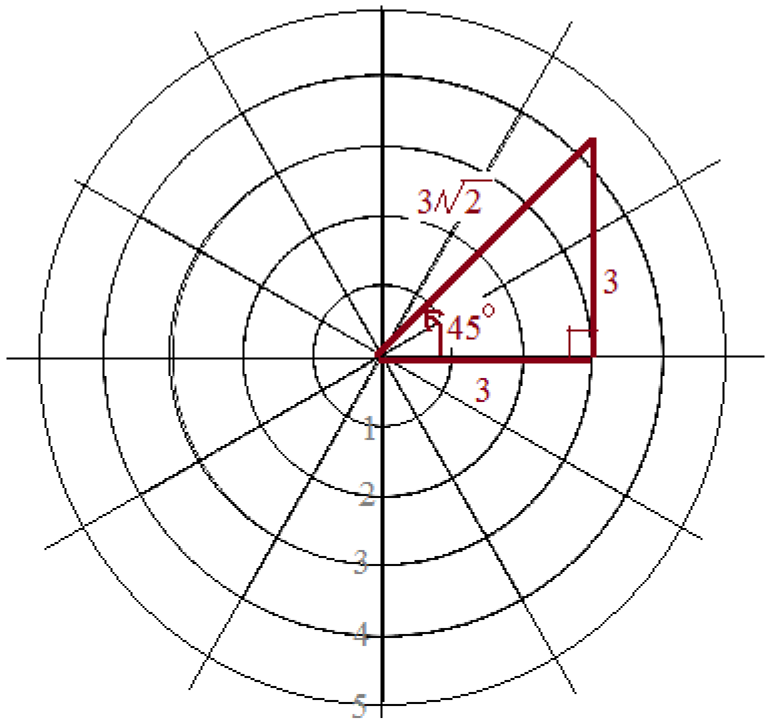
Algebra II/Trigonometry

Working with Polar and Rectangular Coordinates

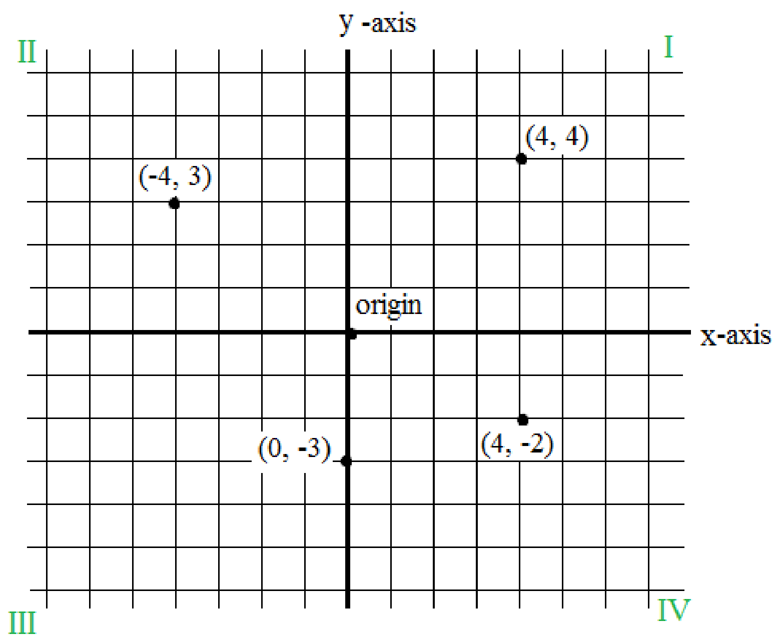


Brief Notes, Examples, and Practice Quiz (and Solutions)

Mathplane.com

Different Planes can be a Pain!

The Cartesian Plane

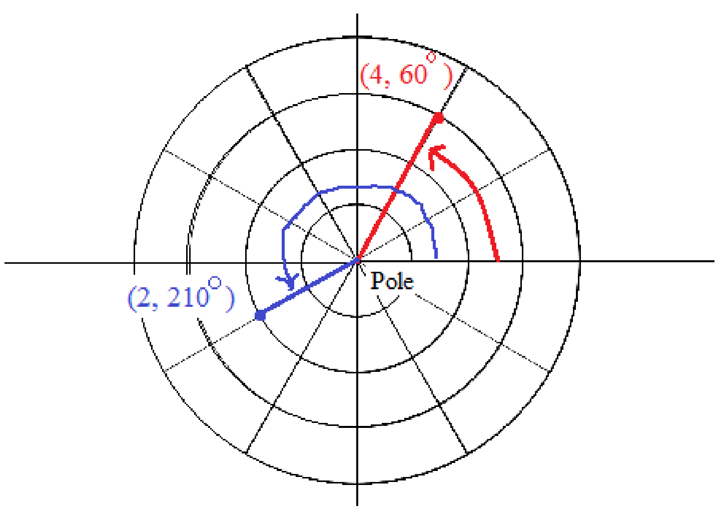


Origin: (0, 0)

Quadrants I, II, III, and IV

Points: (x, y)

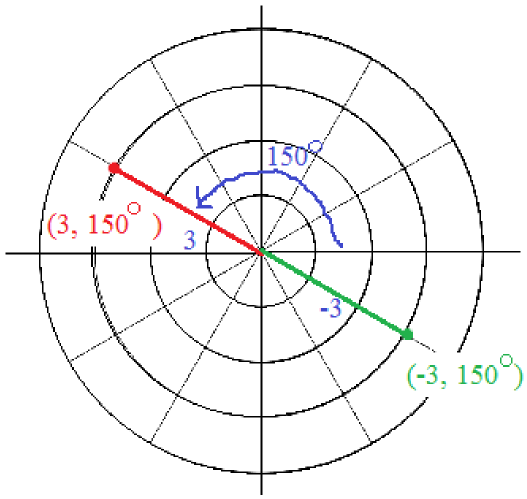
Polar Coordinate System (Plane)



"Origin" or "Pole" : $(0, \theta)$

Points: (r, θ)

Polar Coordinate System (continued)

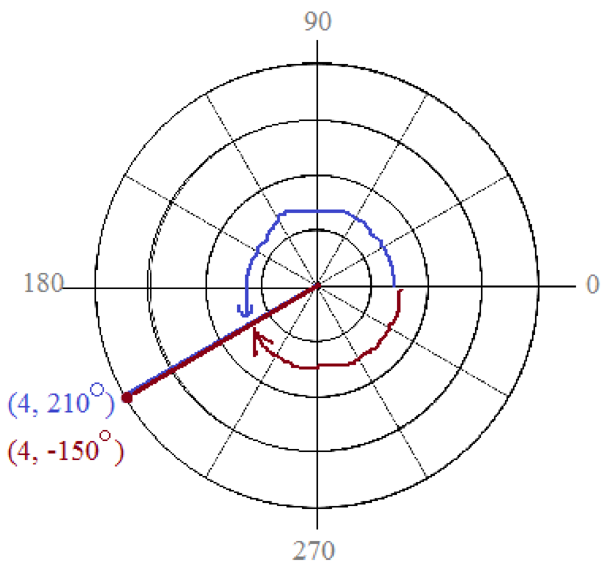


Note the difference!

(3, 150°)

vs.

(-3, 150°)



Note the similarity!

(4, 210°)

and

(4, -150°)

Note: Consider all the coterminal angles and (-r)

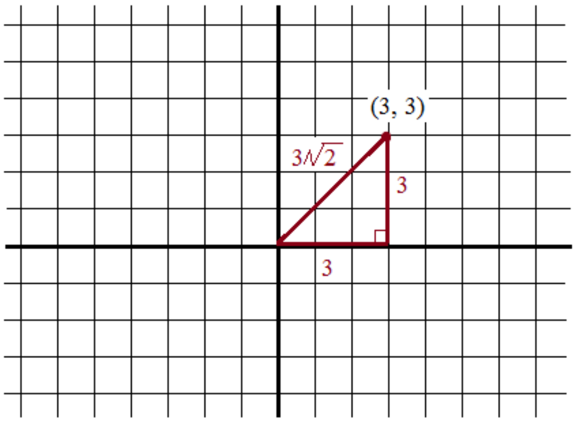
Example: (4, 210°)

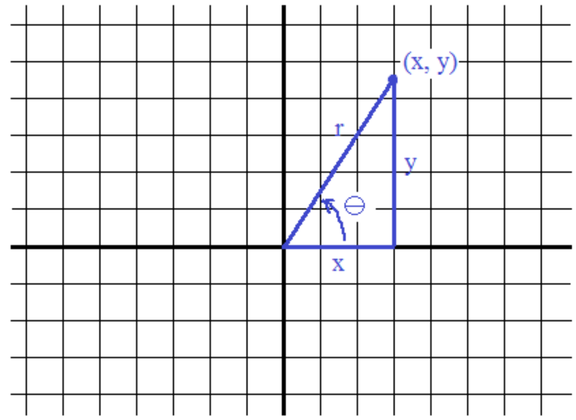
$(4, 210^\circ) = (4, 360^\circ n + 210^\circ)$

$= (-4, 360^\circ n + 30^\circ)$

(n is any integer)

Comparing Rectangular and Polar Coordinates



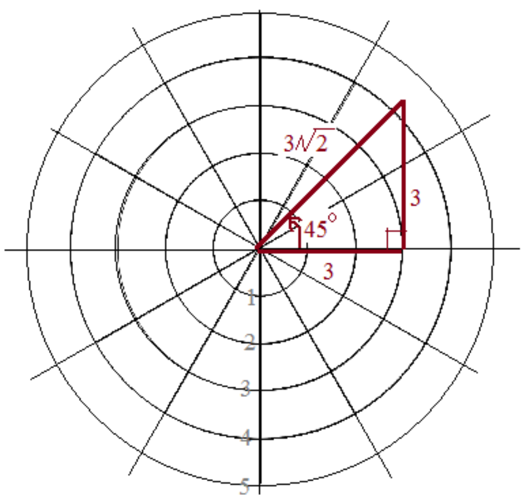


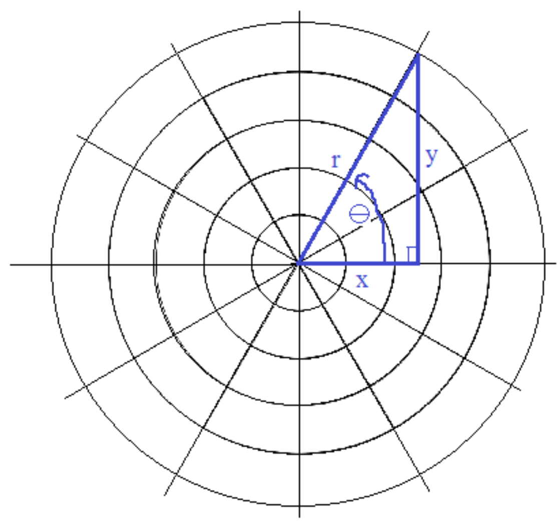
Rectangular: (3, 3)

Polar: ($3\sqrt{2}$, 45°)

Rectangular Coordinates: (x, y)

Polar Coordinates: (r, θ)





Important Implications: To convert from Rectangular to Polar coordinates,

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad x^2 + y^2 = r^2$$

Or, to convert from polar to rectangular,

$$x = r\cos\theta \qquad y = r\sin\theta$$

Polar Coordinates vs. Rectangular Coordinates

Example: Convert rectangular coordinates (3, 7) into polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

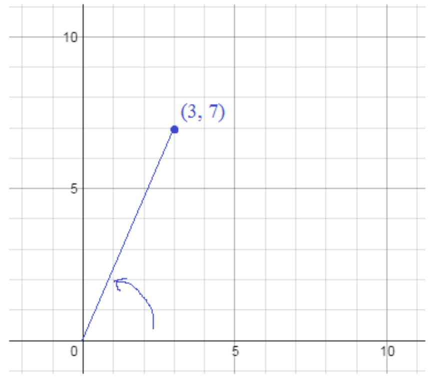
$x^2 + y^2 = r^2$

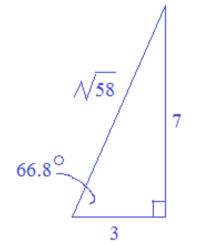
$9 + 49 = 58 \qquad r = \sqrt{58}$

$\tan\theta = \frac{y}{x} = \frac{7}{3}$

$\theta = 66.8^\circ$

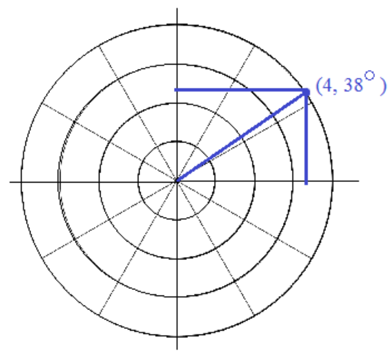
$(r, \theta) = (\sqrt{58}, 66.8^\circ)$

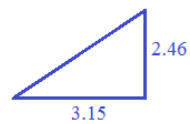




Example: Convert polar coordinates $(4, 38^\circ)$ into rectangular coordinates

$x = r\cos\theta$	$x = 4\cos(38^\circ)$	$x = 3.15$
$y = r\sin\theta$	$y = 4\sin(38^\circ)$	$y = 2.46$





Example: Change (-6, 11) into polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

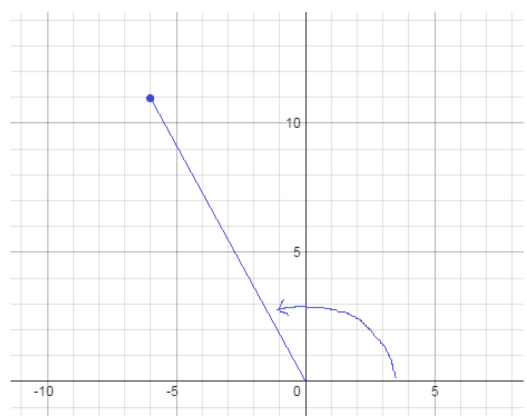
$(-6)^2 + (11)^2 = r^2$

$36 + 121 = 157 \qquad r = \sqrt{157}$

$\tan\theta = \frac{y}{x} = \frac{11}{-6} = -61.3^\circ$ and, since it is in Quadrant II,

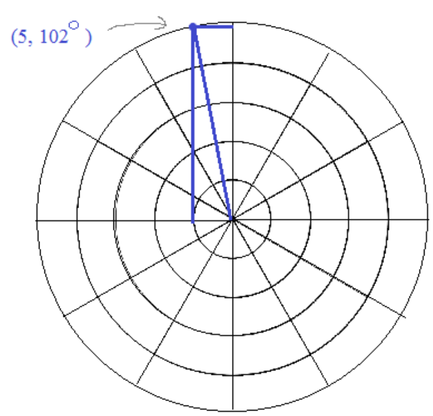
$\theta = 118.7^\circ$

$(r, \theta) = (\sqrt{157}, 118.7^\circ)$



Example: Express $(5, 102^\circ)$ as rectangular coordinates

$x = r\cos\theta$	$x = 5\cos(102^\circ)$	$x = -1.04$
$y = r\sin\theta$	$y = 5\sin(102^\circ)$	$y = 4.89$

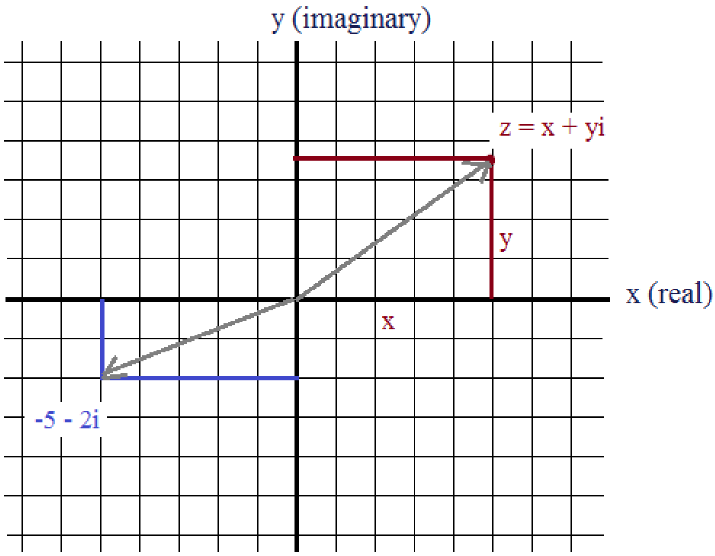


Imaginary Plane of Complex Numbers

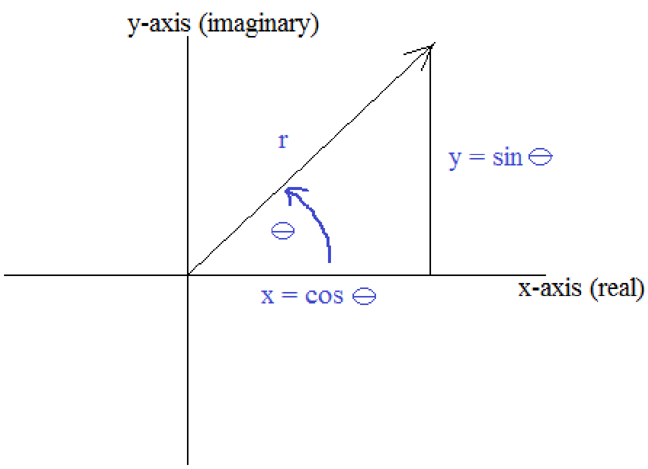
"Imaginary" Number: $i = \sqrt{-1}$

$i^2 = -1$

"Complex" Number: $z = x + yi$ or $z = (x, y)$
where x is the real component
y is the imaginary component



Rectangular Form: $x + yi$



Polar Form:

$z = r(\cos\theta + i \sin\theta)$

or

$r\text{Cis}\theta$

Complex Numbers: Polar and Rectangular
Random Notes and Formulas

$r(\cos\theta + i\sin\theta) \Longrightarrow r\,\text{Cis}\,\theta$

$Z_1Z_2 = r_1r_2\,\text{Cis}(\theta_1+\theta_2)$

$\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2}\,\text{Cis}\,(\theta_1-\theta_2)$

Polar Form (r, θ)

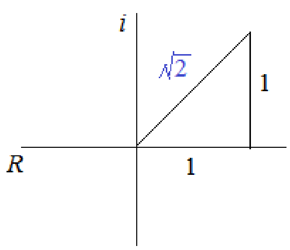
Rectangular Form (x, y)

(Complex) Polar Form $r\text{Cis}\,\theta$

(Complex) Rectangular Form $a + bi$

Converting rectangular to polar using a graph:

Examples: Convert $1 + i$ into polar

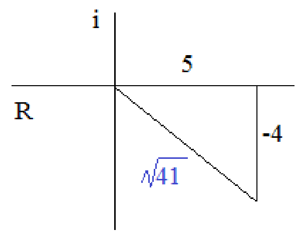


a = 1 (real term)
b = 1 (imaginary term)

45-45-90 triangle

$r = \sqrt{2}$
$\theta = 45$

Convert $5 - 4i$ into polar



a = 5
b = -4

$r = \sqrt{41}$
(pythagorean theorem)

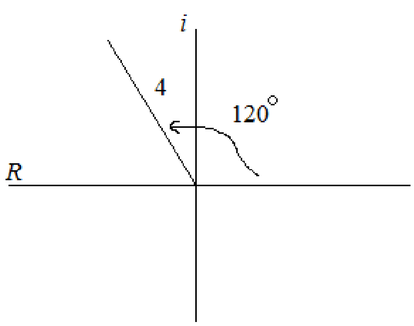
$\sqrt{41}\ \text{Cis}(321.35^\circ)$

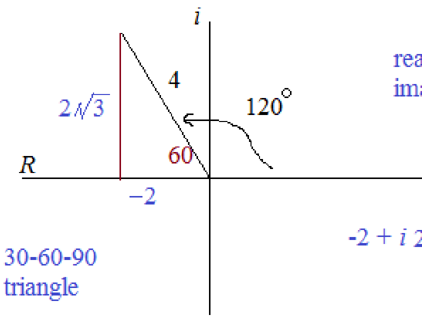
$\cos\theta = \dfrac{5}{\sqrt{41}} = 38.65^\circ$

$360 - 38.65 = 321.35$

Converting Polar to Rectangular using the graph:

Example: Convert $4\text{Cis}120^\circ$ into Rectangular





real term a = -2
imaginary term b = $2\sqrt{3}$

30-60-90 triangle

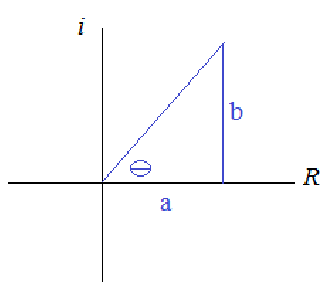
$-2 + i\,2\sqrt{3}$

For $Z = a + bi$

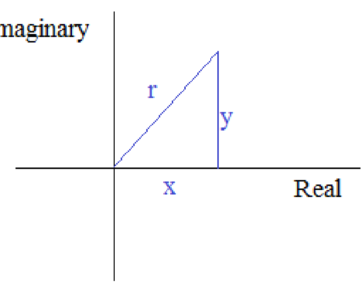
$|Z|$ is 'magnitude' of Z (length of r) $= \sqrt{a^2+b^2}$

$a = r\cos\theta$

$b = r\sin\theta$



Also, expressed as $Z = x + iy$ (complex Plane or Argand Diagram)



CIS and Component Vectors

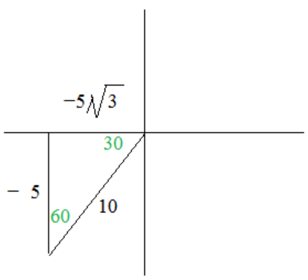
Example: $z_1 = -5\sqrt{3} - 5i$ Find $z_1 z_2$ and $\frac{z_1}{z_2}$ Identify $|z_1|$ and $|z_2|$

$z_2 = 2\sqrt{3} + 2i$

Method 1: Using Cis

$z_1 = -5\sqrt{3} - 5i$

10Cis(210)



$x = r\cos\theta$

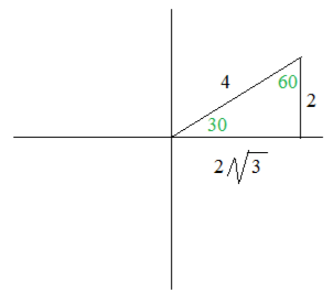
$-5\sqrt{3} = 10\cos\theta$

$\cos\theta = \frac{-5\sqrt{3}}{2}$

$\theta = 210^\circ$ (quadrant III)

$z_2 = 2\sqrt{3} + 2i$

4Cis(30)



$y = r\sin\theta$

$2 = 4\sin\theta$

$\sin\theta = \frac{2}{4}$

$\theta = 30^\circ$ (quadrant I)

$z_1 z_2 = 10\text{Cis}(210) \cdot 4\text{Cis}(30) = 40\text{Cis}(240)$

$\frac{z_1}{z_2} = 10\text{Cis}(210) / 4\text{Cis}(30) = \frac{5}{2}\text{Cis}(180)$

Method 2: Using component vector

$z_1 = -5\sqrt{3} - 5i$ $z_2 = 2\sqrt{3} + 2i$

$z_1 z_2 = (-5\sqrt{3} - 5i)(2\sqrt{3} + 2i)$ FOIL

$-30 - 10\sqrt{3}\,i - 10\sqrt{3}\,i + 10$

$-20 - 20\sqrt{3}\,i$

40Cis(240)

$\frac{z_1}{z_2} = \frac{-5\sqrt{3} - 5i}{2\sqrt{3} + 2i} \cdot \frac{(2\sqrt{3} - 2i)}{(2\sqrt{3} - 2i)}$

$\frac{-30 + 10\sqrt{3}\,i - 10\sqrt{3}\,i - 10}{12 + 4}$

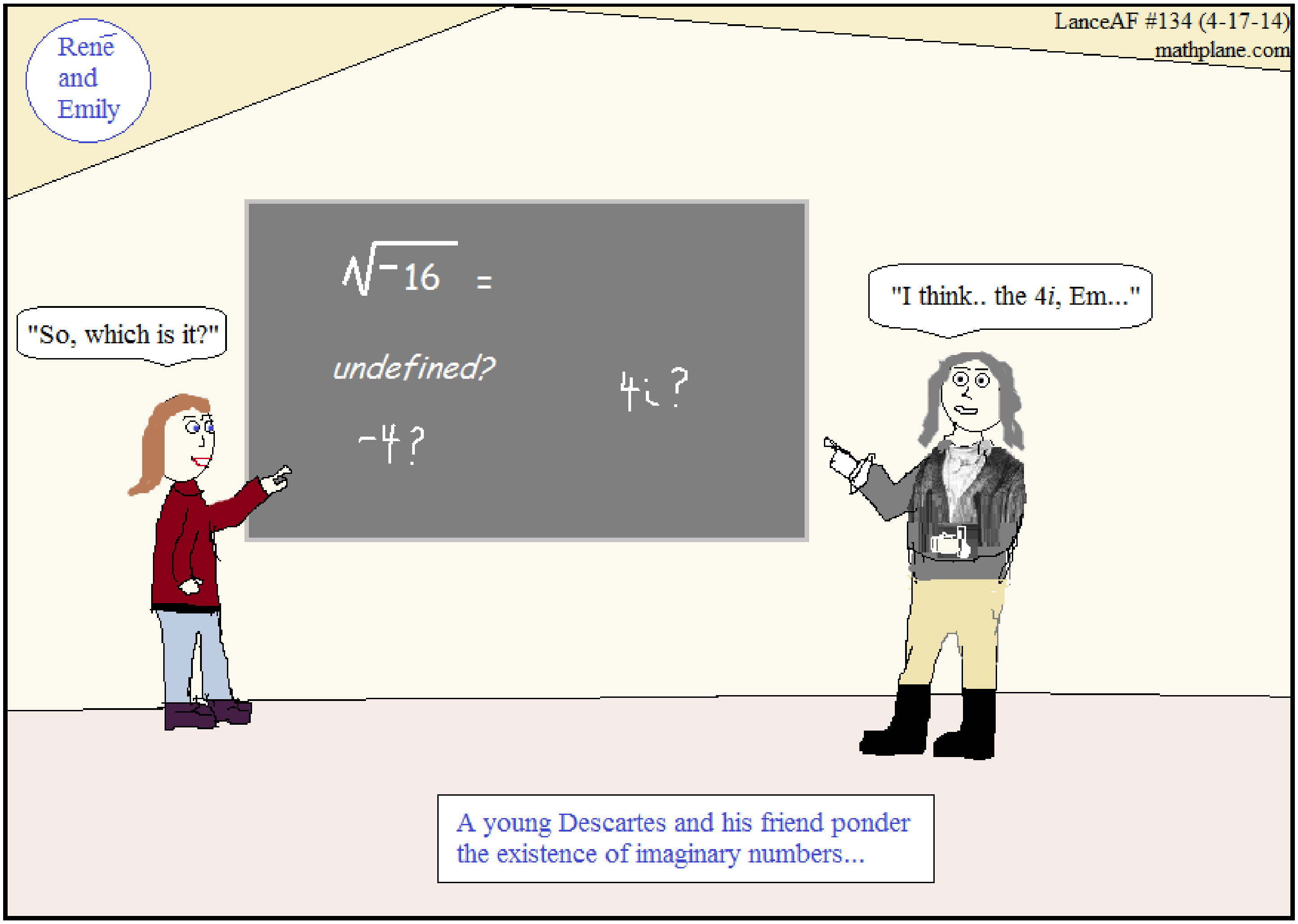
$\frac{-40 + 0i}{16} = \frac{-5}{2} + 0i$

$\frac{5}{2}\text{Cis}(180)$

$z_1 = -5\sqrt{3} - 5i \implies |z_1| = \sqrt{(-5\sqrt{3})^2 + (-5)^2} = \sqrt{75 + 25} = 10$

$z_2 = 2\sqrt{3} + 2i \implies |z_2| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{12 + 4} = 4$

Note: These are the same measures as each hypotenuse in the above graphs

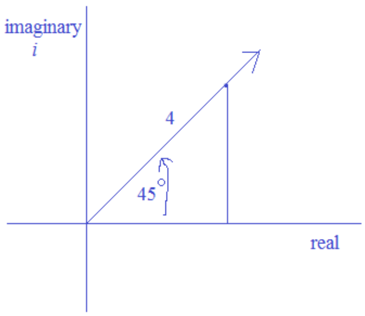


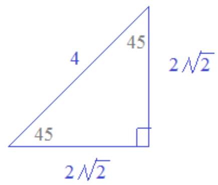
More Examples-→

Examples: Convert to rectangular coordinates and a + b*i* complex form

$4cis45^\circ$

We know the radius is 4 units
and the angle is 45 degrees





Using the formulas:

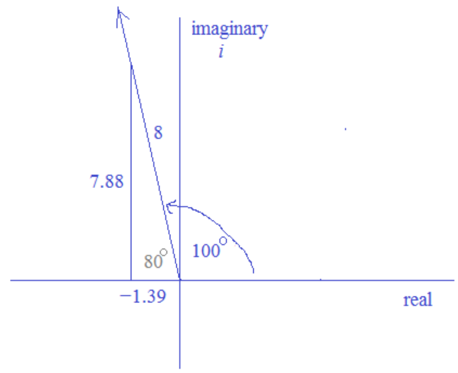
$y = r\sin\theta$ $x = r\cos\theta$

$y = 4\sin(45^\circ)$ $x = 4\cos(45^\circ)$

$= 2\sqrt{2}$ $= 2\sqrt{2}$

$(2\sqrt{2}\ ,\ 2\sqrt{2})$ $2\sqrt{2} + 2\sqrt{2}\,i$

$8cis100^\circ$



Using the formulas:

$y = r\sin\theta$ $x = r\cos\theta$

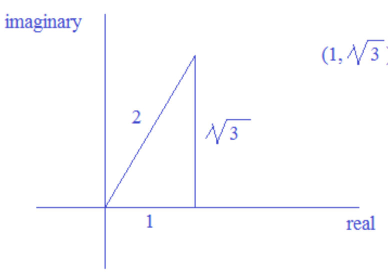
$y = 8\sin(100^\circ)$ $x = 8\cos(100^\circ)$

$= 7.88$ $x = -1.39$

$(7.88, -1.39)$ $-1.39 + 7.88i$

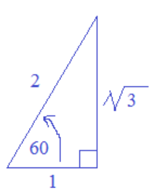
Examples: Identify the related coordinates and convert to polar form $rcis\ \theta$

$1 + i\sqrt{3}$



$(1, \sqrt{3})$

30-60-90
right
triangle

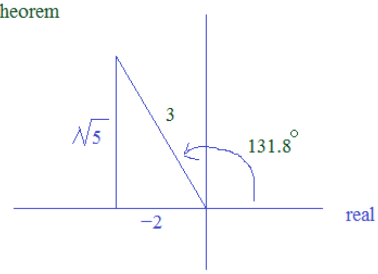


$2cis60^\circ$

$-2 + i\sqrt{5}$

Pythagorean Theorem
radius is 3

$(-2, \sqrt{5})$



$x = r\cos\theta$

$-2 = 3\cos\theta$

$\theta = \cos^{-1}(-2/3)$

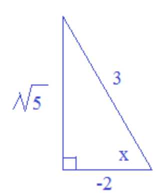
$y = r\sin\theta$

$\sqrt{5} = 3\sin\theta$

$\theta = \sin^{-1}(\sqrt{5}/3)$

in Quadrant II

131.8 degrees



$\cos x = \frac{2}{3}$

$x = 48.2^\circ$

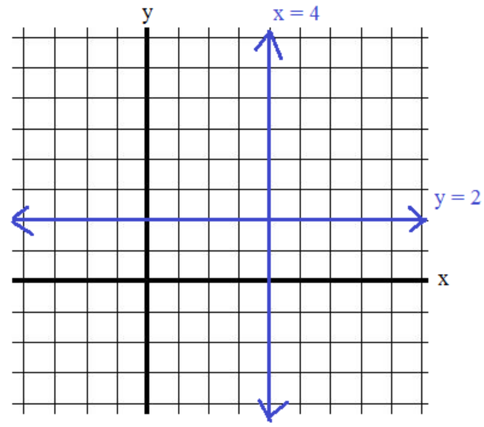
reference angle

$3cis131.8^\circ$

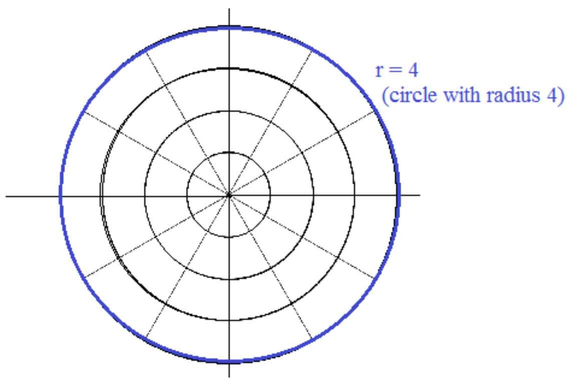
$x = r\cos\theta$ $y = r\sin\theta$

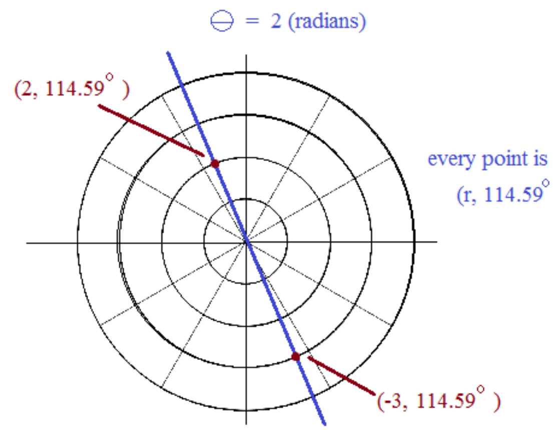
$x^2 + y^2 = r^2$

Example: On a rectangular xy-plane, graph $x = 4$ and $y = 2$



On a polar coordinate (Argand) plane, graph $r = 4$ and $\theta = 2$





Example: For the line $y = 2$, what is the equation in polar coordinates?

$\sin\theta = \dfrac{y}{r}$

$y = r\sin\theta$

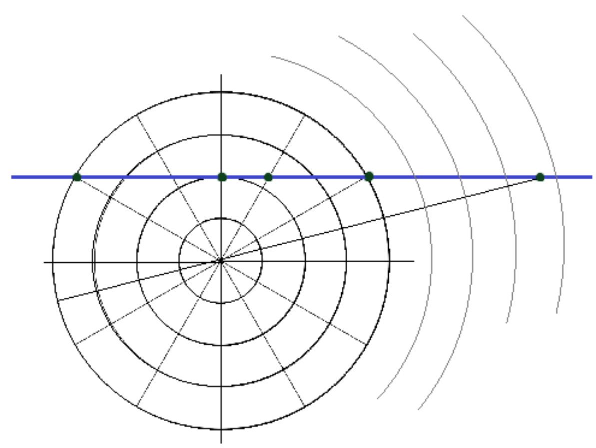
substitue $y = 2$

$2 = r\sin\theta$

$r = \dfrac{2}{\sin\theta}$

$r = 2\csc\theta$

θ	$\csc\theta$	r
30	2	4
60	$\frac{2}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$
90	1	2
150	2	4
195	-3.86	-7.72



Example: Find $(5\text{cis}30^\circ)(2\text{cis}60^\circ)$

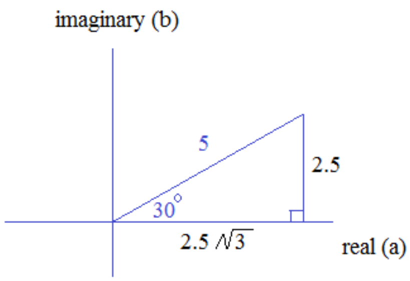
Method 1: Use the formula

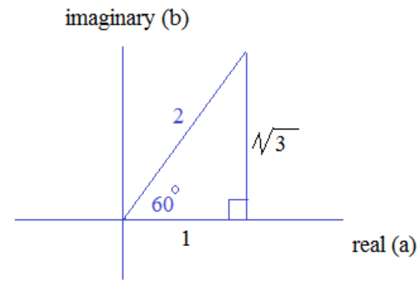
$(5)(2)\text{cis}(30^\circ + 60^\circ) = 10\text{cis}90^\circ$ ← $Z_1Z_2 = r_1r_2\text{Cis}(\theta_1 + \theta_2)$

Method 2: Change to complex number form and solve

$(5\text{cis}30^\circ) = 2.5\sqrt{3} + 2.5i$
a + b*i*

$(2\text{cis}60^\circ) = 1 + \sqrt{3}\,i$
a + b*i*





$(5\text{cis}30^\circ)(2\text{cis}60^\circ) = (2.5\sqrt{3} + 2.5i)(1 + \sqrt{3}\,i)$

FOIL

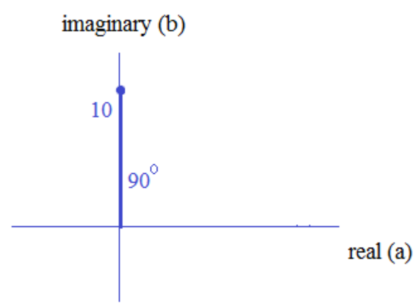
$2.5\sqrt{3} + 7.5i + 2.5i + 2.5\sqrt{3}\,i^2$

$2.5\sqrt{3} + 10i - 2.5\sqrt{3}$

$0 + 10i$

change back to polar form

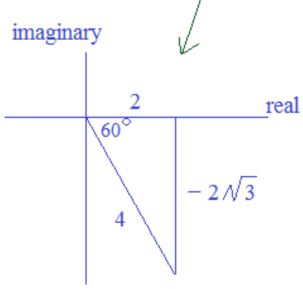
$10\text{cis}90^\circ$



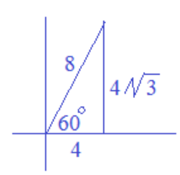
Example: $\left(8\text{cis}(\frac{\pi}{3})\right)\left(\frac{1}{2}\text{cis}(\frac{-2\pi}{3})\right)$

$r_1r_2\text{Cis}(\theta_1 + \theta_2)$

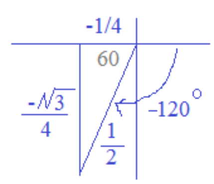
$(8)(\frac{1}{2})\,cis(\frac{\pi}{3} + \frac{-2\pi}{3}) = 4\text{cis}(\frac{-\pi}{3})$



$8\text{cis}60^\circ$ ---> $4 + 4\sqrt{3}\,i$



$\frac{1}{2}\text{cis}(-120^\circ)$ ---> $\frac{-1}{4} - \frac{\sqrt{3}}{4}\,i$



$(4 + 4\sqrt{3}\,i)(\frac{-1}{4} - \frac{\sqrt{3}}{4}\,i)$

$-1 - \sqrt{3}\,i - \sqrt{3}\,i - 3i^2$

$2 - 2\sqrt{3}\,i$

Example: Convert $r = -5\sin\theta$

$r = -5\,\frac{y}{r}$

$r^2 = -5y$

$x^2 + y^2 = -5y$

$x^2 + y^2 + 5y = 0$ (circle)

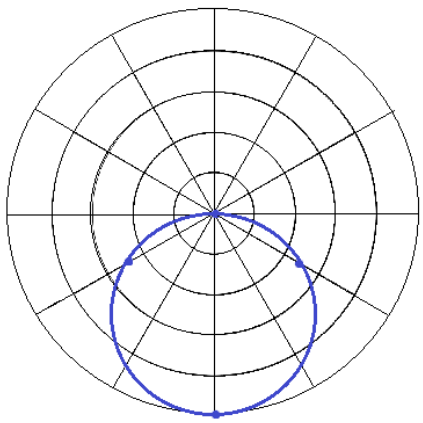
$y = r\sin\theta \longrightarrow \sin\theta = \frac{y}{r}$

$x^2 + y^2 = r^2$

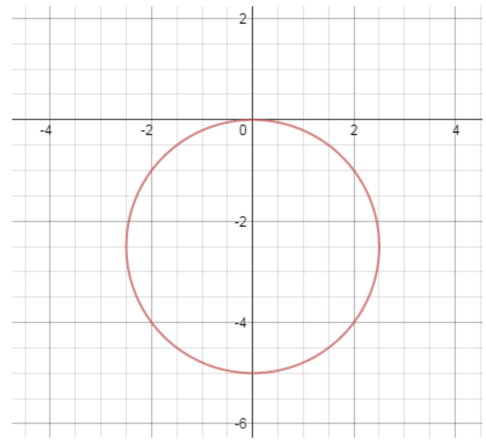
$x^2 + y^2 + 5y = 0$ complete the square

$x^2 + y^2 + 5y + \frac{25}{4} = 0 + \frac{25}{4}$

$x^2 + (y + \frac{5}{2})^2 = \frac{25}{4}$ standard form of circle with center $(0, -5/2)$



$r = -5\sin\theta$



Example: Where do $r = 1 + \sin\theta$ and $r = 2\sin\theta$ intersect?

Solve by substitution:

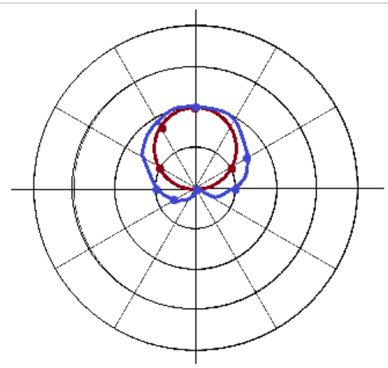
$1 + \sin\theta = 2\sin\theta$

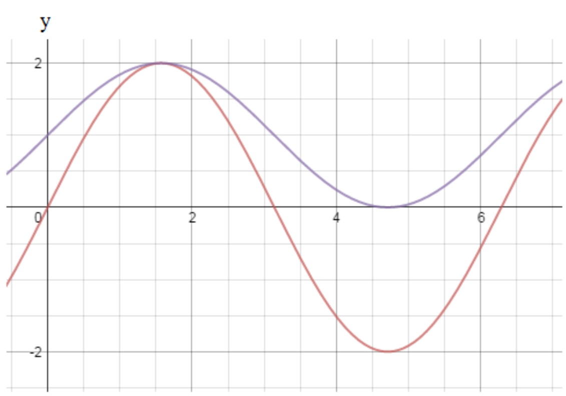
$1 = \sin\theta$

$\theta = 90^\circ$

NOTE: The graphs intersect at $(2, 90^\circ)$

They also pass through the origin, but at different times!





$r = 1 + \sin\theta$

θ	$\sin\theta$	$r = 1 + \sin\theta$
0	0	1
30	1/2	3/2
90	1	2
120	$\sqrt{3}/2$	1.86
180	0	1
210	-1/2	1/2
270	-1	0
330	-1/2	1/2

$r = 2\sin\theta$

θ	$r = 2\sin\theta$
0	0
30	1
90	2
120	1.73
180	0
210	-1
270	-2
330	-1

How many "petals" are on the (polar coordinate) graph of $r = -3\sin 5\theta$?

Polar Coordinate Graph

The graph of $y = -3\sin 5x$ is periodic
with maximum and minimum values of 3 and -3

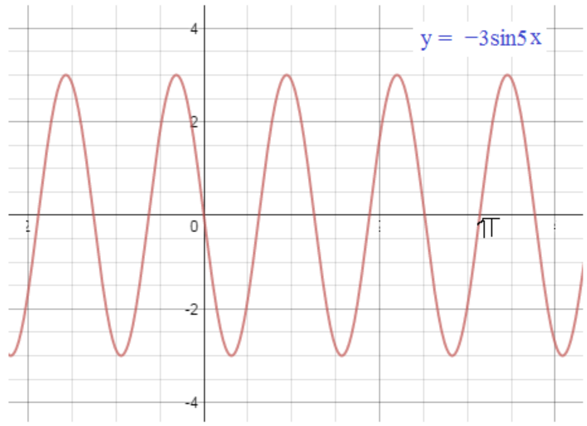
Therefore, to determine the "tips" of the petals, solve

$3 = -3\sin 5\theta$

and

$-3 = -3\sin 5\theta$

(i.e. find values of θ where r = 3 or -3)



$3 = -3\sin 5\theta$

$-1 = \sin 5\theta$

Let $5\theta = A$

$-1 = \sin A$

$A = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}, \frac{19\pi}{2} \ldots$

5θ

therefore,

$\theta = \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{11\pi}{10}, \frac{15\pi}{10}, \frac{19\pi}{10} \ldots$

$-3 = -3\sin 5\theta$

$1 = \sin 5\theta$

Let $5\theta = A$

$1 = \sin A$

$A = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2} \ldots$

5θ

therefore,

$\theta = \frac{\pi}{10}, \frac{5\pi}{10}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10} \ldots$

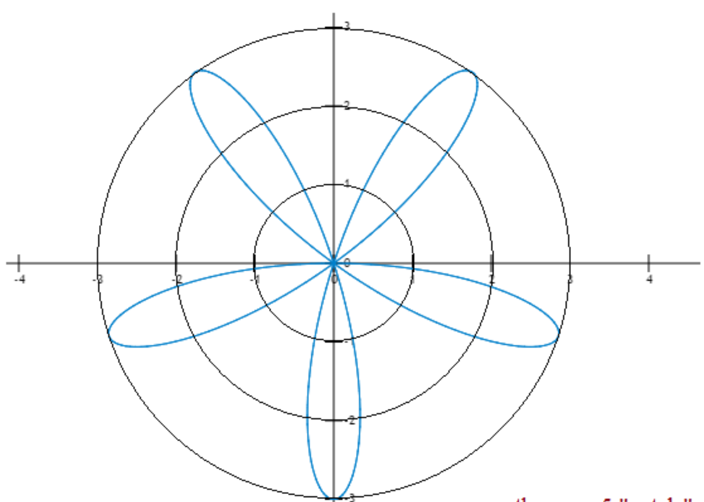
These values of θ are where the tips of the petals occur...
Now, we have to determine which ones overlap....

$r = -3\sin 5\theta$

(r, θ)

$(3, \frac{3\pi}{10})$	$(-3, \frac{\pi}{10})$
$(3, \frac{7\pi}{10})$	$(-3, \frac{\pi}{2})$
$(3, \frac{11\pi}{10})$	$(-3, \frac{9\pi}{10})$
$(3, \frac{3\pi}{2})$	$(-3, \frac{13\pi}{10})$
$(3, \frac{19\pi}{10})$	$(-3, \frac{17\pi}{10})$

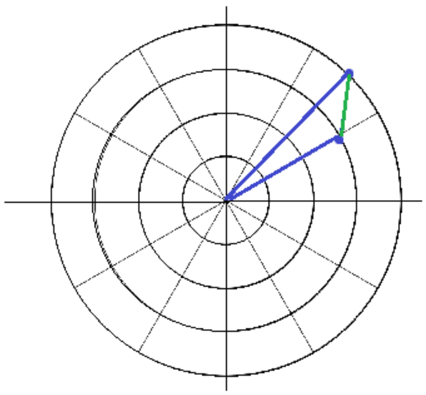
Same points!



there are 5 "petals"...

Find the distance between $(4, \frac{\pi}{4})$ and $(3, \frac{\pi}{6})$

Method 1: Using Law of Cosines...



Law of Cosines: $c^2 = a^2 + b^2 - 2ab\text{Cos}(c)$

$c^2 = (3)^2 + (4)^2 - 2(3)(4)\text{Cos}(\frac{\pi}{12})$

$c = 1.348$ (approx.)

Method 2: Using Rectangular Coordinates

$x = r\cos\theta \quad y = r\sin\theta$

$(4, \frac{\pi}{4}) \Rightarrow (2\sqrt{2}, 2\sqrt{2})$

$(3, \frac{\pi}{6}) \Rightarrow (\frac{3\sqrt{3}}{2}, \frac{3}{2})$

Distance Formula: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$d = \sqrt{(.23)^2 + (1.33)^2}$

$d = 1.349$ (approx.)

Products and roots of complex numbers (De Moivre's Theorem)

Complex Product Formula

$$(r_1 \, cis\,\theta_1)(r_2 \, cis\,\theta_2) = r_1 r_2 \, cis(\theta_1 + \theta_2)$$

Example: $(5cis30)(6cis40) = 30cis70$

De Moivre's Theorem (Product Formula)

$$[\, r cis\,\theta\,]^n = r^n \, cis(n\theta)$$

$$[\, r(\cos\theta + i\sin\theta)\,]^n = r^n(\cos n\theta + i\sin n\theta)$$

Example: $(5cis30)^4 = 5^4 cis(30 \text{ x } 4) = 625cis120$

Example: Use Demoive's Theorem to rewrite $(1-i)^{10}$ in polar form and complex form

convert $(1-i)$ into polar form: $\sqrt{2}\; cis(-45^\circ)$

$$[\sqrt{2}\; cis(-45^\circ)]^{10} = \sqrt{2}^{10}\; cis(10 \cdot -45^0)$$

$$32cis(-450^\circ) \Rightarrow 32cis(270^0) \Rightarrow 0 - 32i$$

polar complex

Complex Roots Theorem

$$r^{\frac{1}{n}}\left(\cos\theta + \frac{360^\circ k}{n} + i\sin\theta + \frac{360^\circ k}{n}\right)$$

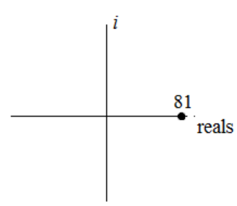
Notice how it connects roots and complex numbers with trigonometry!

Example: What are the fourth roots of 81? Verify using the complex roots theorem.

$\sqrt[4]{81} = 3$ and $\sqrt[4]{81} = -3$ But, if you introduce complex numbers, then $3i$ and $-3i$ are fourth roots of 81

Express the term: $81 + 0i$

convert to polar coordinates: $(81, 0^\circ)$



Apply root theorem:

$$r^{\frac{1}{n}}\left(\cos\theta + \frac{360^\circ k}{n} + i\sin\theta + \frac{360^\circ k}{n}\right)$$

$81^{\frac{1}{4}}(\cos(0) + i\sin(0)) = 3cis(0^\circ)$ $3 + 0i$

$81^{\frac{1}{4}}(\cos(90) + i\sin(90)) = 3cis(90^\circ)$ $0 + 3i$

$81^{\frac{1}{4}}(\cos(180) + i\sin(180)) = 3cis(180^\circ)$ $-3 + 0i$

$81^{\frac{1}{4}}(\cos(270) + i\sin(270)) = 3cis(270^\circ)$ $0 - 3i$

1, 3, 9, 27, 81 (sequence with common ratio 3)

1, $3i$, -9, $-27i$, 81 (sequence with common ratio $3i$)

1, -3, 9, -27, 81 (sequence with common ratio -3)

1, $-3i$, -9, $27i$, 81 (sequence with common ratio of $-3i$)

Example: Solve the following $x^4 - 16 = 0$

Method 1: Factoring

$(x^2 - 4)(x^2 + 4) = 0$

$(x + 2)(x - 2)(x + 2i)(x - 2i) = 0$

$x = -2, 2, 2i, -2i$

Method 2: Simplifying

$x^4 = 16$

$x^2 = \pm 4$

$x^2 = 4 \qquad x^2 = -4$

$x = \pm 2 \qquad x = \pm 2i$

Method 3: Applying DeMoivre's Theorem

$x = \sqrt[4]{16}$ convert 16 into polar/cis form $\Longrightarrow$ 16cis0

Then, apply for 1/4 root

$r^{\frac{1}{n}} \Longrightarrow 16^{\frac{1}{4}} = 2$

1/4 root ----> $\frac{2\pi}{4} = \frac{\pi}{2}$

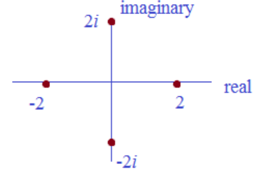
2cis0

2cis $\frac{\pi}{2}$

2cis π

2cis $\frac{3\pi}{2}$

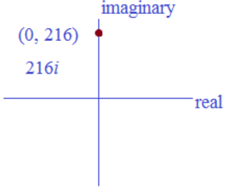
convert back to complex number



Example: Solve $x^3 - 216i = 0$

$x = \sqrt[3]{216i} \Longrightarrow x = (216i)^{\frac{1}{3}}$

$(216\text{cis}\frac{\pi}{2})$ or 216cis90



apply formula (DeMoivre's Theorem)

$(216\text{cis}\frac{\pi}{2})^{\frac{1}{3}}$ or $(216\text{cis}90)^{\frac{1}{3}}$

$\sqrt[3]{216} = 6$

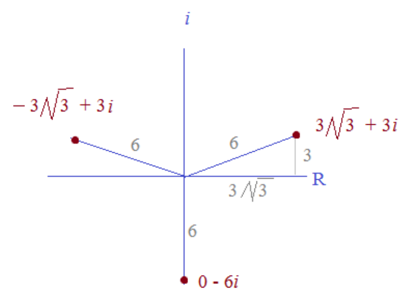
$\frac{1}{3} \cdot 90^\circ = 30^\circ$

since $360^\circ/3 = 120^\circ$

6cis30

6cis150

6cis270



Example: Find 4 fourth roots of -81

write in complex polar form -81 -----> $81cis180^\circ$

$$r^{\frac{1}{n}}\left(\cos\theta + \frac{360^\circ k}{n} + i\sin\theta + \frac{360^\circ k}{n}\right)$$

apply formula

$r = 81 \qquad \theta = 180 \qquad n = 4$

$r^{\frac{1}{4}} = 3 \qquad \frac{\theta}{4} = 45 \qquad \frac{360}{4} = 90$

3cis45	3cis135	3cis225	3cis315

quick check: (3cis45)----> $\left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\right)$

$(3\text{cis}45)^2 = \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\right) \times \left(\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\right)$

$\frac{9}{2} + \frac{9}{2}i + \frac{9}{2}i - \frac{9}{2} = 9i$

$(3\text{cis}45)^2(3\text{cis}45)^2 = 9i \times 9i = -81$ ✓

$81\text{cis}(180) = -81$

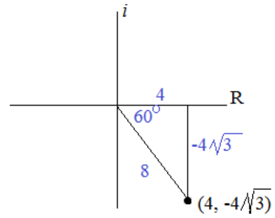
Example: Find 3 cubic roots of $4 - 4\sqrt{3}\,i$

Step 1: Convert to Polar (complex) Coordinates

$4 - 4\sqrt{3}\,i \Longrightarrow (8, -60^\circ)$

$r(\cos\theta + i\sin\theta)$ or $r(cis\,\theta)$

$8cis(-60^\circ)$



Step 2: Apply Formula

$r = 8 \qquad r^{\frac{1}{3}} = 2$

$\frac{-60^\circ}{3} = -20^\circ$

$\frac{360}{3} = 120^\circ$

$2(\cos(-20^\circ) + i\sin(-20^\circ))$

$2(\cos(100^\circ) + i\sin(100^\circ))$

$2(\cos(220^\circ) + i\sin(220^\circ))$

2*cis*(-20)
2*cis*(100)
2*cis*(220)

Step 3: Convert back to Rectangular (complex) Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

2*cis*(-20)	$r\cos\theta = 1.88$	$r\sin\theta = -.68$	$1.88 - .68i$
2*cis*(100)	$r\cos\theta = -.35$	$r\sin\theta = 1.97$	$-.35 + 1.97i$
2*cis*(220)	$r\cos\theta = -1.53$	$r\sin\theta = -1.29$	$-1.53 - 1.29i$

Step 4: (Optional) Quick Check

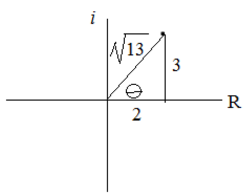
$(2cis(-20))^3 \Longrightarrow 2^3 cis(3 \times (-20)) = 8cis(-60)$ ✓

$(-.35 + 1.97i)^3 \Longrightarrow (-.35 + 1.97i)(-.35 + 1.97i)(-.35 + 1.97i) = 4 - 4\sqrt{3}\,i$ ✓

Example: Expand the following $(2 + 3i)^5$

Method 1: DeMoivre's Theorem

Step 1: convert into polar *cis* form



$r = \sqrt{13}$

$\theta = 56.3099$

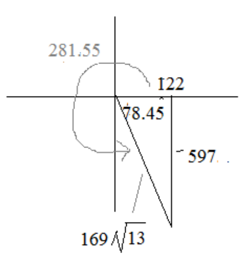
$\sqrt{13}\ cis\ 56.3099^\circ$

Step 2: apply DeMoivre's Theorem

$[r cis\theta]^n = r^n\ cis(n\theta)$

$[\sqrt{13}\ cis\ 56.3099]^5 = 169\sqrt{13}\ cis\ 281.55$

Step 3: convert to rectangular complex form



$121 - 597i$

Method 2: Binomial Expansion Theorem

$(2 + 3i)^5$

Step 1: Apply first part of binomial expansion

$2^5(3i)^0 + 2^4(3i)^1 + 2^3(3i)^2 + 2^2(3i)^3 + 2^1(3i)^4 + 2^0(3i)^5$

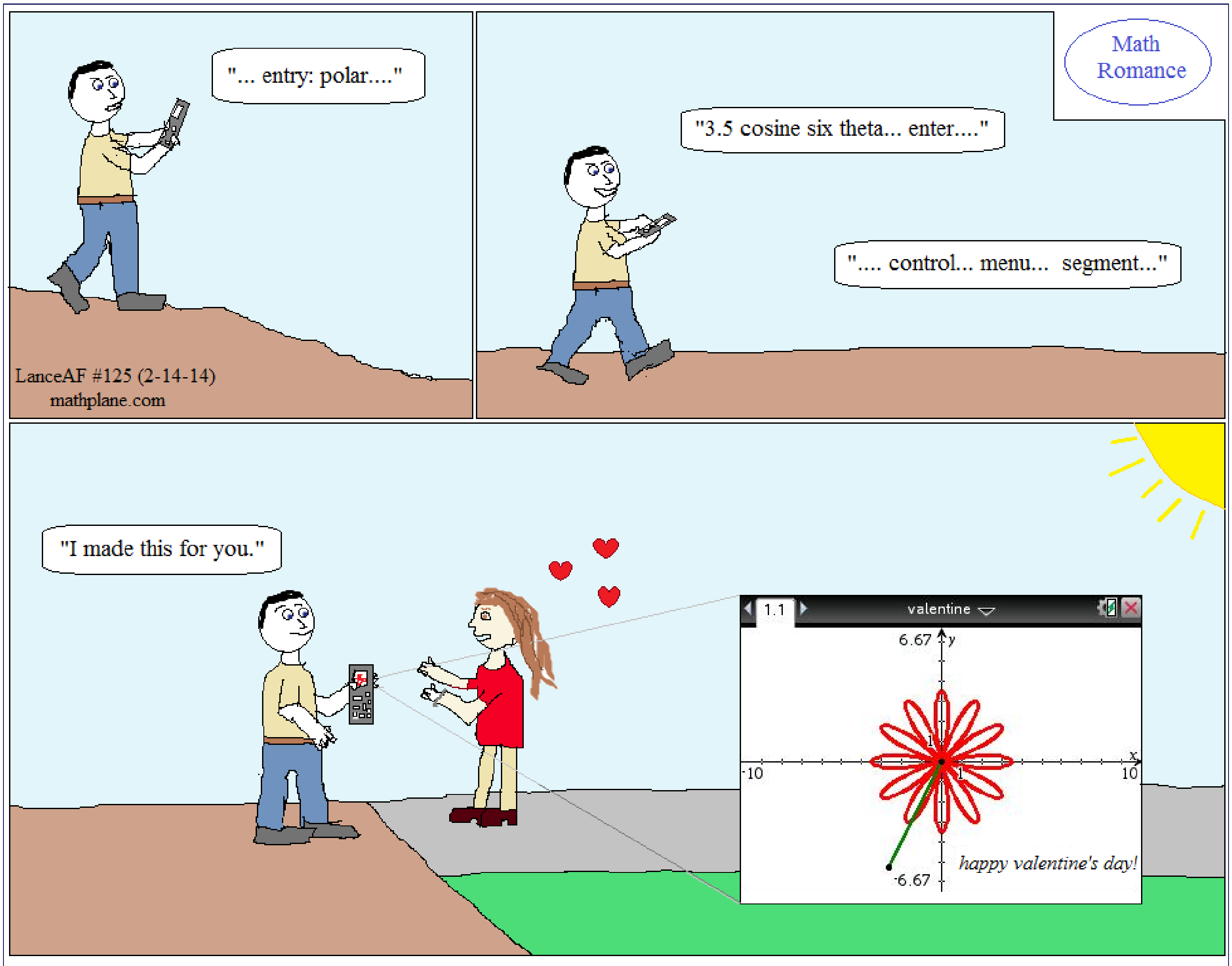
Step 2: Add the coefficients (using Pascal's triangle or combinations)

$\binom{5}{0}2^5(3i)^0 + \binom{5}{1}2^4(3i)^1 + \binom{5}{2}2^3(3i)^2 + \binom{5}{3}2^2(3i)^3 + \binom{5}{4}2^1(3i)^4 + \binom{5}{5}2^0(3i)^5$

$32 + 80(3i) + 80(9i^2) + 40(27i^3) + 10(81i^4) + 243i^5$

$32 + 240i - 720 - 1080i + 810 + 243i$

$121 - 597i$



A Valentine's Day Flower that lasts forever... (as long as you recharge the batteries!)

Practice Quiz-→

Polar and Rectangular Quick Quiz

I. Convert the following:

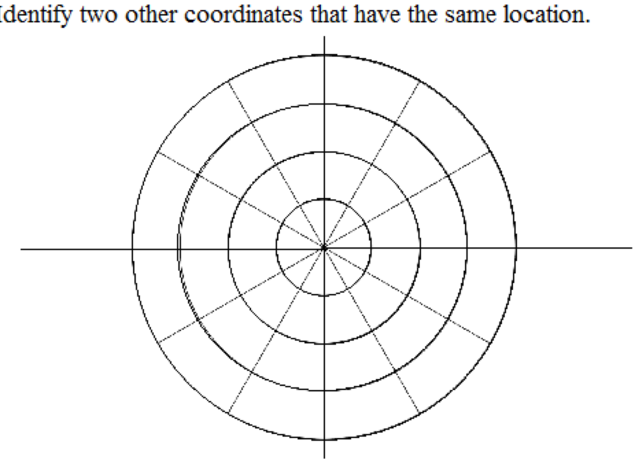
1) Rectangular to Polar

A) $(3, 3)$ B) $(0, -2)$ C) $(-1, \sqrt{3})$

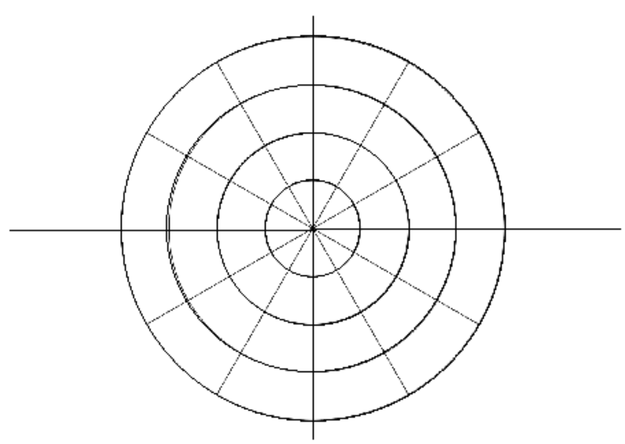
2) Polar to Rectangular

A) $(6, 90^\circ)$ B) $(8, \pi)$ C) $(-2, 60^\circ)$

II. Plot $(3, 120^\circ)$ on the graph. Identify two other coordinates that have the same location.



III. Sketch $r = 1 + \sin\theta$ Give the rectangular equation.



Polar and Rectangular Quick Quiz (continued)

IV: Complex Numbers

1) $Z_1 = 3 - i$ $Z_2 = 4 + 4i$

A) Express Z_1 and Z_2 in polar form

B) Find $Z_1 Z_2$

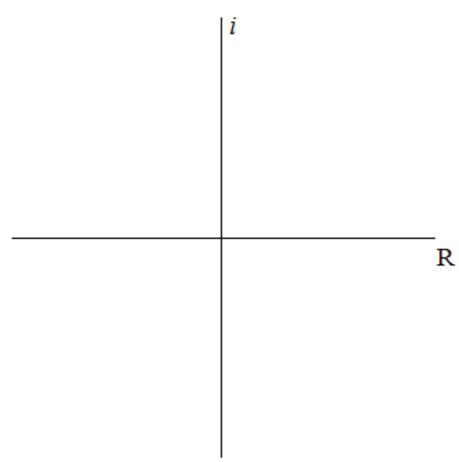
C) Determine $|Z_1|$ and $|Z_2|$

2) $Z = 2\text{Cis}120^\circ$

A) Find Z^2

B) Find Z^5

C) Express the answers in A) and B) in Complex form; and, graph.



Polar and Rectangular Quick Quiz (continued)

Express each product in polar and rectangular form.

A) $(2\text{Cis}115^\circ)(3\text{Cis}65^\circ)$

B) $(8\text{Cis}60^\circ)(\frac{1}{2}\text{Cis}(-120^\circ))$

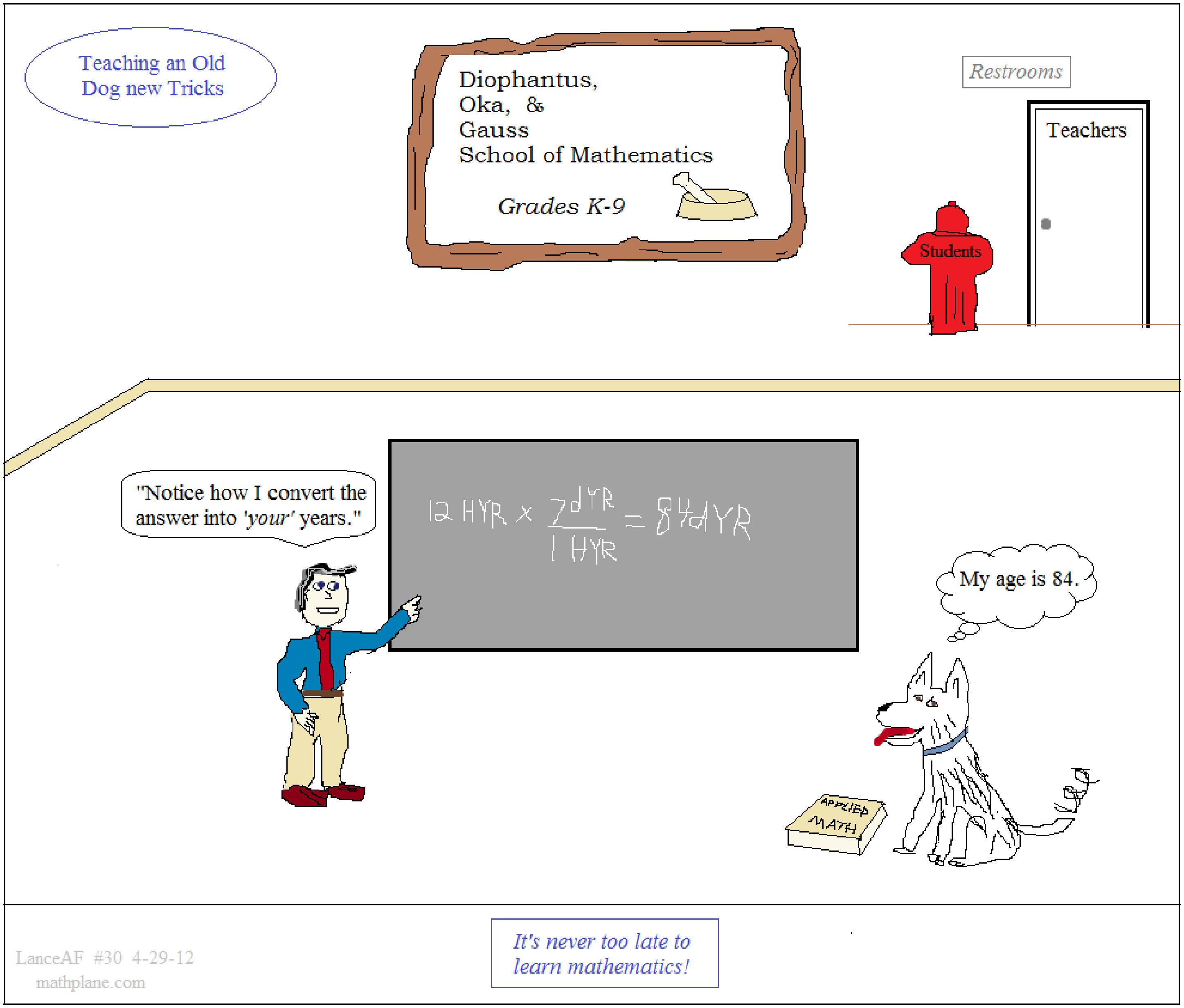
V. Compute using 2 methods — Verify solutions from A) and B) are equivalent!

A) $(1 - i\sqrt{3})(1 - i\sqrt{3})$

B) Convert to polar form (CIS) and solve.

A) $\dfrac{6\text{Cis}30^\circ}{3\text{Cis}150^\circ}$

B) Convert to Complex/Rectangular Form a + b*i*. then, divide to confirm the answer in A)


Teaching an Old Dog new Tricks
Diophantus,
Oka, &
Gauss
School of Mathematics
Grades K-9
Restrooms
Teachers
Students
"Notice how I convert the answer into 'your' years."
12 HYR x 7 dYR / 1 HYR = 84 dYR
My age is 84.
APPLIED MATH
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It's never too late to learn mathematics!

Solutions -→

Polar and Rectangular Quick Quiz

SOLUTIONS

I. Convert the following:

1) Rectangular to Polar

A) (3, 3)

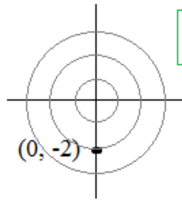
$x^2 + y^2 = r^2$ $\quad \tan\theta = \frac{y}{x}$

$9 + 9 = r^2$ $\quad = \frac{3}{3}$

$r = 3\sqrt{3}$ $\quad \theta = 45°$

$(3\sqrt{3}, 45°)$

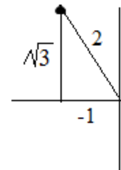
B) (0, −2)



$(2, 270°)$

also, $(-2, 90°)$

C) $(-1, \sqrt{3})$



(30-60-90 triangle)

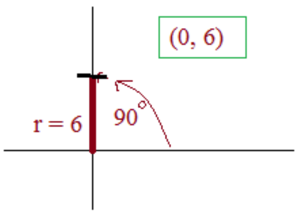
$(2, 120°)$

$x = r\cos\theta$

$(-1) = (2)\cos 120°$ ✓

2) Polar to Rectangular

A) $(6, 90°)$



(0, 6)

B) $(8, \pi)$

$x = r\cos\theta$

$x = 8(-1) = -8$

$y = r\sin\theta$

$y = 8(0) = 0$

(-8, 0)

C) $(-2, 60°)$

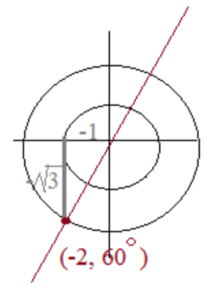
$x = -2\cos 60$

$= -2(1/2) = -1$

$y = -2\sin 60$

$= -2(\sqrt{3}/2) = -\sqrt{3}$

$(-1, -\sqrt{3})$



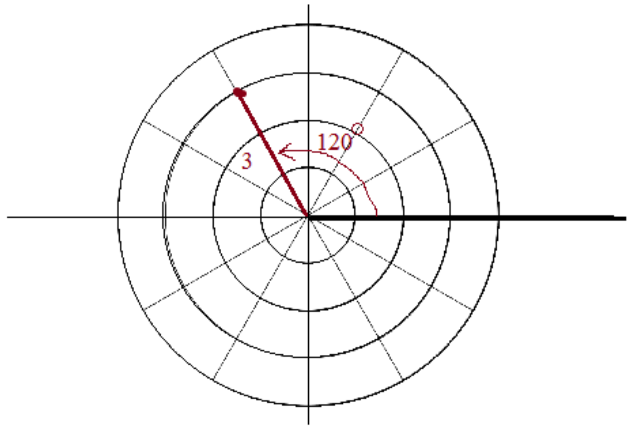
II. Plot $(3, 120°)$ on the graph. Identify two other coordinates that have the same location.

$(3, 480°)$

$(-3, -60°)$

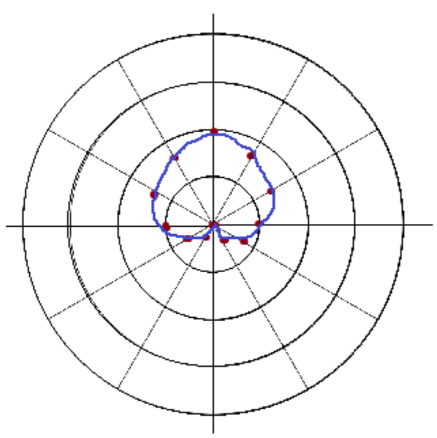
$(3, -240°)$

are 3 possibilities....



III. Sketch $r = 1 + \sin\theta$ Give the rectangular equation.

θ	r
0	1
30	3/2
60	$(2+\sqrt{3})/2$
90	2
120	$(2+\sqrt{3})/2$
150	3/2
180	1
210	1/2
240	$(2-\sqrt{3})/2$
270	0
330	1/2
360	1



$\sin\theta = \frac{y}{r}$

$r^2 = x^2 + y^2$

$r = \sqrt{x^2 + y^2}$

(substitute and simplify)

$r = 1 + \frac{y}{r}$

$r^2 = r + y$

$x^2 + y^2 = \sqrt{x^2 + y^2} + y$

Polar and Rectangular Quick Quiz (continued)

SOLUTIONS

IV: Complex Numbers

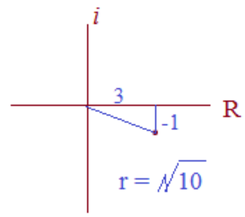
1) $Z_1 = 3 - i$ $\quad$ $Z_2 = 4 + 4i$

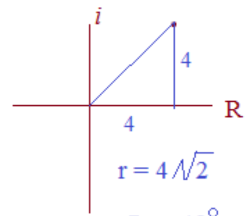
(45-45-90 triangle)

A) Express Z_1 and Z_2 in polar form

$Z = \sqrt{10}\text{Cis}(341.6^\circ)$

$Z = 4\sqrt{2}\text{Cis}45^\circ$





$\theta = 45^\circ$

B) Find $Z_1 Z_2$

$4\sqrt{20}\text{Cis}(386.6^\circ) =$

$8\sqrt{5}\text{Cis}(26.6^\circ)$

$\tan\theta = -1/3$

$\theta = -18.4^\circ$

$= 341.6^\circ$

C) Determine $|Z_1|$ and $|Z_2|$

$|Z_1| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$ $\qquad$ $|Z_2| = 4\sqrt{2}$

2) $Z = 2\text{Cis}120$

A) Find Z^2

$(2\text{Cis}120^\circ)(2\text{Cis}120^\circ) = (2\text{x}2)\text{Cis}(120+120) = 4\text{Cis}240^\circ$

B) Find Z^5

$2^5\ \text{Cis}(5 \text{ x } 120) = 32\text{Cis}(600^\circ) = 32\text{Cis}240^\circ$

C) Express the answers in A) and B) in Complex form; and, graph.

2Cis120

$= (-1, -\sqrt{3})$

$r = 4$

$x = r\cos 240$

$x = 4\ (-1/2) = -2$

$y = r\sin 240$

$y = 4\ (-\sqrt{3}/2) = -2\sqrt{3}$

$(-2, -2\sqrt{3})$

$r = 32$

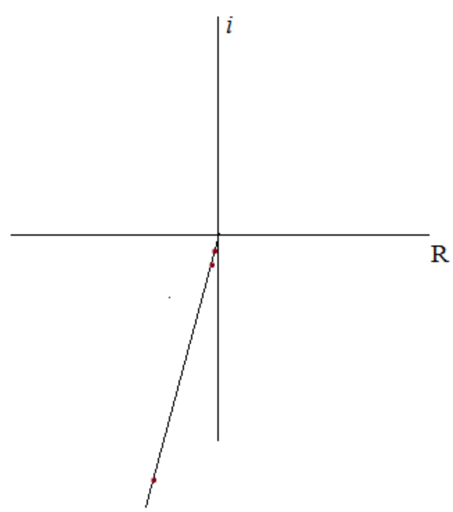
$x = 32\cos 240$

$x = 32(-1/2) = -16$

$y = 32\sin 240$

$y = 32\ (-\sqrt{3}/2) = -16\sqrt{3}$

$(-16, -16\sqrt{3})$



Polar and Rectangular Quick Quiz (continued)

SOLUTIONS

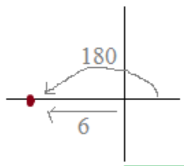
Express each product in polar and rectangular form.

$Z_1 Z_2 = r_1 r_2 \text{Cis}(\theta_1 + \theta_2)$

A) $(2\text{Cis}115^\circ)(3\text{Cis}65^\circ)$

$2 \cdot 3 \text{ Cis}(115 + 65) =$

$6\text{Cis}(180^\circ)$ (Polar)



$-6 + 0i = -6$ (rectangular)

B) $(8\text{Cis}60^\circ)(\frac{1}{2}\text{Cis}(-120^\circ))$

$8 \cdot \frac{1}{2} \text{ Cis}(60 + (-120)) =$

$4\text{Cis}(-60) =$

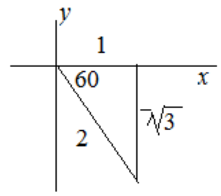
$4\text{Cis}(300^\circ)$

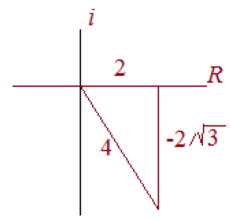
(note: 300 is the coterminal angle of -60 that is between 0 and 360)

$4(\cos 300 + i \sin 300) =$

$4(\frac{1}{2} - i\frac{\sqrt{3}}{2})$

$2 - i\,2\sqrt{3}$





V. Compute using 2 methods — Verify solutions from A) and B) are equivalent!

A) $(1 - i\sqrt{3})(1 - i\sqrt{3})$

"FOIL" $1 - i\sqrt{3} - i\sqrt{3} + i^2(3)$

$1 - 2i\sqrt{3} - 3$

$-2 - i\,2\sqrt{3}$

B) Convert to polar form (CIS) and solve.

$1 - i\sqrt{3}$

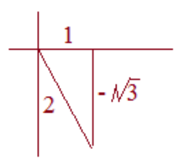
$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$

$\theta = 300^\circ$

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{\sqrt{3}}{1}$

$1 - i\sqrt{3} = 2\text{Cis}300^\circ$



$(2\text{CIS}300)(2\text{CIS}300) =$

$4\text{CIS}600^\circ = -360^\circ$

$4\text{Cis}240^\circ$

$4(\cos 240 + i \sin 240)$

$4(-1/2 - i\sqrt{3}/2)$

$-2 - i\,2\sqrt{3}$

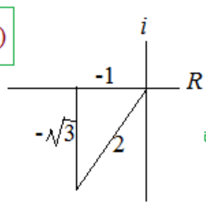
A) $\frac{6\text{Cis}30^\circ}{3\text{Cis}150^\circ}$

$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \text{CIS}(\theta_1 - \theta_2)$

$= \frac{6}{3} \text{CIS}(30 - 150)$

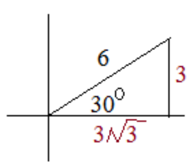
$= 2\text{Cis}(-120)$

$= 2\text{Cis}(240^\circ)$



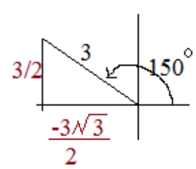
B) Convert to Complex/Rectangular Form a + b*i*. then, divide to confirm the answer in A)

$6\text{Cis}30 =$



$3\sqrt{3} + 3i$

$3\text{Cis}150 =$



$\frac{-3\sqrt{3}}{2} + \frac{3i}{2}$

$\frac{1}{2}(-3\sqrt{3} + 3i)$

$\frac{3\sqrt{3} + 3i}{\frac{1}{2}(-3\sqrt{3} + 3i)}$ multiply by $\frac{2}{2}$

$\frac{6\sqrt{3} + 6i}{(-3\sqrt{3} + 3i)} \cdot \frac{(-3\sqrt{3} - 3i)}{(-3\sqrt{3} - 3i)}$ mult. by conjugate/conjugate

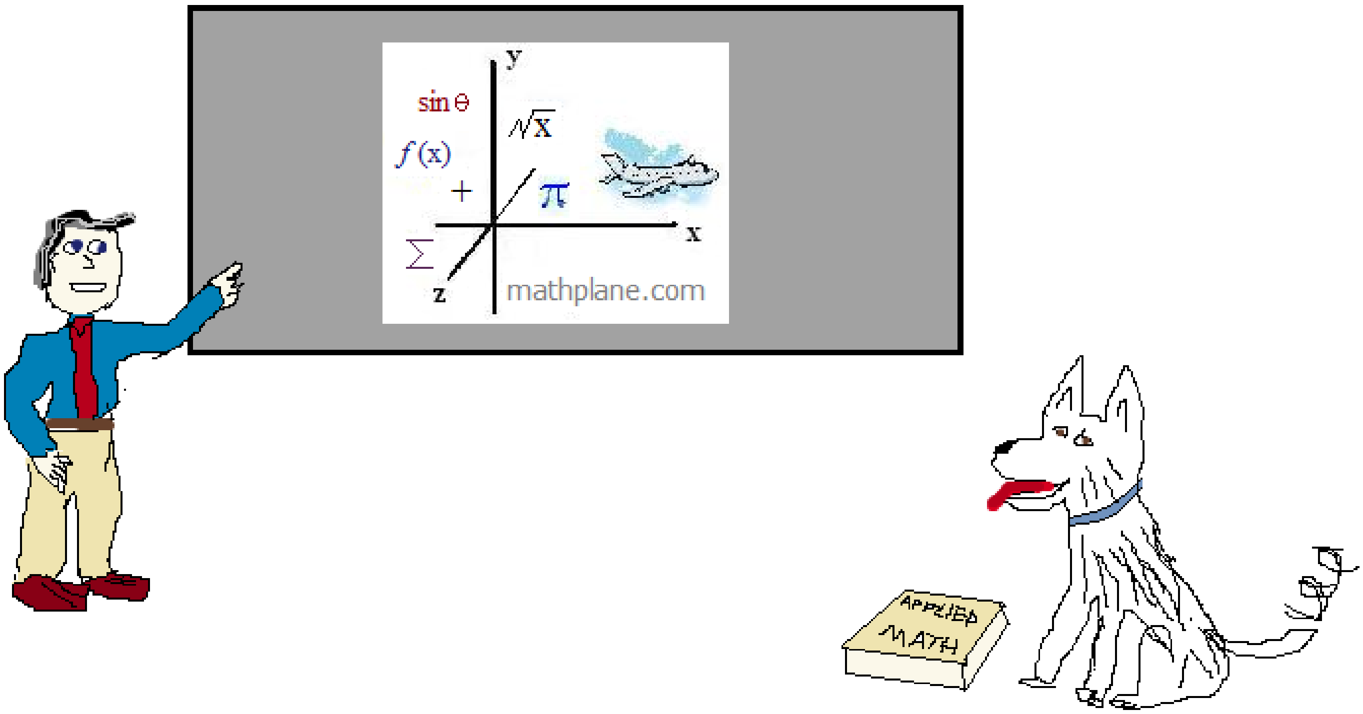
$\frac{-54 + 18 - 18\sqrt{3}\,i - 18\sqrt{3}\,i}{27 + 9}$

$-1 - \sqrt{3}\,i$

Thanks for downloading this packet. (Hope it helps!)

If you have questions, suggestions, or feedback, let us know.

Cheers



Also, at TES, and TeachersPayTeachers

Mathplane *Express* for mobile at Mathplane.org

Imaginary and Complex Numbers Topics

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

a + bi

a is the 'real' part

bi is the 'imaginary' part

$(4 + 3i) - (2 - 5i)$ Adding/Subtracting complex numbers

$2 + 8i$

$(2 + 5i)(3 - i)$ Multipying complex numbers

$6 - 2i + 15i - 5i^2$

$6 + 13i - 5(-1)$

$11 + 13i$

$\frac{4}{2 - 3i}$ Dividing or Simplifying complex rational expression

multiply by conjugate to simplify into a + bi form

$$\frac{4}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{8+12i}{4+6i-6i-9i^2} = \frac{8+12i}{4-9i^2} = \frac{8+12i}{13} \Rightarrow \frac{8}{13} + \frac{12}{13}i$$

i^{17} Reducing i^n to its lowest term

$i^{16} \cdot i^1$

$1 \cdot i$

i

$x^2 + 2x + 7 = 0$ Solving equations with Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(7)}}{2(1)} = \frac{-2 \pm \sqrt{-24}}{2} = \frac{-2 \pm 2i\sqrt{6}}{2} \Rightarrow -1 \pm i\sqrt{6}$$

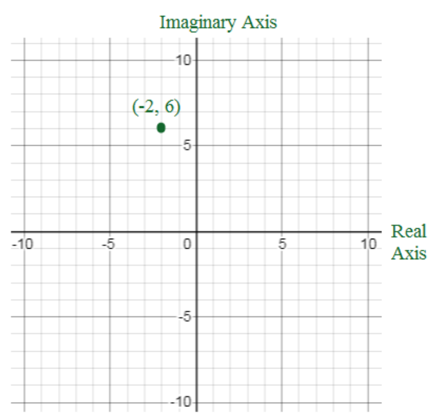
$4x^2 + 27 = 11$ Solving algebraic equations

$4x^2 = -16$

$x^2 = -4 \Rightarrow \sqrt{x^2} = \sqrt{-4}$

$x = \pm 2i$

Plot $-2 + 6i$ on the Complex Plane



Imaginary & Complex Numbers: Quick Quiz

Part I: Simplify

1) $i^2 =$

2) $i^{51} =$

3) $i^8 =$

4) $i^{-5} =$

Part II: Simplify

1) $\sqrt{-25} =$

2) $\sqrt{-72} =$

3) $\sqrt[3]{-8} =$

4) $\sqrt{-4ab^3} =$

Part III: Complex numbers

Given: $w = 3i + 7$
$v = 2i - 5$

Find: 1) $w + v$

2) $3w$

3) vw

Solutions must be in standard form: $a + bi$

4) w^2

5) $\frac{1}{v}$

6) v^3

Part IV: Solve

1) $x^2 + 3x + 10 = 0$

2) $3(x + 8)^2 = -15$

3) $\frac{3i + 4}{4i - 9} =$

4) $(5i - 6)^2 =$

5) $(7 - 8i)(7 + 8i) =$

Imaginary & Complex numbers: Quick Quiz

SOLUTIONS

Part I: Simplify

1) $i^2 = -1$

2) $i^{51} = i^{48} \cdot i^3 = 1 \cdot i^3 = -i$

3) $i^8 = 1$

4) $i^{-5} = i^{-8} \cdot i^3 = \frac{1}{i^8} \cdot i^3 = \frac{1}{1} \cdot -i = -i$

Part II: Simplify

1) $\sqrt{-25} = 5i$

2) $\sqrt{-72} = \sqrt{(-1)(2)(36)}$

$6i\sqrt{2}$

3) $\sqrt[3]{-8} = -2$

$(-2)(-2)(-2) = -8$

4) $\sqrt{-4ab^3} = 2bi\sqrt{ab}$

Part III: Complex numbers

Given: $w = 3i + 7$
$v = 2i - 5$

Solutions must be in standard form: a + bi

Find:

1) $w + v$

$$\begin{array}{r} 3i + 7 \\ 2i - 5 \\ \hline \boxed{5i + 2} \end{array}$$

2) $3w$

$3(3i + 7)$

$\boxed{9i + 21}$

3) vw

$(2i - 5)(3i + 7)$

$6i^2 - 15i + 14i - 35$

$6(-1) - i - 35 = \boxed{-41 - i}$

4) w^2

$(3i + 7)(3i + 7)$

$9i^2 + 21i + 21i + 49$

$\boxed{40 + 42i}$

5) $\frac{1}{v}$

$\frac{1}{(2i - 5)} \cdot \frac{(2i + 5)}{(2i + 5)} =$

$\frac{2i + 5}{4i^2 - 25} = \frac{5 + 2i}{-29} =$

$\boxed{\frac{-5}{29} - \frac{2}{29}i}$

6) $v^3 = (2i - 5)(2i - 5)(2i - 5)$

$(2i - 5)(2i - 5) = -4 - 20i + 25$

$= 21 - 20i$

then, $(2i - 5)(-20i + 21)$

$-40i^2 + 100i + 42i - 105$

$= 40 + 142i - 105 = \boxed{-65 + 142i}$

Part IV: Solve

1) $x^2 + 3x + 10 = 0$

(use quadratic formula)

$\frac{-3 \pm \sqrt{9 - 4(1)(10)}}{2(1)} =$

$\boxed{\frac{-3 \pm i\sqrt{31}}{2}}$

2) $3(x + 8)^2 = -15$

$(x + 8)^2 = -5$

$(x + 8) = \pm\sqrt{-5}$

$\boxed{x = -8 \pm i\sqrt{5}}$

3) $\frac{3i + 4}{4i - 9} =$

$\frac{3i + 4}{4i - 9} \cdot \frac{4i + 9}{4i + 9} =$

$\frac{12i^2 + 16i + 27i + 36}{16i^2 - 81} =$

$\frac{24 + 43i}{-97} = \boxed{\frac{-24}{97} - \frac{43i}{97}}$

4) $(5i - 6)^2 =$

$(5i - 6)(5i - 6) =$

$25i^2 - 30i - 30i + 36 =$

$-25 - 60i + 36 =$

$\boxed{11 - 60i}$

5) $(7 - 8i)(7 + 8i) =$

$49 - 56i + 56i - 64i^2 =$

$49 + 64 = \boxed{113}$