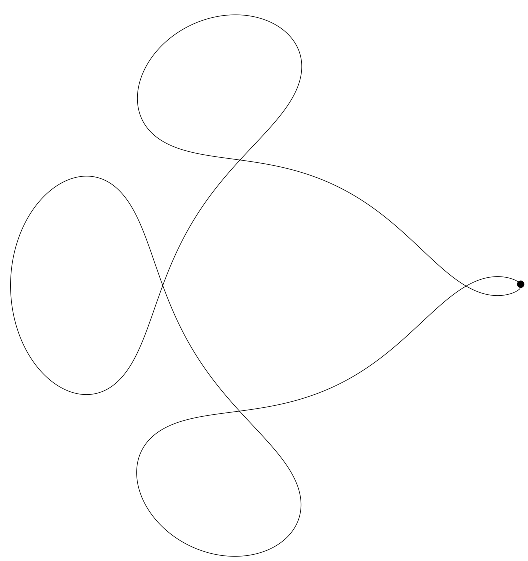
DAE Solver Examples

daesolver@icloud.com

Astronomical Orbit



The classical three body problem for planar motion of a body of negligible mass around two objects of very different masses, a and b. The figure shows how the body coordinates, represented by u_1 and u_2, change in time.

Equations

$$\frac{du_1}{dt} = u_3$$

$$\frac{du_2}{dt} = u_4$$

$$\frac{du_3}{dt} = u_1 + 2u_4 - b\frac{u_1 + a}{D_1} - a\frac{u_1 - b}{D_2}$$

$$\frac{du_4}{dt} = u_2 - 2u_3 - b\frac{u_2}{D_1} - a\frac{u_2}{D_2}$$

Variable Definitions

$$D_1 = ((u_1 + a)^2 + u_2^2)^{3/2}$$

$$D_2 = ((u_1 - b)^2 + u_2^2)^{3/2}$$

Parameters

$$a = 0.012277471$$

$$b = 1 - a$$

Initial Conditions

$$u_1(0) = 0.994$$

$$u_2(0) = 0$$

$$u_3(0) = 0$$

$$u_4(0) = -2.00158510637908252240537862224$$

Autocatalytic Scheme

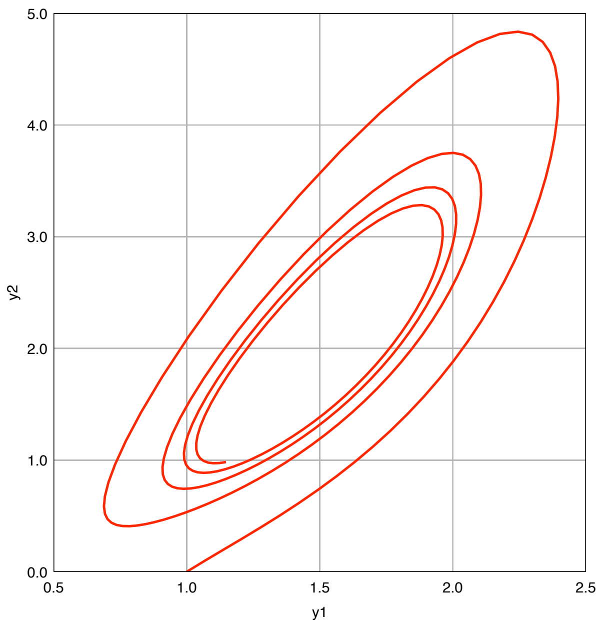
Dynamic behavior of an autocatalytic reaction scheme in an isothermal continuous stirred tank reactor. Species concentrations are represented by y_1, y_2, and y_3, with the second and third species combining in the autocatalytic step that produces the third species. Example from the book *Practical Bifurcation and Stability Analysis* by Rüdiger Seydel.

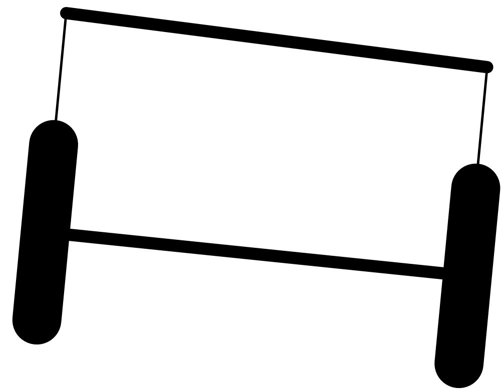
Equations

$$\frac{dy_1}{dt} = \psi - y_1 - \lambda y_1 y_3$$

$$\frac{dy_2}{dt} = y_1 - y_2 y_3$$

$$\frac{dy_3}{dt} = y_2 y_3 - \lambda y_1 y_3$$





Parameters

$\psi = 3$

$\lambda = 1.33$

Initial Conditions

$y_1(0) = 1$

$y_2(0) = 0$

$y_3(0) = 1$

Car Axis

An index-3 differential algebraic system simulating the behavior of a car axis on a bumpy road. The left wheel travels on a flat surface while the right wheel follows a sinusoidal surface. In the diagram, x_l and y_l represent the coordinates of the left point of the car body while x_r and y_r represent the coordinates of right point of the car body. These are also p_1 through p_4 in the canonical equations describing the problem. The left and right points must remain apart by a fixed distance L. This constraint is the source of one of the algebraic equations in the model. The car axis has a fixed point, the origin, as its left point, with the coordinates of its right point being x_b and y_b. This point is the one exposed to the bumpy road and it also must remain at a fixed distance L from the origin.

Equations

$$\frac{dp_1}{dt} = q_1$$

$$\frac{dp_2}{dt} = q_2$$

$$\frac{dp_3}{dt} = q_3$$

$$\frac{dp_4}{dt} = q4$$

$$\epsilon^2 \frac{M}{2} \frac{dq_1}{dt} = f_1$$

$$\epsilon^2 \frac{M}{2} \frac{dq_2}{dt} = f_2$$

$$\epsilon^2 \frac{M}{2} \frac{dq_3}{dt} = f_3$$

$$\epsilon^2 \frac{M}{2} \frac{dq_4}{dt} = f_4$$

$$0 = x_l x_b + y_l y_b$$

$$L^2 = (x_l - x_r)^2 + (y_l - y_r)^2$$

Variable Definitions

$x_l = p_1$

$y_l = p_2$

$x_r = p_3$

$y_r = p_4$

$y_b = h\sin(\omega t)$

$x_b = \sqrt{L^2 - y_b^2}$

$L_l = \sqrt{x_l^2 + y_l^2}$

$Lr = \sqrt{(x_r - x_b)^2 + (y_r - y_b)^2}$

$f_1 = \dfrac{(L_o - L_l)x_l}{L_l} + \lambda_1 x_b + 2\lambda_2(x_l - x_r)$

$f_2 = \dfrac{(L_o - L_l)y_l}{L_l} + \lambda_1 y_b + 2\lambda_2(y_l - y_r) - \epsilon^2 \dfrac{M}{2}$

$f_3 = \dfrac{(L_o - L_r)(x_r - x_b)}{L_r} - 2\lambda_2(x_l - x_r)$

$f_4 = \dfrac{(L_o - L_r)(y_r - y_b)}{L_r} - 2\lambda_2(y_l - y_r) - \epsilon^2 \dfrac{M}{2}$

Parameters

$L = 1$

$L_o = \dfrac{1}{2}$

$\epsilon = 0.01$

$M = 10$

$h = 0.1$

$\tau = \dfrac{\pi}{5}$

$\omega = 10$

Initial Conditions

$p_1(0) = 0$

$p_2(0) = \dfrac{1}{2}$

$p_3(0) = 1$

$p_4(0) = \dfrac{1}{2}$

$q_1(0) = -\dfrac{1}{2}$

$q_2(0) = 0$

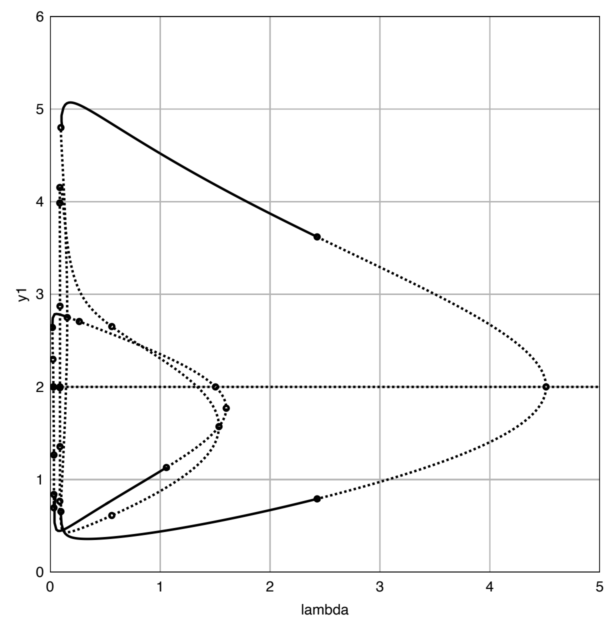
$q_3(0) = -\dfrac{1}{2}$

$q_4(0) = 0$

$\lambda_1(0) = 0$

$\lambda_2(0) = 0$

Coupled Cells



A variant on the Brusselator involving three connected reaction cells, with the coupling coefficient λ as bifurcation parameter. This parameter, with an initial value of 1, is allowed to vary between 0 and 5. Two reaction intermediates, X and Y, have their concentrations represented by y_1 and y_2 in the first cell, y_3 and y_4 in the second cell, and y_5 and y_6 in the third cell. Additional species A, B, D, and E are present, but their concentrations are not tracked since they are assumed to be constant. Rich dynamic behavior is observed in this problem, with multiple bifurcation points. Example from the book

Practical Bifurcation and Stability Analysis by Rüdiger Seydel.

Equations

$$0 = 2 - 7y_1 + y_1^2 y_2 + \lambda(y_3 - y_1)$$

$$0 = 6y_1 - y_1^2 y_2 + 10\lambda(y_4 - y_2)$$

$$0 = 2 - 7y_3 + y_3^2 y_4 + \lambda(y_1 + y_5 - 2y_3)$$

$$0 = 6y_3 - y_3^2 y_4 + 10\lambda(y_2 + y_6 - 2y_4)$$

$$0 = 2 - 7y_5 + y_5^2 y_6 + \lambda(y_3 - y_5)$$

$$0 = 6y_5 - y_5^2 y_6 + 10\lambda(y_4 - y_6)$$

Parameters

$\lambda = 1$

Initial Conditions

$y_1 = 4$

$y_2 = 2$

$y_3 = 1$

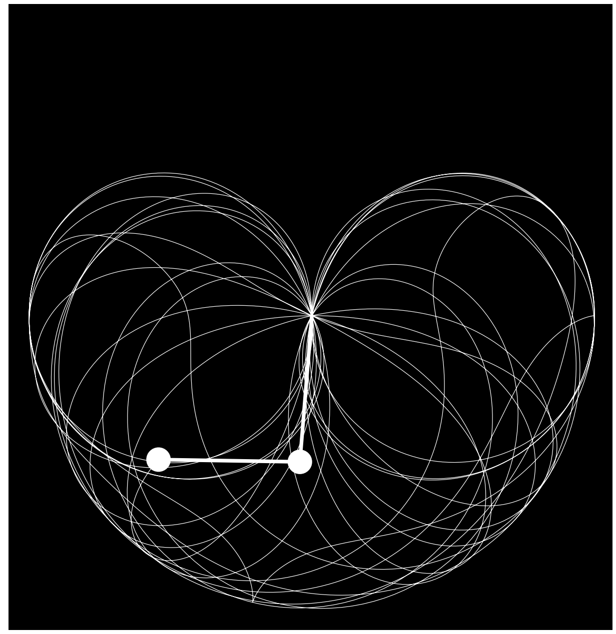
$y_4 = 3$

$y_5 = 0$

$y_6 = 4$

Double Pendulum

Famous deterministic system illustrating chaotic behavior. Coordinates of the first ball are represented by q_1 and q_2 while coordinates of the second ball relative to the first are represented by r_1 and r_2. The two algebraic equations in this system arise from the fact that the first ball must stay at a constant distance from the origin, while the second ball must stay at a constant distance from the first.



Equations

$$0 = \frac{dq_1}{dt} - v_1$$

$$0 = \frac{dq_2}{dt} - v_2$$

$$0 = \frac{dv_1}{dt} + yq_1$$

$$0 = \frac{dv_2}{dt} + yq_2 + g$$

$$0 = 1 - q_1^2 - q_2^2$$

$$0 = \frac{dq_1}{dt} + \frac{dr_1}{dt} - w_1$$

$$0 = \frac{dq_2}{dt} + \frac{dr_2}{dt} - w_2$$

$$0 = \frac{dv_1}{dt} + \frac{dw_1}{dt} + zr_1$$

$$0 = \frac{dv_2}{dt} + \frac{dw_2}{dt} + zr_2 + g$$

$$0 = 1 - r_1^2 - r_2^2$$

Parameters

$g = 1$

Initial Conditions

$q_1(0) = 1$

$q_2(0) = 0$

$v_1(0) = 0$

$v_2(0) = 0$

$y(0) = 0$

$r_1(0) = 1$

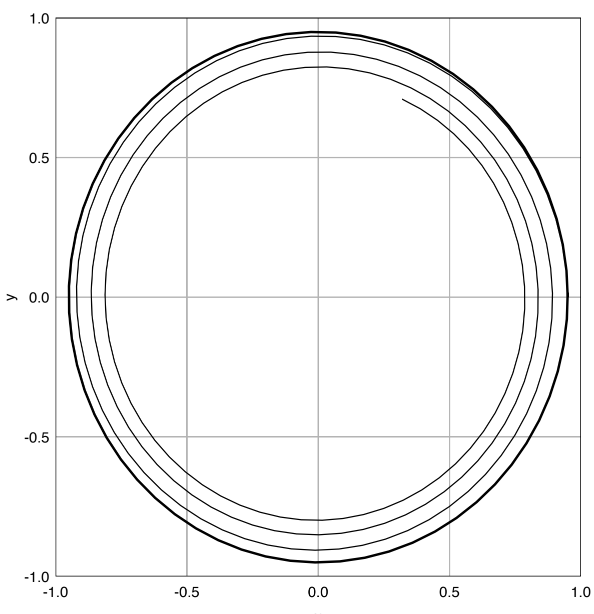
$r_2(0) = 0$

$w_1(0) = 0$

$w_2(0) = 0$

$z(0) = 0$

Drawing Circles



Example illustrating the different results generated by two implicit one-step methods at a fixed step size of 0.02. The Implicit Midpoint method correctly predicts the solution (thick circle) while the Backward Euler method increasingly deviates from that solution (thin spiral).

Equations

$$\frac{dx}{dt} = -y$$

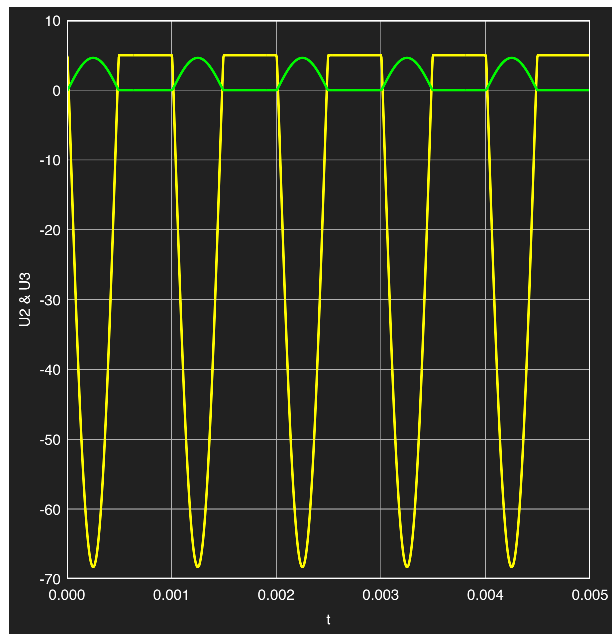
$$\frac{dy}{dt} = x$$

Initial Conditions

$x(0) = 0.95$

$y(0) = 0$

Electric Circuit



Circuit containing 5 resistors, 1 capacitor, 2 power sources, and 2 nonbipolar transistors. Solved using the Third Order Radau method to better handle the nonlinearities introduced by the transistors. System unknowns U_1 through U_5 are voltages at various points across the circuit. Example from the book *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations* by Uri Ascher and Linda Petzold.

Equations

$$0 = \frac{U_1 - U_e}{R_1} - (\alpha - 1) f_{13}$$

$$0 = \frac{U_b - U_2}{R_2} + \frac{U_4 - U_2}{R_4} - \alpha f_{13} + C\left(\frac{dU_4}{dt} - \frac{dU_2}{dt}\right)$$

$$0 = \frac{U_3 - U_o}{R_3} - f_{13} - f_{43}$$

$$0 = (\alpha - 1) f_{43} + \frac{U_b - U_2}{R_2} - \alpha f_{13}$$

$$0 = \frac{U_5 - U_b}{R_5} + \alpha f_{43}$$

Variable Definitions

$U_e = 5\sin(2000\pi t)$

$f_{13} = \beta\left(e^{\frac{U_1 - U_3}{U_F}} - 1\right)$

$f_{43} = \beta\left(e^{\frac{U_4 - U_3}{U_F}} - 1\right)$

Parameters

$\alpha = 0.99$

$\beta = 0.000001$

$R_1 = 200$

$R_2 = 1600$

$R_3 = 100$

$R_4 = 3200$

$R_5 = 1600$

$C = 0.00004$

$U_o = 0$

$U_b = 5$

$U_F = 0.026$

Initial Conditions

$U_1(0) = 0$

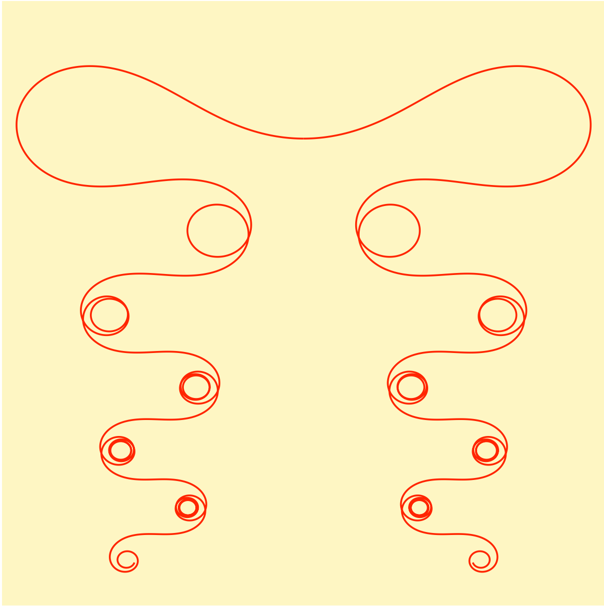
$U_4(0) = 0$

$U_3(0) = 0$

$U_2(0) = U_b$

$U_5(0) = U_b$

Elegant Madness



Parametric curve with arc length as the parameter, represented in this case by t. The right side of the graph shows a curve of y vs x while the left side shows y vs $-x$. Example from the book *Curves for the Mathematically Curious* by Julian Havil.

Equations

$$\frac{dx}{dt} = \cos r$$

$$\frac{dy}{dt} = \sin r$$

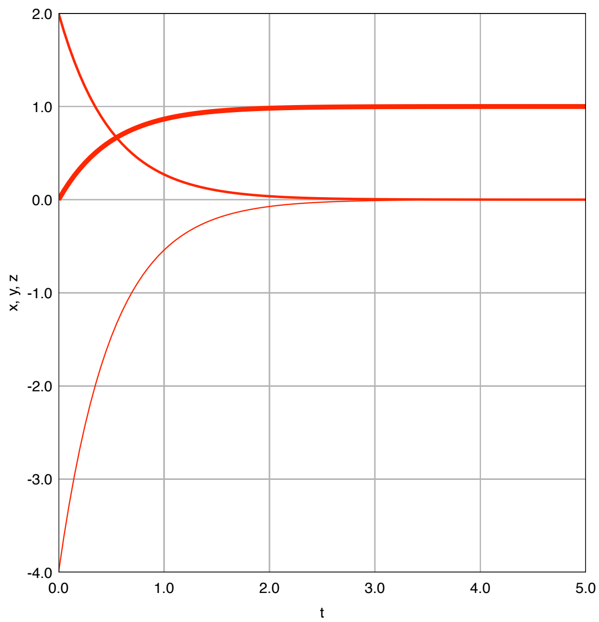
Variable Definitions

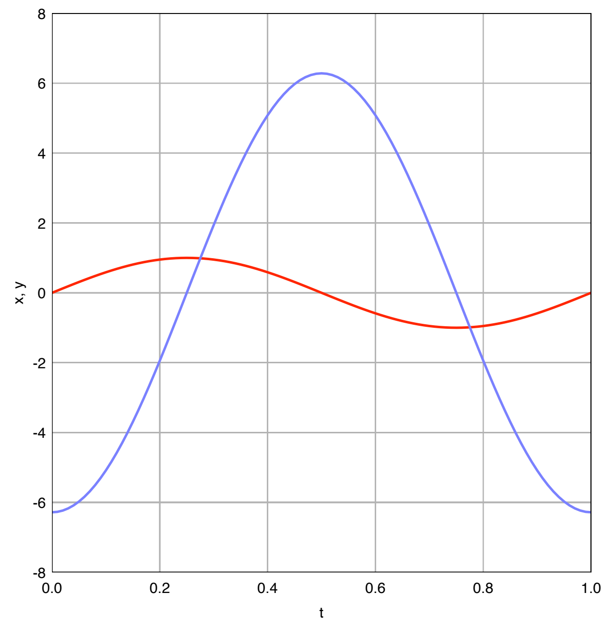
$r = t\cos t$

Initial Conditions

$x(0) = 0$

$y(0) = 0$





Hessenberg Three

Hessenberg form of an index-3 differential-algebraic equation system. Unlike problems with only differential equations, where one has freedom to choose initial values of the unknowns, there is no such freedom in this example, where the initial conditions listed are the only ones satisfying the equations.

Equations

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = z$$

$$x - 1 + e^{-2t} = 0$$

Initial Conditions

$$x(0) = 0$$

$$y(0) = 2$$

$$z(0) = -4$$

Hessenberg Two

Hessenberg form of an index-2 differential-algebraic equation system. Unlike problems with only differential equations, where one has freedom to choose initial values of the unknowns, there is no such freedom in this example, where the initial conditions listed are the only ones satisfying the equations.

Equations

$$\frac{dx}{dt} = -y$$

$$x - \sin(2\pi t) = 0$$

Initial Conditions

$$x(0) = 0$$

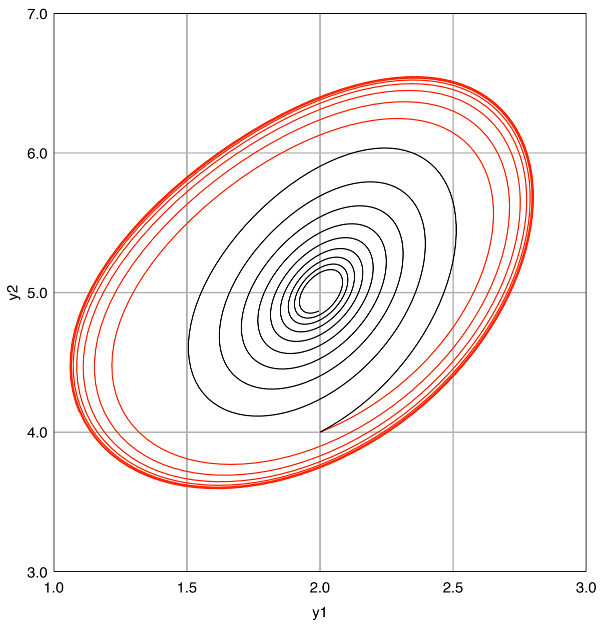
$$y(0) = -2\pi$$

Hopf Bifurcation

Dynamic behavior around a Hopf Bifurcation. When enough time has passed, the system approaches a limit cycle if β is equal to 3 (red curve) or a unique steady state if β is equal to 4 (black curve).

Equations

$$\frac{dy_1}{dt} = \alpha - y_1 \left(1 + 4\frac{y_2}{1 + y_1^2}\right)$$



$$\frac{dy_2}{dt} = \beta y_1 \left(1 - \frac{y_2}{1 + y_1^2}\right)$$

Parameters

$\alpha = 10$

$\beta = 3, 4$

Initial Conditions

$y_1(0) = 2$

$y_2(0) = 4$

Kepler Equation

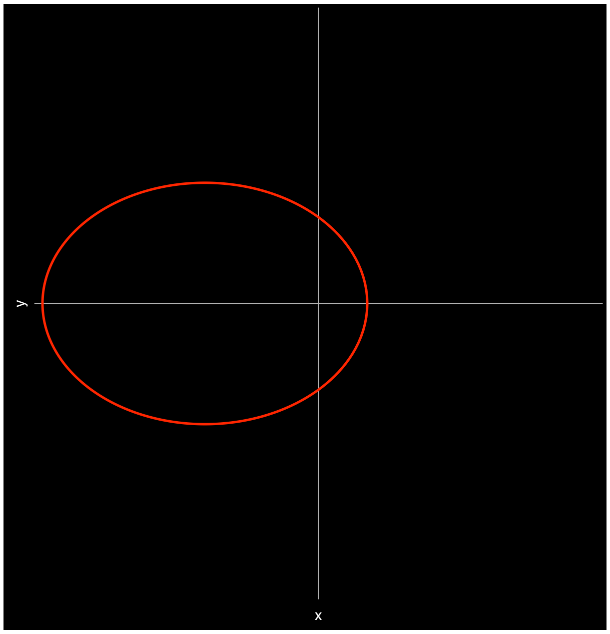
Kepler's equation describes the orbit of a body subject to a central gravitational force located at the point of intersection of the two axes in the figure. The orbit is generated in this case by solving the equation for E using values of t in the range from 0 to 365.

Equation

$M - E + e \sin E = 0$

Variable Definitions

$$M = \frac{2\pi t}{T}$$



$x = \cos E - e$

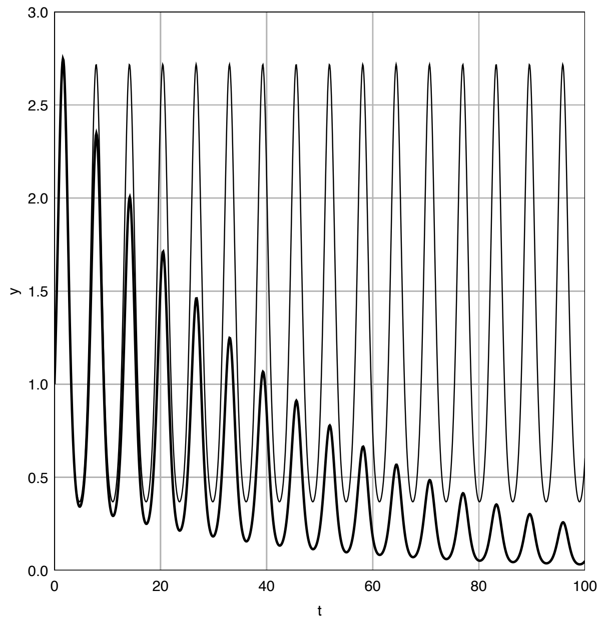
$y = \sqrt{1 - e^2} \sin E$

Parameters

$e = 0.7$

$T = 365$

Long Integration



Example illustrating the danger in using the Euler integration method with a large step size, 0.1 in this case. The numerical solution (thick curve) deviates from the actual analytic solution (thin curve) y_e.

Equation

$$\frac{dy}{dt} = y \cos t$$

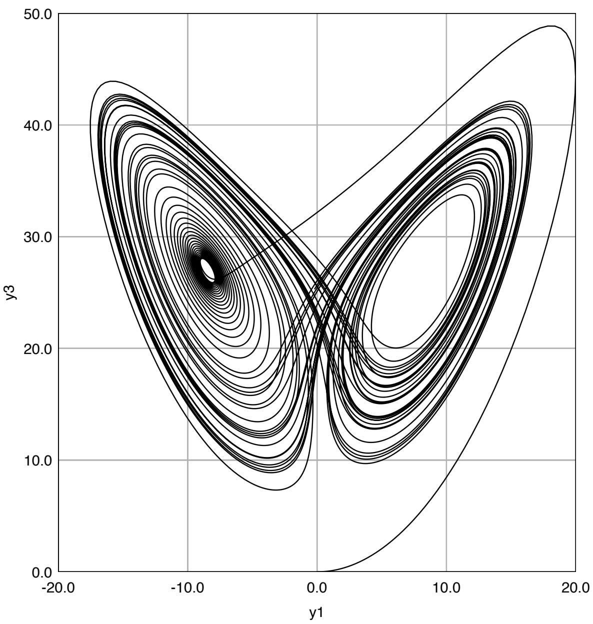
Variable Definition

$$y_e = e^{\sin t}$$

Initial Condition

$$y(0) = 1$$

Lorenz Attractor



The famous Lorenz Attractor to illustrate the dynamic behavior of chaotic systems.

Equations

$$\frac{dy_1}{dt} = \sigma(y_2 - y_1)$$

$$\frac{dy_2}{dt} = \rho y_1 - y_2 - y_1 y_3$$

$$\frac{dy_3}{dt} = y_1 y_2 - \beta y3$$

Parameters

$$\sigma = 10$$

$$\beta = \frac{8}{3}$$

$$\rho = 28$$

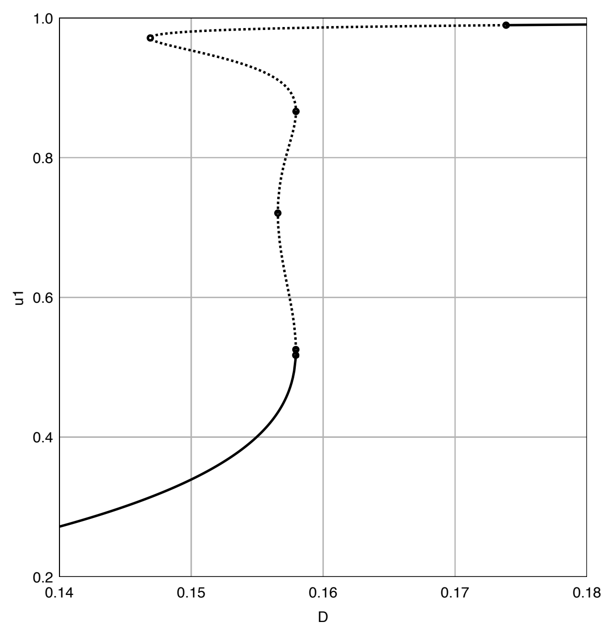
Initial Conditions

$$y_1(0) = 0$$

$$y_2(0) = 1$$

$$y_3(0) = 0$$

Nonisothermal in Series



Bifurcation analysis of a non-isothermal continuous stirred tank reactor with two reactions in series using the Damkohler number D as continuation parameter. This parameter, with an initial value of 0.14, is allowed to vary between 0.14 and 0.18.

Equations

$$0 = -u_1 + D(1 - u_1)e^{u_3}$$

$$0 = -u_2 + D_a(1 - u_1)e^{u_3} - D\sigma u_2 e^{u_3}$$

$$0 = -u_3 - \beta u_3 + DB(1 - u_1)e^{u_3} + DB\alpha\sigma u_2 e^{u_3}$$

Parameters

$\alpha = 1$

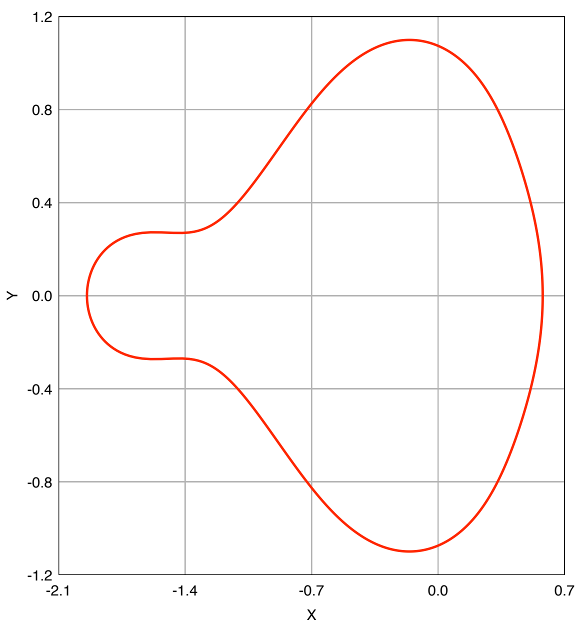
$\beta = 1.25$

$\sigma = 0.04$

$B = 8$

$D = 0.14$

Pear Curve



Parametric plot generated as the solution of an equation in r for values of t in the range from 0 to 1.

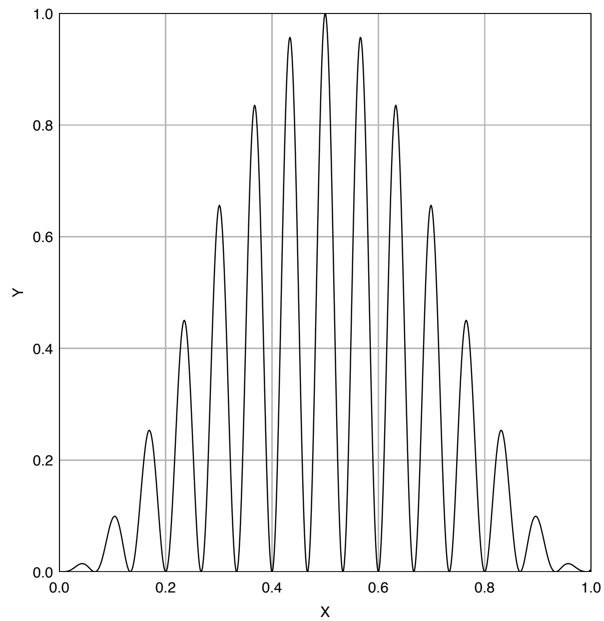
Equation

$$(x^2+y^2)(1+2x+5x^2+6x^3+6x^4+4x^5+x^6-3y^2-2xy^2+8x^2y^2+8x^3y^2+3x^4y^2+2y^4+4xy^4+3x^2y^4+y^6)-2=0$$

Variable Definitions

$x = r\cos(2\pi t)$

$y = r\sin(2\pi t)$



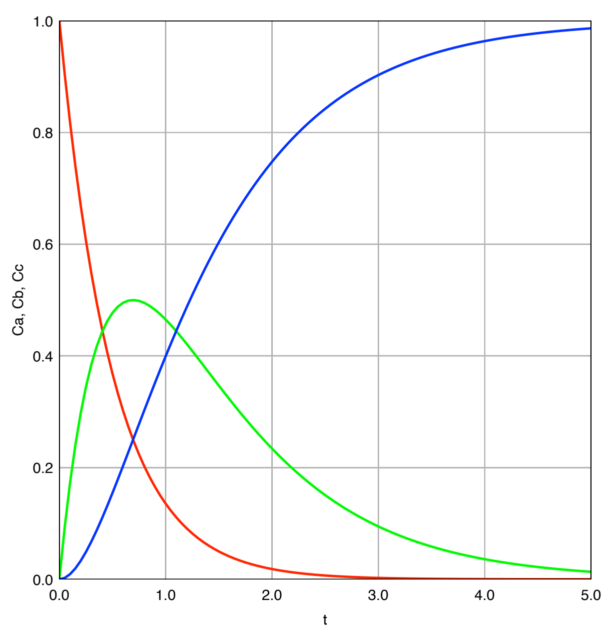
Plain Graph

Simple example illustrating how to create function plots.

Equation

$y = \sin(\pi t)\sin^2(15\pi t)$

Reactions in Series



Model of a batch reactor where species A is converted into species B, followed by the conversion of species B into species C. Initially, only species A is loaded in the reactor.

Equations

$$\frac{dC_a}{dt} = -R_1$$

$$\frac{dC_b}{dt} = R_1 - R_2$$

$$\frac{dC_c}{dt} = R_2$$

Variable Definitions

$$R_1 = k_1 C_a$$

$$R_2 = k_2 C_b$$

Parameters

$$k_1 = 2$$

$$k_2 = 1$$

Initial Conditions

$$C_a(0) = 1$$

$$C_b(0) = 0$$

$$C_c(0) = 0$$

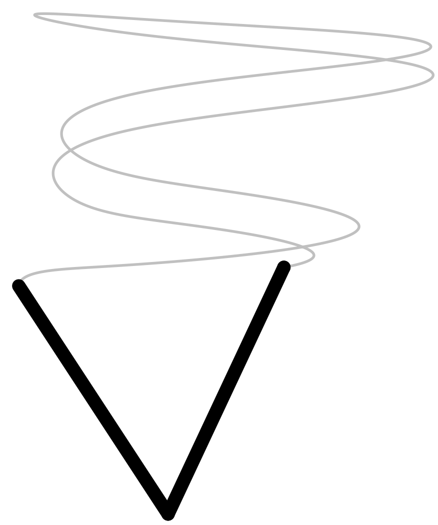
Robotic System

Euler-Lagrange equations describing a two-link planar robotic system. The first link connects the origin to the point with coordinates x_1 and y_1 while the second link connects this point with the point at x_2 and y_2, The final point satisfyies the constraint: $y_2 = \sin^2(t/2)$. Such constraint is thus responsible for the algebraic equation in the model.

Equations

$$\frac{dq_1}{dt} = v_1$$

$$\frac{dq_2}{dt} = v_2$$



$$0 = M_{11}\frac{dv_1}{dt} + M_{12}\frac{dv_2}{dt} - f_1 + x_1 z$$

$$0 = M_{21}\frac{dv_1}{dt} + M_{22}\frac{dv_2}{dt} - f_2 + l_2 c_{12} z$$

$$y_2 = \sin^2(t/2)$$

Variable Definitions

$$T = \frac{t}{\pi}$$

$$c_1 = \cos q_1$$

$$c_2 = \cos q_2$$

$$s_1 = \sin q_1$$

$$s_2 = \sin q_2$$

$$c_{12} = \cos(q_1 + q_2)$$

$$s_{12} = \sin(q_1 + q_2)$$

$$x_1 = l_1 c_1$$

$$y_1 = l_1 s_1$$

$$x_2 = x_1 + l_2 c_{12}$$

$$y_2 = y_1 + l_2 s_{12}$$

$$M_{11} = m_1 l_1^2/3 + m_2\left(l_1^2 + \frac{l_2^2}{3} + l_1 l_2 c_2\right)$$

$$M_{12} = m_2\left(\frac{l_2^2}{3} + \frac{l_1 l_2 c_2}{2}\right)$$

$$M_{21} = M_{12}$$

$$M_{22} = \frac{m_2 l_2^2}{3}$$

$$f_1 = -\frac{m_1 g l_1 c_1}{2} - m_2 g\left(l_1 c_1 + \frac{l_2 c_{12}}{2}\right) + \frac{m_2 l_1 l_2 s_2}{2}\left(2v_1 v_2 + v_2^2\right)$$

$$f_2 = -\frac{m_2 g l_2 c_{12}}{2} - \frac{m_2 l_1 l_2 s_2 v_1^2}{2}$$

$$p_{drift} = y_2 - \sin^2(t/2)$$

$$v_{drift} = x_2 v_1 + (x_2 - x_1)v_2 - 2w\sin(wt)\cos(wt)$$

Parameters

$$w = 0.5$$

$$l_1 = 1$$

$$l_2 = 1$$

$$m_1 = 36$$

$$m_2 = 36$$

$$g = 9.81$$

$$q_{1o} = 0$$

$$q_{2o} = -\pi$$

Initial Conditions

$$q_1(0) = q_{1o}$$

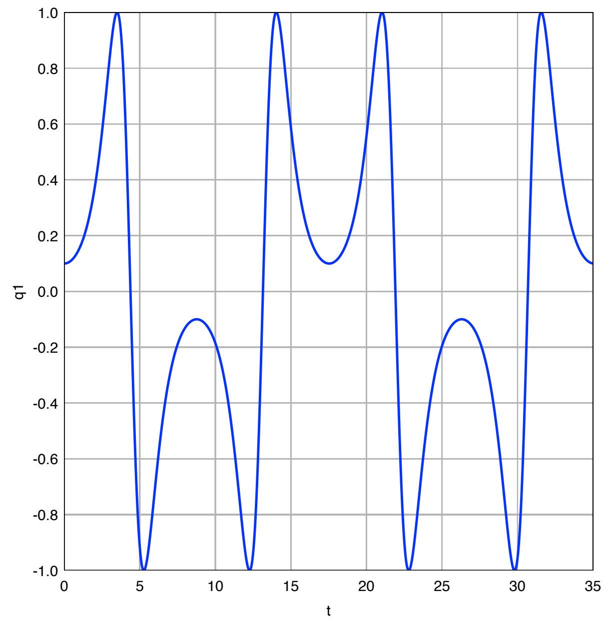
$$q_2(0) = q_{2o}$$

$$v_1(0) = 0$$

$$v_2(0) = 0$$

$$z(0) = 0$$

Simple Pendulum



An index-3 differential algebraic system simulating the behavior of a simple pendulum with coordinates q_1 and q_2 for the ball. The only algebraic equation in this system describes the fixed distance from the origin that the ball must keep.

Equations

$$\frac{dq_1}{dt} = v_1$$

$$\frac{dq_2}{dt} = v_2$$

$$\frac{dv_1}{dt} = -zq_1$$

$$\frac{dv_2}{dt} = -zq_2 - g$$

$$1 = q_1^2 + q_2^2$$

Parameters

$$g = 1$$

Initial Conditions

$$q_1(0) = 0.1$$

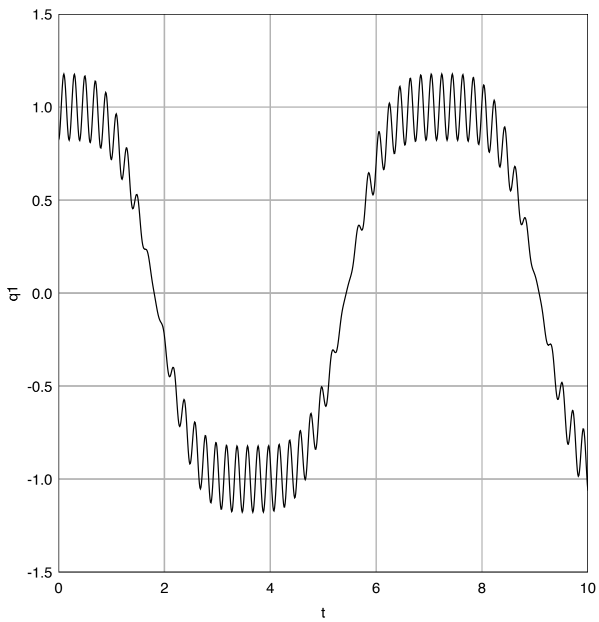
$q_2(0) = 0.9$

$v_1(0) = 0$

$v_2(0) = 0$

$z(0) = 0$

Spring Pendulum



An index-1 differential algebraic system simulating the behavior of the pendulum with coordinates q_1 and q_2 for the ball. The ball is attached to a stiff spring so its distance to the origin is not fixed but varies as the spring expands and contracts.

Equations

$$\frac{dq_1}{dt} = v_1$$

$$\frac{dq_2}{dt} = v_2$$

$$\frac{dv_1}{dt} = -\frac{zq_1}{r}$$

$$\frac{dv_2}{dt} = -\frac{zq_2}{r} - g$$

$$0 = \epsilon z - r + 1$$

Variable Definitions

$$r = \sqrt{q_1^2 + q_2^2}$$

Parameters

$g = 1$

$\epsilon = 0.001$

Initial Conditions

$q_1(0) = 1 - e^{0.25}$

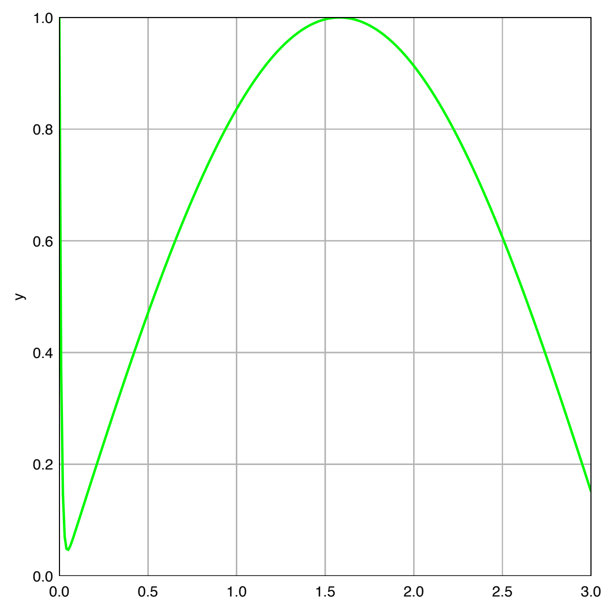
$q_2(0) = 0$

$v_1(0) = 0$

$v_2(0) = 0$

$z(0) = 0$

Stiff Problem



Rapid initial decrease in y makes this a stiff system that implicit linear multistep integration methods are better at handling.

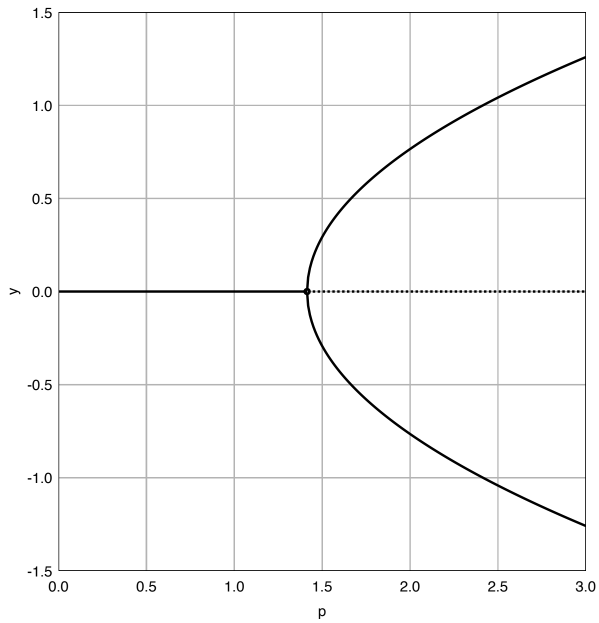
Equations

$$\frac{dy}{dt} = -100(y - \sin t)$$

Initial Conditions

$y(0) = 1$

Supercritical Pitchfork



Simplest example of a supercritical pitchfork bifurcation with the horizontal stable and unstable branches meeting the two stable branches of the parabola at $p = \sqrt{2}$. The continuation parameter is p. This parameter, with an initial value of 0, is allowed to vary between 0 and 3.

Equations

$0 = y(p - \sqrt{2} - y^2)$

Parameters

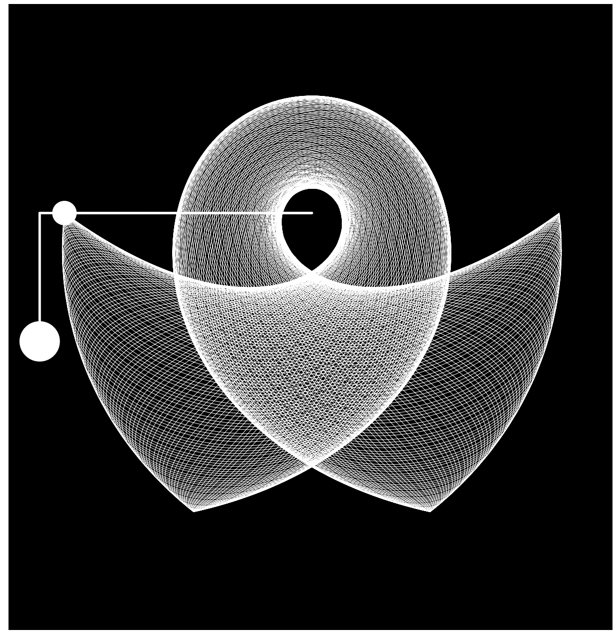
$p = 0$

Initial Conditions

$y = 0$

Swinging Atwood Machine

Atwood machine with the larger mass moving vertically and the smaller mass moving in two dimensions.



Equations

$$\frac{dr}{dt} = \frac{P_r}{M + m}$$

$$\frac{dP_r}{dt} = \frac{P_a^2}{mr^3} - Mg + mg\cos a$$

$$\frac{da}{dt} = \frac{P_a}{mr^2}$$

$$\frac{dP_a}{dt} = -mgr\sin a$$

Variable Definitions

$x = r\sin a$

$y = -r\cos a$

Parameters

$g = 1$

$k = 4.5$

$m = 1$

$M = km$

Initial Conditions

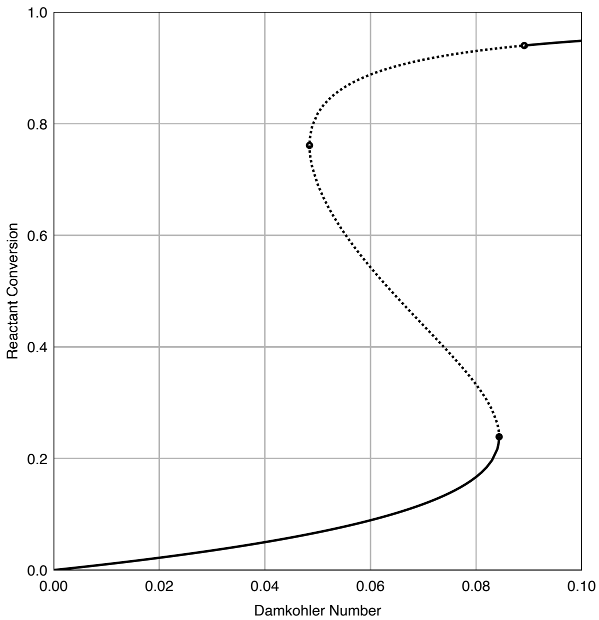
$r(0) = 1$

$P_r(0) = 0$

$a(0) = \frac{\pi}{2}$

$P_a(0) = 0$

Tank Reactor



First order irreversible reaction in a non-isothermal continuous stirred tank reactor using the Damkohler number D_a as continuation parameter. This parameter, with an initial value of 0, is allowed to vary between 0 and 0.1.

Equations

$$\frac{dx_1}{dt} = -x_1 + D_a(1 - x_1)e^{x_2}$$

$$\frac{dx_2}{dt} = -x_2 + BD_a(1 - x_1)e^{x_2} - \beta x_2$$

Parameters

$B = 22$

$D_a = 0$

$\beta = 3$

Initial Conditions

$x_1 = 0.9$

$x_2 = 5.0$