PreCalculus: Minimum/Maximum Value

Given a, b, and c are real numbers, and $a + 2022 = be^{c-b}$ (b>0)
Find the minimum value of $(a + 2022)bc$

$\because$ $(a + 2022)bc = be^{c-b} \cdot bc = b^2c \cdot e^{c-b} = \frac{b^2}{e^b} \cdot c \cdot e^c$ (b>0)

Let $f(x) = \frac{x^2}{e^x}$ (x>0)

We have $f(x) = \frac{2xe^x - x^2e^x}{(e^x)^2} = \frac{2x - x^2}{e^x}$

$\therefore$ $f(x)$is monotonically increasing on interval (0,2), and monotonically decreasing on interval (2,+∞)

$\because$ $f(x) > 0$

$\therefore$ $f(x)_{max} = f(2) = \frac{4}{e^2}$

Let g(x) = xe^x, then $g'(x) = (x + 1)e^x$. Therefore, g(x) is monotonically decreasing on interval (-∞,-1) and monotonically increasing on interval (-1,+∞)

$\therefore$ $g(x)_{min} = g(-1) = -\frac{1}{e}$

Thus, $(a + 2022)bc = f(b) \cdot g(c) \geq \frac{4}{e^2} \cdot \left(-\frac{1}{e}\right) = -\frac{4}{e^3}$

If and only if b=2 and c=-1, $(a + 2022)bc = -\frac{4}{e^3}$

PreCalculus: Minimum/Maximum Value

Given $a^2 + b^2 = 1$, (a>0, b>0), find the minimum value of $\frac{a^4+b^4}{a+b}$

Method I:

Let $a = \cos\theta$, $b = \sin\theta$, $and\ 0 < \theta \le \frac{\pi}{2}$

$\therefore\ a^2 + b^4 = \cos^4\theta + \sin^4\theta = 1 - 2\cos^2\theta\sin^2\theta = 1 - \frac{1}{2}\sin^2\theta \ge \frac{1}{2}$

$\because\ a + b = \cos\theta + \sin\theta \le \sqrt{2}$

$\therefore\ \frac{a^4+b^4}{a+b} \ge \frac{\frac{1}{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$

Method II:

Let $a = \cos\theta$, $b = \sin\theta$, $and\ 0 < \theta \le \frac{\pi}{2}$

Let $\sin\theta + \cos\theta = t$

We have $t\epsilon[-\sqrt{2}, \sqrt{2}]$

$\therefore\ \frac{a^4+b^4}{a+b} = \frac{1-\frac{1}{2}(t^2-1)^2}{t} = \frac{1-\frac{1}{2}t^4+t^2-\frac{1}{2}}{t} = -\frac{1}{2}t^3 + t + \frac{1}{2t}$

$\therefore\ y' = \frac{-3t^4+2t^2-1}{2t^2}$

$\because\ 3t^4 - 2t^2 + 1 > 0\ is\ true$

$\therefore\ y' < 0$

$\therefore\ y = f(x)$ is monotonically decreasing on interval $[-\sqrt{2}, \sqrt{2}]$

$\therefore\ f(t)_{min} = f(\sqrt{2}) = \frac{\sqrt{2}}{4}$

$\therefore\ \frac{a^4+b^4}{a+b} \ge \frac{\sqrt{2}}{4}$

Method III:

Since $a^2 + b^2 \ge 2ab$ (a>0, b>0)

$\therefore\ 2(a^2 + b^2) \ge (a + b)^2$

That is $a^2 + b^2 \ge \frac{(a+b)^2}{2}$

$\therefore\ a^4 + b^4 \ge \frac{(a^2+b^2)^2}{2} = \frac{1}{2}$

$\because\ a^2 + b^2 \ge \frac{(a+b)^2}{2}$ and $a + b \le \sqrt{2}$ (a>0, b>0)

$\therefore\ \frac{a^4+b^4}{a+b} \ge \frac{\frac{1}{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$ (if and only if a=b, $\frac{a^4+b^4}{a+b} = \frac{\sqrt{2}}{4}$)