

SOLUTIONS

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

11) $\displaystyle\int_0^{1/2} \cos^{-1} x\, dx$

There's no direct definition for the antiderivative of inverse cosine; but, we do know its derivative.
So, we'll try integration by parts.

$$\int \cos^{-1} x\,(1)\,dx$$

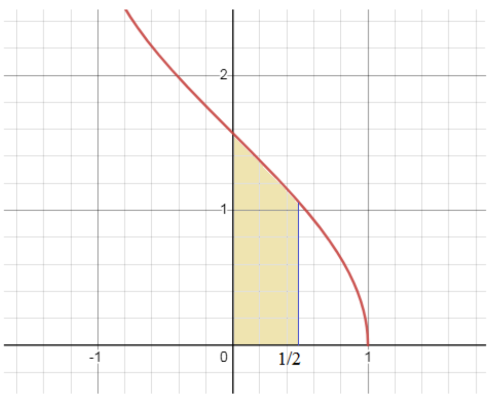
Assign the parts..

$u = \cos^{-1} x \qquad du = \dfrac{-1}{\sqrt{1-x^2}}$

$dv = 1 \qquad v = x$

$$\int \cos^{-1} x\,(1)\,dx = \underset{u\ \ v}{x \cos^{-1} x} - \int \underset{v}{x}\, \underset{du}{\frac{-1}{\sqrt{1-x^2}}} = x\cos^{-1}x - \frac{1}{2}\int 2x\,\frac{-1}{\sqrt{1-x^2}}$$

$$= x\cos^{-1}x - \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{1/2}$$



$$\Longrightarrow\ x\cos^{-1}x - \sqrt{1-x^2}\,\Big|_0^{1/2} = (1/2)\left(\frac{\pi}{3}\right) - \sqrt{\frac{3}{4}} - \left(0\cos^{-1}(0) - \sqrt{1}\right)$$

$$= \boxed{\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1} \quad \text{or } .658 \text{ (approx)}$$

12) $\displaystyle\int x^2 \ln x\, dx$

Apply Integration by Parts

$u = \ln x \qquad dv = x^2\,dx$

$du = \dfrac{1}{x} \qquad v = \dfrac{x^3}{3}$

$$\int x^2 \ln x\,dx = \underset{uv}{\frac{x^3}{3}\ln x} - \int \underset{v\ du}{\frac{x^3}{3}\,\frac{1}{x}}\,dx$$

$$\frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx$$

$$\boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

SOLUTIONS

13) $\displaystyle\int_0^1 \frac{3x-2}{x+1}\,dx$

First rewrite with long division...

$$\begin{array}{r|l} & 3 - \frac{5}{x+1} \\ \hline x+1 & 3x - 2 \\ & \underline{-\,3x + 3} \\ & -5 \end{array}$$

$$\int_0^1 3 - \frac{5}{x+1}\,dx$$

$$3x - 5\ln|x+1| \Big|_0^1 = 3 - 5\ln 2 - 0 + 5\ln 1 = \boxed{3 - 5\ln 2}$$

14) $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

Use partial fractions

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$5x+1 = A(x-1) + B(2x+1)$$

Let $x = 1$ $\quad 5(1)+1 = A(0) + B(3) \quad B = 2$

Let $x = -1/2$ $\quad 5(-1/2)+1 = A(-3/2) + B(0) \quad A = 1$

or, solve with system of equations...

$$5x+1 = Ax - A + 2Bx + B$$

$$5x+1 = (A+2B)x + (-A+B)$$

$A + 2B = 5$

$-A + B = 1$

$B = 2$ and $A = 1$

$$\int \frac{1}{2x+1} + \frac{2}{x-1}\,dx$$

$$\boxed{\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C}$$

15) $\displaystyle\int \tan^3 x \sec^4 x\,dx$

$U = \tan^2 x$

$U = \sec^2 x - 1$

$dU = 2\tan x \sec^2 x\,dx$

$dx = \dfrac{dU}{2\tan x \sec^2 x}$

$$\int U \cdot \cancel{\tan x} \cdot \sec^{\cancel{4}\,2} x \;\frac{dU}{2\cancel{\tan x}\,\cancel{\sec^2 x}}$$

$$\int U \sec^2 x\,\frac{dU}{2}$$

$$\frac{1}{2}\int U(U+1)\,dU \implies \frac{1}{2}\left(\frac{U^3}{3} + \frac{U^2}{2}\right)$$

$$\boxed{\frac{1}{2}\left(\frac{\tan^6 x}{3} + \frac{\tan^4 x}{2}\right) + C}$$

16) $\displaystyle\int_0^{\pi} x\cos^2 x \, dx$

We'll apply trig identities and integration by parts....

$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$1 + \cos(2x) = 2\cos^2 x \Rightarrow \boxed{\cos^2 x = \frac{1+\cos(2x)}{2}}$

$\displaystyle\int \frac{1+\cos(2x)}{2}\, dx = \int \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1}{2}x + \frac{1}{4}\sin(2x)$

$u = x \qquad dv = \cos^2 x \, dx$

$du = 1\, dx \qquad v = \frac{1}{2}x + \frac{1}{4}\sin(2x)$

$$\int_0^{\pi} \underbrace{x}_{u}\underbrace{\cos^2 x \, dx}_{dv} = \underbrace{x}_{u}\underbrace{\left(\frac{1}{2}x + \frac{1}{4}\sin(2x)\right)}_{v} - \int \underbrace{\frac{1}{2}x + \frac{1}{4}\sin(2x)}_{v}\underbrace{dx}_{du}$$

$$= x\left(\frac{1}{2}x + \frac{1}{4}\sin(2x)\right) - \left(\frac{1}{4}x^2 - \frac{1}{8}\cos(2x)\right)\Bigg|_0^{\pi}$$

$$\pi\left(\frac{1}{2}\pi + \frac{1}{4}\sin(2\pi)\right) - \left(\frac{1}{4}\pi^2 - \frac{1}{8}\cos(2\pi)\right) - \left[0\left(\frac{1}{2}\cdot 0 + \frac{1}{4}\sin(0)\right) - \left(\frac{1}{4}(0)^2 - \frac{1}{8}\cos(0)\right)\right]$$

$$\frac{\pi^2}{2} + 0 - \frac{\pi^2}{4} + \frac{1}{8} - \left[0 + 0 - \left(0 - \frac{1}{8}\right)\right] = \boxed{\frac{\pi^2}{4}}$$

17) $\displaystyle\int_0^{\frac{\pi}{4}} \sec(x)\, dx$

Method 1:

$\displaystyle\int \sec(x)\, \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\, dx$

$\displaystyle\int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\, dx$

Let $U = \sec(x) + \tan(x)$

$\frac{dU}{dx} = \sec(x)\tan(x) + \sec^2(x)$

$\displaystyle\int \frac{dU}{U} = \ln|U|$

$$\ln|\sec(x) + \tan(x)|\Bigg|_0^{\pi/4} = \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$$

$$\ln|\sqrt{2} + 1| - \ln|1 + 0|$$

$$= \boxed{\ln(\sqrt{2} + 1)}$$

Method 2:

Change using algebra and trig identities

$$\int \sec(x)\, dx \Rightarrow \int \frac{1}{\cos(x)} = \frac{\cos x}{\cos^2 x} \Rightarrow \int \frac{\cos x}{1 - \sin^2 x}\, dx$$

Apply U-substitution

$U = \sin x$

$dU = \cos x \, dx$

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{dU}{1 - U^2}$$

Using Partial Fractions....

$$\frac{1}{1 - U^2} = \frac{A}{1+U} + \frac{B}{1-U}$$

$$1 = A(1-U) + B(1+U)$$

$A + B = 1$

$-A + B = 0$

$A = 1/2 \quad B = 1/2$

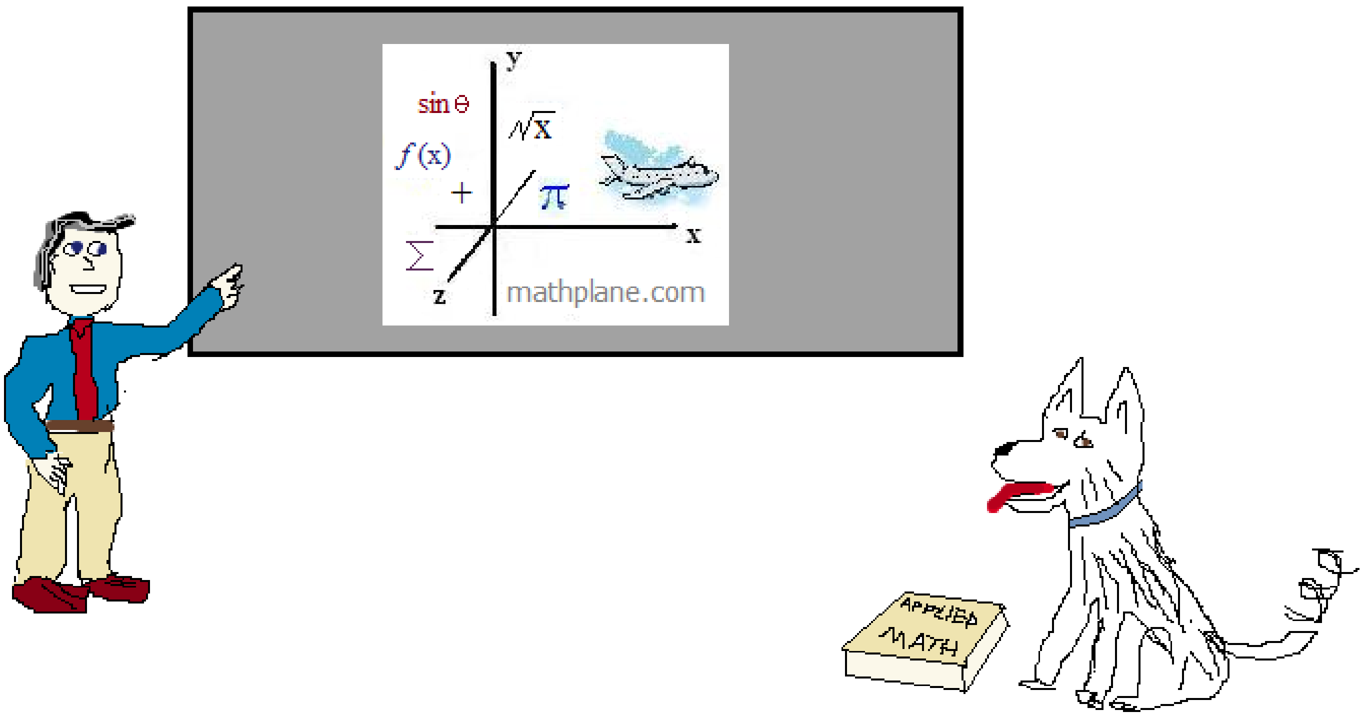
$$\int_0^{\frac{\sqrt{2}}{2}} \frac{1/2}{1+U} + \frac{1/2}{1-U}\, dU = \frac{1}{2}\ln|1+U| - \frac{1}{2}\ln|1-U|\Bigg|_0^{\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{2}\ln\left|\frac{1+U}{1-U}\right|\Bigg|_0^{\frac{\sqrt{2}}{2}} = \frac{1}{2}\ln\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right| - \frac{1}{2}\ln|1| \quad \boxed{= .88137 \text{ (approx)}}$$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers.



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