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Nat

ural approach

to arithmetic and math

at the primary school level

in context of Gestalt-Dialektik,

a -philosophical approach- to

holistic learning, ages 2-18,

any subject in all areas

according to

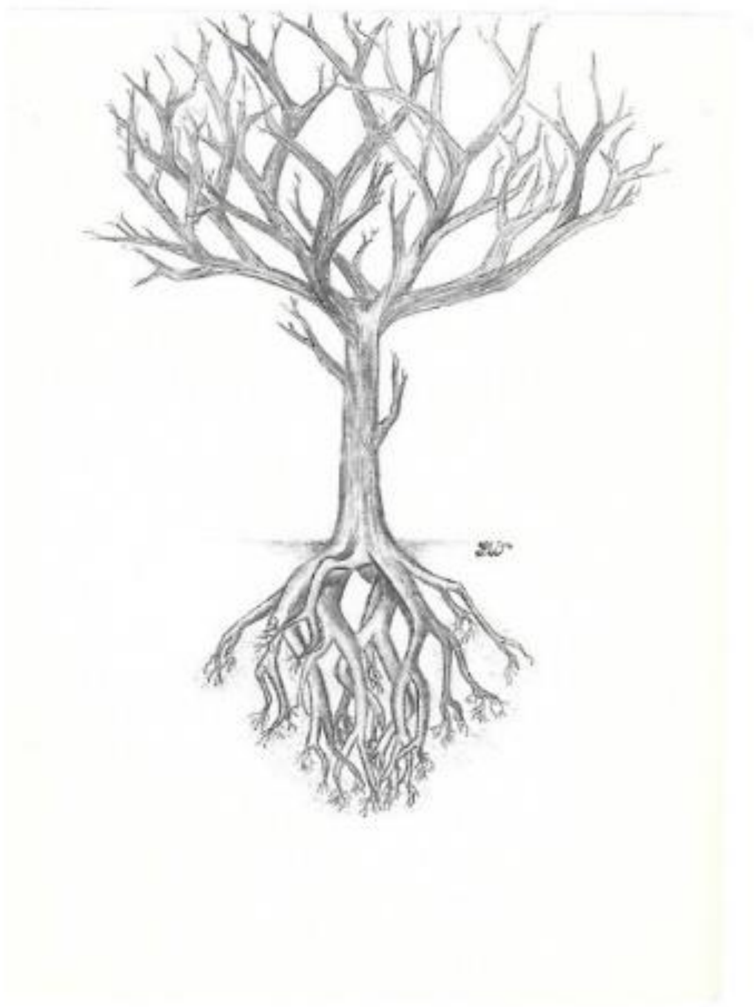
Azul Celeste ad Princess Esther

By Gustavo Vieyra

To Princess Esther[1]

May your voice be an echo of your heart, not a resonance of foreign thoughts. Inside of you is your true clock, a parameter of your innermost feelings, permutations of a many sleepless night, transcending your soul into self-preserving actions, in hopes of finding the effervescent radiance, the self-propelling and infinite prowess . . . Thus, inside of you, you will find the tintinnabulation that eludes the intellect and preludes an afterlife of never-ending happiness. Inside you will find your true self, your true, immortal entity.

Close your eyes and see yourself in the new life that permeates all your dreams, and, above all, let not that soul within you be cut off from that everlasting mighty force of the Deity who hath created thee in all yourself, in all your true and transcendental self.



[1] Tree illustrated by Lisa Phyllis Walter

Proposal to the parents in any community throughout the USA and abroad

I hereby propose a German school, ages 2-18, in order to promote

the best methods in the world to teaching any skill and any academic subject

according to Gestalt-Dialektik, a holistic philosophy of education.

In essence, I propose a holistic approach to teaching and learning via oral language development in terms of health and wellness as well as music and the arts, especially theater arts, singing, sports, drawing and painting, cooking, permaculture and different social skills in a bilingual setting, either in English-German or English-Spanish. In due time, our high school students will be able to pass ***das Abitur***, the German high school exit exam so that they may study any university career practically **for free**. German university education is offered to any student in the world for free, who can pass **das Abitur**. My objective is to establish a ***Gymnasium***, a German high school, so that our students graduate from preschool to elementary and from there to ***das Gymnasium*** to be called **Friedrich Gauß Gymnasium. Gauß is known as the "Prince of Mathematics"** and thus, our students will become princes and princesses according to:

GD
Gestalt-Dialektik
A dialectical approach to education via

Azul Celeste ad Princess Esther

💛💛💛💛💛 💙💙💙💙💙 💛💛💛💛💛

A philosophical, poetic, artistic, music continuum

from preschool up to the college level

through love and inspiration

and in a multilingual

individualized

setting

by
∞
all

And thus,
in this paradigm,
Vieyra hereby postulates
Azul Celeste ad Princess Esther
as the philosophical means to an end in
order to develop the most significant methods
in the world to literacy, mathematics, second language
acquisition, music & singing, drawing and all other academic
areas in several languages, especially English, Spanish and German.
Essentially, Azul Celeste becomes reflected and refracted via the spoken word
within a poetic-music continuum ad infinitum as the alpha and omega of all learning!

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Introduction

Gestalt-Dialektik (GD) is a holistic philosophy of education in order to promote literacy skills in the native language for both parents and children as a bridge towards ever-increasing levels of second language acquisition into Spanish, English and German as L1, L2, L3 or L4 (languages one through four). GD postulates a poetic-music pedagogy based on the tenets of Gestalt psychology, (the Berlin school of thought[2]), the sense of belonging according to Alfred Adler (*das Gemeinschaftsgefühl*), a pedagogical integration of the family and other holistic principles. For example, the zones of proximal development in context of the conceptual-developmental theory by Lev S. Vygotsky as well as the sensitive periods from Maria Montessori and the anthroposophy according to Rudolf Steiner will play a foundational role in any GD program. Furthermore, the music school of rhythmic and movement according to Emile Jaques-Dalcroze and the music-pedagogical approach from Carl Orff and Zoltán Kodály will help us to guide our artistic pedagogy ad infinitum.

In essence, GD embraces the reform-pedagogical embodiment of the turn of the 20th century in Germany. Based on the application of this reform-pedagogical approach, the common ground of which is the connection of artistic and holistic processes, the *Gestalt-Dialektik* team, within the context of our current circumstances, is to develop a corresponding strategy, *an academy of the arts for every child.*

Furthermore, with our literacy and math methods, children with an advanced oral language development may learn to read and write at significant levels in L1 or L2 in context of family literacy vis-à-vis a poetic-music continuum. We postulate a significant pedagogy, especially with children who can speak their native language fluently at the preschool level. With a corresponding infrastructure it will be possible for example that, around the age of five, they will not only learn how to read and write at above expectations, but also master a second or third language in due time at native-like proficiencies.

In essence, *Gestalt-Dialektik* postulates a poetic-music continuum in order to transpose poetry and music into the center of all curricular processes, especially beginning at age two or three for linguistically gifted children, who would be acquiring a second or third language. GD is therefore not just a holistic pedagogy (gestalt psychology, Maria Montessori, Rudolf Steiner, Adler, Vygotsky, etc.), but also a poetic-music continuum, beginning with very simple, but clear and transparent forms of *Gestaltqualitäten* (gestalt-qualities), via which the children, through poetic, rhythmical and melodic pathways, internalize and put into practice everything that they do so want to learn.

This new holistic philosophy of teaching and learning has shown excellent results in California with Spanish-speaking minority children and with German children in Kassel, Germany in terms of its initial reading and writing pedagogy.

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[2] According to Wolfgang Köhler (1971) the whole is not just bigger, but different than the sum of its parts:

Köhler, W. (1971): <u>Die Aufgabe der Gestaltpsychologie</u> ["The mission of gestalt psychology"]. Berlin: Walter de Gruyter.

Katz, D. (1944): <u>Gestaltpsychologie</u> [„Gestalt psychology“]. Basel: Benno Schwabe & Co Verlag.

1.0 A Dialectical Approach via the Spoken Word as the Alpha and Omega of all Learning

Dialektik as in Gestalt-Dialektik implies an interactive and transformative process, beginning with the **spoken word** in all its extensions, especially in terms of music, singing, poetry, theater and drawing. The **spoken word** is the starting and ending point of the learning process, **the alpha and omega**, the beginning and the end of most of our "teaching units." For example, we may read a story to a group of children. From the point of view of the children, they are listening to a story. They are not reading it, but just listening to what is being spoken. Eventually, the children may illustrate the story, learn new phrases, new words, but with the objective of transforming the story into their own interpretation, that is, in terms of translating the story into their **"own spoken words."** In other words, the first objective of reading aloud stories is for children to retell the same stories, but in their own words. Thus, we start with the spoken word, that is, the teacher reading aloud or dramatizing a story to a group of children and we may end up with the children dramatizing or retelling the story in their own words. Thus, the teaching unit begins with the spoken word and ends with the spoken word. Here is an example of four- and five-year old children in Kassel, Germany in 1998 interpreting a story that I had dramatized for them during the previous 5 weeks:

https://www.youtube.com/watch?v=gDeNQyVM5Dc.

***Dialektik* means on the one hand** the simultaneous acquisition of two abilities and/or skills. Within this paradigm change a child, for example, may be able to learn how to sing and at the same time acquire Spanish, English or German (as a second or third language) in that he sings for example a German song with the objective of analyzing the grammar of the text of the song. In essence, the *Gestalt-Dialektik* philosophy is dialectic in nature: it seeks to interconnect several skills and cultural systems[3] into one poetic continuum. **On the other hand** *Dialektik* means the transformation of the child through the internalization of certain cultural systems (reading, writing, a foreign language, singing, mathematics, music, drawing, painting, dancing, etc.) as cultural values, which we want to promote in context of the aforementioned five dialectical leaps..

In order to internalize a cultural value such as initial reading and writing, GD starts with a particular whole such as a story or a song. In other words, GD is a holistic pedagogy in which the elements should not be interpreted outside of its whole. For example, the meaning of a word does not depend on the denotational meaning of a dictionary, but rather on its connotation in a real authentic social interaction. The objective consists in postulating the context of **a whole** in order to define its parts. In some literacy methods, the word may be considered as the whole to a syllable or to a phoneme. However, I may argue that a word does not necessarily have to be the "targeted whole" because it can also be perceived as a part of a sentence. So, which one is the whole, the word or the sentence? Furthermore, I could even argue that there is a greater whole above the sentence: that of a paragraph, and yet again, the paragraph may also be analyzed as a part, as a scene of a story. Is the story then the most significant and transcendental whole? I would even go above and beyond any particular story in order to integrate other holistic factors such those dealing with the affective domain; for example, the motivation to listen to a story or the child´s self-confidence and personal interests, etc. reflect realities that are greater than a story itself. From a socio-psychological point of view, certain holistic factors dealing with the affective domain may be perceived as being above and beyond a particular story, i.e., as factors that transcend the story itself because they deal with the child as a whole. For example, if a child loves storytelling time, if he´s inspired by how a teacher dramatizes the

[3] For example, drawing, painting, arithmetic, writing, reading, theater arts, retelling short stories in the children´s own words, the acquisition of a second and if possible, a third language, etc.

story, then that love and inspiration transcend the story itself; they are above and beyond the particular story, because they deal with the entire child as a whole and how he reacts to books and stories.

Furthermore, GD´s open-ended holistic philosophy of teaching and learning is also based on *patterns of thinking* through a poetic-music continuum in order to promote the transformational process of the human spirit and consciousness from a lower ***dialectical leap*** into the next higher one within the context of child and human development. Such a transformational process may best be measured by a specific set of cultural standards as a means to evaluate any significant progress in all areas of social, personality, academic and intellectual development as the child progresses from preschool to elementary and from there on to secondary and college education.

In essence, GD subscribes to certain dialectical transformations as measured by a specific set of cultural and academic standards into the next higher transformation in terms of psycholinguistic development. For example, in terms of oral language development one set of cultural standards may be defined as the normal psycholinguistic level of a child, that is, the expected syntactical, morphological and semantic level that a child of any age is expected to master. In this case, it is widely recognized that a child begins to speak his first words around the first birthday and that there is a general grammatical explosion by the middle of the second year of life. By age four, the average child should be able to master most of the syntactic and morphological structures of the native language and by age six, psycholinguistic mastery should be completed by most children. In other words, they may be able to express almost anything they want to say without any semantical, syntactical, morphological and other grammatical errors.

1.1 Gestalt-Dialektik implies a dialectical process approach to teaching and learning in accordance to Vygotskyan psychology

On the one hand, a dialectical process may be defined, from the Vygotskyan point of view (Vygotsky, 1978, 1999), as the transformation of a child from a lower level of consciousness to a higher level via the internalization of cultural values. For example, once a child learns how to sing, or how to read and write or even learns a second language, that child is no longer the same child as before. There´s been a social-linguistic and cognitive transformation in his or her consciousness and personality. **On the other hand,** a dialectical process, may be defined as the ability of a teacher to combine at least two academic factors in the same lesson, be it for example storytelling and singing or drawing and calligraphy. In my personal opinion, this dialectical approach is highly significant in the art of teaching children several academic areas, especially literacy, arithmetic and foreign language skills from preschool to high school. How do we integrate at least two factors **"dialectically"** into one another? Here is an excellent example of a storytelling lesson, which integrates singing according to the Kodaly method:

https://www.youtube.com/watch?v=oNowXOLkyBA.

This dialectical approach can help us develop the **best holistic methods in the world at the preschool and elementary school level**. Thus, children could be learning for example how to sing while they learn Spanish or German as a second language. Learning a second language with an emphasis on singing as opposed to grammar would be very interesting if done properly. In this case, we would be integrating singing and second language acquisition in a very dialectical format.

In essence, *Gestalt-Dialektik* (GD) is a holistic philosophy of education in order to promote literacy skills in the native language for both parents and children as a bridge towards ever increasing levels of second language acquisition into Spanish, English and German as L2 or L3 (language two or three). GD postulates a poetic-musical pedagogy based on the tenets of Gestalt psychology, (the Berlin school of thought[4]), Alfred Adler, a pedagogical integration of the family and other holistic principles. For example, parallelisms _to the pedagogical principles from Lev S. Vygotsky, Maria Montessori, the anthroposophy according to Rudolf Steiner, the music school of rhythmic and movement according to Emile Jaques-Dalcroze as well as the *language of music* based on the music-pedagogical approach from Carl Orff and Zoltán Kodály _originate themselves out of the reform-pedagogical embodiment of the turn of the 20th century in Germany. Based on the application of this reform-pedagogical approach, the common ground of which is the connection of artistic and pedagogical processes, the *Gestalt-Dialektik* team, within the context of our current circumstances, is to develop a corresponding strategy, *an academy of the arts for every child.*

Furthermore, in the GD literacy and math methods, children with an advanced oral language development may learn to read and write at significant levels in the native language in context of family literacy vis-à-vis a poetic-music continuum. We postulate a significant pedagogy, especially with children who can speak their native language fluently at the preschool level. With a corresponding infrastructure it will be possible for example that, around the age of five, they will not only learn how to read and write at above expectations, but also master a second or third language in due time at native-like proficiencies.

[4] According to Wolfgang Köhler (1971) the whole is not just bigger, but different than the sum of its parts.

Vygotskyan psychology in the GD dialectical approach

Vygotsky was a product of a Marxist-Hegelian philosophy. He sought to explain child development in terms of a dialectical materialism. According to this thesis, all phenomena must be studied from the perspective of processes of motion, and change. In child development for example, the teacher needs to analyse how elementary psychological process “become” transformed into complex ones. However, how do we trace the qualitative changes in behaviour that occur in the course of development?

He focused his attention on how a person changes nature (his surroundings) and in doing so changes his own behaviour via the mediating effect of tool use, which he expanded to the symbolic activity of the child. Speech as a socio-historical symbolic by-product becomes the major and most significant “tool” in the transformation of the elementary functions into their higher psychological counterparts. That is to say, between man and his physical world, a dialectical unity takes place as soon as the child acquires language. Man is no longer a passive actor in a natural environment, but a transformer of his surrounding nature. Animals change nature by their presence, but humans go beyond the simple physical objects in order to reach an objective or solve a problem. Time and space become transcended with the empowering force of the language function.

Thus, speech becomes the most transcendental moving force, the “mediating tool” which at first is social in nature (interpersonal) and later intrapersonal. Man is changed from **the outside in** via the social forces and via **the internalization of sign systems** (language, numbers, music, etc.) as opposed to from **inside out** as implied in the principles of constructivism such as that of Piaget and Bruner.

On the other hand, practical intelligence and language may act independently from each other, especially in the preverbal stage of ontogenesis, but when these two converge, then a dialectical unity arises which changes forever the nature of human kind:

> The most significant moment in the course of intellectual development, which gives birth to the purely human forms of practical and abstract intelligence, occurs when speech and practical activity, two previously completely independent lines of development, converge. Although children`s use of tools during their preverbal period is comparable to that of apes, as soon as speech and the use of signs are incorporated into any action, the action becomes transformed and organized along entirely new lines. The specifically human use of tools is thus realized, going beyond the more limited use of tools possible among the higher animals. (Vygotsky, 1978, p. 24-25)

Sign use such as speech was not just any form of human communication, but also a tool that would help the child master his-her surroundings (social speech) in order to master his/her own behaviour. **In effect, as the child internalizes a sign system, such as language, writing, number, he ascends into a dominion based on symbols and his reality is no longer dictated by the crude physical world.** Things and objects are viewed differently. Objects are not simple objects of any shape and form, but rather “categories” within a semantic structure. Such systems in turn transform human behaviour as well as his consciousness, which in turn affect other higher level psychological processes such as attention, voluntary memory, deductive thinking and the like.

Vygotsky, having lived within a Marxist regime had no other choice than to view everything from the point of view of dialectical and historical materialism. A sign system such as language thereby became another tool, a psychological tool one may add, but nevertheless a tool.

> Like tool systems, sign systems (language, writing, number systems) are created by societies over the course of human history and change with the form of society and the level of its cultural development. Vygotsky believed that the internalization of culturally produced sign systems brings about behavioural transformations and forms the bridge between early and later forms of individual development. Thus, for Vygotsky, in the tradition of Marx and Engels, the mechanism of individual developmental change is rooted in society and culture. (Vygotsky, 1978, p. 7)

Thus, the use of speech becomes a psychological tool. Children must speak in order to act and the more complex the situation, the greater role that is accorded to the use of speech. In essence, speech and action form a dialectical unity in the solutions of problems and in the mastery of a child´s social and physical surroundings. A child must speak in order to solve a problem or reach any objective:

> These observations lead me to the conclusion that children solve practical tasks with the help of their speech, as well as their eyes and hands. This unity of perception, speech, and action, which ultimately produces internalization of the visual field, constitutes the central subject matter for any analysis of the origin of uniquely human form of behaviour. (Vygotsky, 1978, p. 26)

An animal on the other hand is incapable of making such a dialectical unity between perception, speech and action. The experiments conducted by Köhler proved that apes are not capable of any symbolic activity. They have to rely on their immediate environment. Their actions depend on the actual physical surroundings conducive to a practical task. For the ape, the tools must be perceived at once in order to be of any practical use. The physical configuration dictates to the ape the limited range of possibilities, but for a child, precisely because of the use of speech, the field of possibilities do not depend on the physical configuration of a set of stimuli.

On the other hand, the emotional and communicative functions of speech are expanded by its planning organizational factor. . ., which allows the child to plan ahead and thus transcend time:

> Unlike the ape, which Köhler tells us is “the slave of its own visual field,” children acquire an independence with respect to their concrete surroundings; they cease to act in the immediately given and evident space. Once children learn how to use the planning function of their language effectively, their psychological field changes radically. A view of the future is now an integral part of their approaches to their surroundings. (Vygotsky, 1978, p. 28)

A child thus becomes human precisely because of his sign using ability, transcending time and space. His is capable of organizing the environment and mastering it at will, changing the focus of attention at any time. Thus, the world is perceived not just in a mechanical formulation, or via the “schemata” that Piaget suggested, but it has meaning and sense:

> I do not merely see something round and black with two hands; I see a clock and I can distinguish one hand from the other. Some brain-injured patients say, when they see a clock, that they are seeing something round and white with two thin steel strips, but they do not know it is a clock; such people have lost their real relationship with objects. These observations suggest that all human perception consists of categorized rather than isolated perception. (Vygotsky, 1978, p. 33)

In other words, the children not only can perceive objects, but they are capable of sensing and internalizing a "category" out of it, that is a relative meaning in accordance to their psychological development. The world is sensed via semantic categories of all types which affect perception and attention in complex behavioural systems. The child may for example structure his own sensory field, create new structural centers or fantasize about some future event in accordance to the emotional, communicative and social stage of development.

The meaning of a word reflects a generalization, i.e., a thinking process according to Vygotsky

In Vygotskyan psychology, one must find the appropriate method. His method was analytic in nature as opposed to the elementaristic approach to any phenomenon. In this regard one could regard his research as an integrative philosophy, which was capable of to synthesize the functional units of any organism. His main approach was based on the analytic segmentation of a complex whole into its corresponding units. In other words, his analytic method dealt with an analytic result, that in comparison to the elements, possesses all the fundamental qualities corresponding to a whole, thereby representing those living parts of this uniform whole that are no longer divisible (Vygotskij, 2002, p. 47).

Above all, it dealt with those units that are no longer divisible because they preserve the characteristic qualities of their complex whole:

> (informal translation from the German text)
> What is now such a unit that is no longer divisible, in which the characteristics of thought and language as a whole are preserved? We mean that such a unit can be found in the interior side of a word, in its meaning.
>
> …From the word we only know its exterior part, the one facing us. Its interior side, the meaning, remained and persists as before – as the other side of the moon – unexplored and unknown. Precisely in this other side is the possibility hidden in order to solve the problems of interest concerning the relationship between thought and language, because exactly in the meaning of a word is the knot of that whole tied, that which we call *thought and language*.
> (Vygotskij, 2002, S. 48)

And what is the nature of the "word-meaning"? In his research with Lenin´s conception about language, Vygotsky came to the conclusion that all words represent a generalization of reality. All words generalize and as such are acts of thinking. A word must be conceived as an abstraction of reality. A word does not relate to a particular object, but rather to a whole group of objects. Therefore, every word represents a hidden generalization, each word generalizes already. From a psychological perspective, the meaning of a word represents a generalization. It´s easy to recognize that we´re dealing with a very special act of thinking, that which reflects reality completely different than direct sensations and perceptions (Vygotskij, 2002, S. 48-49).

In this regard one would postulate that objects in the surroundings, for example a table, a tree, the house, etc. represent generalizations and as such should be regarded as acts of thinking at the subconscious level. **A word without meaning,** writes Vygotsky, is no longer a word, but rather an empty sound. In essence, a word meaning must belong to the sphere of speaking as well as to the sphere of thinking, because it represents the unity between language and thinking. In this regard, Vygotsky was able to develop a method in order to delineate properly the relationship between thinking and speaking as a major aspect of his area of research.

Furthermore, the act of speaking also unites the function of communication and thinking, and as such, implies an intention to communicate thoughts and feelings, which according to Vygotsky was oversimplified in the former psychology. We assumed for example that a sign[5], a word and its sound are the means of communication. In this sense, it was postulated that a sound could be associated arbitrarily with any feeling and substance of psychic reality and be able to communicate this feeling or substance with another person. Vygotsky was able to research the complexity of communication with great precision and showed that communication without signs is impossible just like communication without meaning, which implies a generalization.

The primary function of speech is communication is social intercourse according to Vygotsky, which is not possible in the animal kingdom, because real communication requires meaning:

> A frightened goose suddenly aware of danger and rousing the whole flock with its cries does not tell the others what it has seen but rather contaminates them with its fear.
> Rational, intentional conveying of experience and thought to others requires a mediating system, the prototype of which is human speech born of the need of intercourse during work. (Vygotski 2012, p. 6)

Furthermore, he postulated that:

> Thus, true human communication presupposes a generalizing attitude, which is an advanced stage in the development of word meanings. The higher forms of human intercourse are possible only because man´s thought reflects conceptualized actuality. That is why certain thoughts cannot be communicated to children even if they are familiar with the necessary words. The adequately generalized concept that alone ensures full understanding may still be lacking. Tolstoy, in his educational writings, says that children often have difficulty in learning a new word not because of its sound but because of the concept to which the word refers. There is a word available nearly always when the concept has matured.

The conception of word meaning as a unit of both generalizing thought and social interchange is of incalculable value for the study of thought and language. It permits true causal genetic analysis, systematic study of the relations between the growth of the child´s thinking ability and his social development. (Vygotski 2012, p. 7)

[5] https://www.thoughtco.com/sign-semiotics-1692096

Zone of Proximal Development

In order to nurture the learning process at an optimal level, **the Zone of Proximal Development** according to Vygotsky will play a central role. This is the zone just outside of cognition in which the learning strategies are not yet visible. The potential exists to learn something new, which can be manifested with a little bit of help by an adult. When a child, for instance, is not yet able to write a word it does not mean that he will not be able to write it in a couple of days with some support. The timeline in which a child learns a new skill with assistance until he´s able to do it independently, all by himself, represents **the Zone of Proximal Development (ZPD)** as postulated by Vygotsky (1978, 2002). With babies this hypothesis very is obvious. At first, they can only speak single words or syllables (lala, mama, papa, etc.) at the morphological level. As time goes by, they eventually experience the morphology, syntax and semantics from the adults. They acquire the native language via the ZPD, first through imitation as postulated by Piaget as a process of assimilation[6]. They will, just like Maria Montessori indicated, be sensibilized for the next mental ability.

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[6] **Assimilation** is a cognitive process that manages how we take in new information and incorporate that new information into our existing knowledge. This concept was developed by **Jean Piaget**, a Swiss developmental psychologist who is best known for his theory of cognitive development in children. For example, when a young child learns the word dog for the family pet, he eventually begins to identify every similar-looking canine as a dog. The child has extended his learning, or assimilated, the concept of dog to include all similar 4-footed friends.
Source: https://study.com/academy/lesson/assimilation-and-piaget-definition-theory-process.html

Both assimilation and accommodation are essential to how organisms build schemas about the world (1952; see also Wadsworth, 2004). While assimilation deals with keeping existing knowledge and schemas intact and finding a new place to store information, accommodation involves actually changing one's existing knowledge of a topic (Tan et al., 2017).
Source: https://www.simplypsychology.org/what-is-accommodation-and-assimilation.html

2. Addition, Subtraction, Multiplication and Division according to GD

A Dialectical Interpretation of Factual Knowledge in Vygotskyan Terms

vs.

Bloom´s Taxonomy as Interpreted by the Teaching Staff

at

75th Street Elementary School, LAUSD

What follows is an email from Bill Myers, the Pathway Leader in Psychology in 2009 for the Post-Graduate Certificate in Education at the University of Wolverhampton in the U.K regarding a critique on Bloom´s Taxonomy vis-à-vis the multiplication facts. He thought that the critique was **"very thought provoking"**. Thus, we are presenting an edited version of the same critique on Sections 2.1 through 2.4 so that you may come up with your own conclusions. Here is his email written on August 14, 2009 for your own analysis:

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Dear Gustavo Vieyra,
I am the Pathway Leader in Psychology for the Post-Graduate Certificate in Education at the University of Wolverhampton in the U.K. and have found your article on A Dialectical Interpretation of Factual Knowledge in Vygotskyan Terms vs. Bloom's Taxonomy as interpreted by the Teaching Staff **very thought provoking.** In this light I would like to use the pdf version on the Virtual Learning Environment which is a closed site only accessed by students on the psychology PGCE course. To do this I need your permission, thus my request.
I look forward to hearing from you.
Regards
Bill Myers

****

Link: http://www.wlv.ac.uk/default.aspx?page=22179

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School of Education Research Staff Profile
Name: Bill Myers Title: Pathway Leader PGCE Psychology[7]
Tel. 01902 32 2873 E-Mail: B.Myers@wlv.ac.uk

[7] This was in 2009 and thus Bill Myers may not be available anymore.

2.1 Thesis: Factual Knowledge Exists at a Relatively Low Level of Abstraction

On July 20, 2004 during staff development at 75th Street Elementary School (Los Angeles Unified School District), teachers were told that factual information is not based on abstract information. According to this thesis, factual information is based on concrete basic elements of a content area. Thus, any facts given to the pupils such as 5 x 5 = 25 are considered to be statements that do not require much abstraction.

The teachers were told that such factual information, in this case 5 x 5 = 25, **is the basic building block for other more complex mathematical operations.** Thus, factual information is categorized as one of the lower levels of knowledge in accordance to Bloom´s Taxonomy. It was further explained that because 5 x 5 doesn´t change (it is always 25), this and other similar multiplication facts are considered to be of low levels of abstraction. This explanation was based on a handout given to all teachers, which was written as follows:

> Thesis: Factual Knowledge Exists at a Relatively Low Level of Abstraction
>
> FACTUAL KNOWLEDGE
>
> Factual knowledge encompasses the basic elements that experts use in communicating about their academic discipline, understanding it, and organizing it systematically. These elements are usually serviceable to people who work in the discipline in the very form in which they are presented; they need little or no alteration from one use or application to another. Factual knowledge contains the basic elements students must know if they are to be acquainted with the discipline or to solve any of the problems in it. The elements are usually symbols associated with some concrete referents, or "strings of symbols" that convey important information. For the most part, **Factual Knowledge exists at a relatively low level of abstraction.**
>
> Because there is a tremendous wealth of these basic elements, it is almost inconceivable that a student could learn all of them relevant to a particular subject matter. As our knowledge increases in the social sciences, sciences, and humanities, even experts in these fields have difficulty keeping up with all the new elements. Consequently, some selection for educational purposes is almost always required. For classification purposes, Factual Knowledge may be distinguished from Conceptual Knowledge by virtue of its very specificity; **that is Factual Knowledge can be isolated as elements or bits of information that are believed to have some value in and of themselves**. The two subtypes of Factual Knowledge are knowledge of terminology (Aa) and knowledge of specific details and elements (Ab).
>
> **Source:** A Handout given to teachers at a Southern California elementary school on July 20, 2004 based on a document titled "A Taxonomy for Learning, Teaching, and Assessing- A Revision of Bloom´s Taxonomy of Educational Objectives" (name of author or year of publication was not given to the teachers).

In essence, this handout (for which the teachers were not given the name of the author or the year of publication) deals with two theoretical postulates. The first one deals with the implication that factual knowledge is based on a relatively low level of abstraction and the second one refers to the **elements or bits of** information **"that are believed to have some value in and of themselves."** In Section 2.2, an antithesis is offered against the first postulate and in Section 2.4, a correction is proposed on the second postulate dealing with the bits of information on an isolated format. An explanation will be given as to why these two postulates are based on **a wrong pedagogical focus** and thus fail to bring new insights into the art of teaching.

However, let us first explain that an interpretation is a very reasonable one based on the current literature of Bloom´s taxonomy. Thus, this essay should not be viewed as a critique on the teaching staff of any particular elementary school, but rather at the interpretation that any other school worldwide would have given, especially if we make further research as to what the experts are writing. Let us for example review a chart of the following article, which was retrieved from the internet on August 18, 2006:

Bloom et al.'s Taxonomy of the Cognitive Domain[8]

LEVEL	DEFINITION	SAMPLE VERBS	SAMPLE BEHAVIORS
KNOWLEDGE	Student recalls or recognizes information, ideas, and principles in the approximate form in which they were learned.	Write List Label Name State Define	The student will define the 6 levels of Bloom's taxonomy of the cognitive domain.
COMPREHENSION	Student translates, comprehends, or interprets information based on prior learning.	Explain Summarize Paraphrase Describe Illustrate	The student will explain the purpose of Bloom's taxonomy of the cognitive domain.
APPLICATION	Student selects, trans-fers, and uses data and principles to complete a problem or task with a mini-mum of direction.	Use Compute Solve Demonstrate Apply Construct	The student will write an instructional objective for each level of Bloom's taxonomy.

[8] Huitt, W. (2004). Bloom et al.'s taxonomy of the cognitive domain. Educational Psychology Interactive. Valdosta, GA: Valdosta State University. Retrieved on August 18, 2006 from http://chiron.valdosta.edu/whuitt/col/cogsys/bloom.html.

ANALYSIS	Student distinguishes, classifies, and relates the assumptions, hypotheses, evidence, or structure of a statement or question.	Analyze Categorize Compare Contrast Separate	The student will compare and contrast the cognitive and affective domains.
SYNTHESIS	Student originates, integrates, and combines ideas into a product, plan or proposal that is new to him or her.	Create Design Hypothesize Invent Develop	The student will design a classification scheme for writing educational objectives that combines the cognitive, affective, and psychomotor domains.
EVALUATION	Student appraises, assesses, or critiques on a basis of specific standards and criteria.	Judge Recommend Critique Justify	The student will judge the effective-ness of writing objectives using Bloom's taxonomy.

If one makes a careful review of Bloom´s taxonomy, then one should come to the conclusion that the interpretation as given to the teaching staff is reasonable because it can be implied from the chart above (Bloom et al.´s Taxonomy of the Cognitive Domain) as given by Huitt (2004). Thus, the school director of any school may reasonably postulate that for the most part, "**factual knowledge exists at a relatively low level of abstraction."**

2.1.2 Antithesis: Factual Knowledge is Abstract

According to Vygotskyan psychology (Vygotsky, 1978, 1999; Vygotskij, 2002), any knowledge is based on a particular level of abstraction. Otherwise, we would not be dealing with knowledge per se. Thus, it is fair to conclude that any concept and any fact such as 5 x 5 = 25 is already a generalized interpretation of reality be it at a conscious or unconscious level. To generalize in this sense means to interpret any phenomenon as a general idea valid outside of its immediate context or as in the case of 5 x 5, it means to transform any instance of reality into a universal concept. All concepts are general statements of some sort. Thus, "knowledge" per se, even at its very concrete stage or factual domain, **is still based on abstract phenomena** contrary to certain interpretations that may be implied based on Bloom´s taxonomy.

Unless a pupil is mindlessly reciting a phrase or a fact, then any thought is based on a particular level of abstraction. Therefore, a "low level of abstraction" or "factual knowledge at a non-abstract level" in accordance to Bloom´s Taxonomy are terms that become "secondary in nature" in a pedagogical philosophy based on Vygotskyan principles. Furthermore, even a fact such as "5 x 5" is a rather complex phenomenon. To begin with, we are dealing with a multiplication fact. That in and of itself should warn the psychologist or the teacher that we are dealing with a "generalization of a generalization" in accordance to Vygotskyan principles. That is to say, we are dealing with a higher order level of cognition rather than with the supposedly non-abstract or low-abstract levels that may be conveyed in any reasonable interpretations of Bloom´s taxonomy such as the one given by the teaching staff.

In accordance to the general theoretical framework of Bloom´s taxonomy, it is normal to postulate for example that "5 x 5 = 25" is a prime example of factual information that is supposedly based on a low (or a zero) level of abstraction[9]. According to this interpretation, the multiplication facts then may become the building blocks of some new complex insights within the California standards of elementary school education. This is a rather misleading interpretation of psychic reality. Indeed, according to the Vygotskyan principles of thought and language (1999, 2002), a statement such as "5 x 5 = 25" even as a multiplication fact is considered to be a rather complex phenomenon. The idea that factual information somehow corresponds to a thinking act based on zero abstraction or on a low level of abstraction reflects a rather **naïve psychology of child development,** especially if it is used to build a higher form of thinking. That would be the equivalent of building a house on sand rather than on a solid foundation (Matthew 7:24-27).

Any concept, even in the form of factual information or factual knowledge, is already a generalized interpretation of reality and thus represents an abstract act of thinking. At some point however, this level of abstraction becomes automatic (that is, it becomes "factual knowledge") in the mind of an individual.

[9] This was in fact postulated during the regular staff development meeting at 75th Street Elementary School (Los Angeles Unified School District) on July 20, 2004.

2.1.3 The Principle of a Generalization of a Generalization in Accordance to Vygotsky

How should one define a concept according to the theory of Vygotsky? A concept can only be defined within the realms of a semantic system as a special "generalization of a generalization." Thus, we are dealing with a multi-facet-type-of-generalization in contrast to a "zero-level-of-generalization" in the animal kingdom. Animals cannot generalize because they are incapable of abstracting their visual field at a symbolic level. In other words, animals do not use a symbolic language in order to communicate. In contrast, even children via their mental processes are capable of symbolic language because they are able to find solutions or even make detours in order to solve a problem outside of their immediate visual and perceptual surroundings. Ideas and concepts become the basic tools in order to come up with the best solutions to a particular problem. Animals on the other hand do not have any mental descriptions of time and space relations. Thus, in the case of animal behavior, we may be justified in speaking about a zero level of abstraction as opposed to the multiple symbolic nature of most human behavior.

However, in the case of small children, we may indeed be dealing with semi-concepts in relation to the optical constellations of perception. Only true concepts may be reflected in a generalization of a generalization from a **dialectic perspective** because they are based not on their visual and perceptive configurations, but rather on their symbolic value within a linguistic and cognitive system. According to Vygotsky (1978, 1999, 2002), **true concepts may be mediated and this mediation is the true nature of a generalization.**

In conceptual development in accordance to Vygotskyan psychology, the field of perception becomes generalized and this generalization becomes itself generalized in the next dialectical level of cognitive development. In other words, any field of perception becomes generalized as soon as a child is able to arbitrarily manipulate the individual objects in his surroundings. In terms of gestalt psychology, what used to be the figure may now become the background and vice versa. Objects become subjective phenomena and thus may be subject to the arbitrary and mediated interpretation of a child who becomes capable of transcending time and space via the use of symbols, especially in relation to human language.

However, at the early stages of conceptual development, objects may become generalized in relation to their optical field of reference in a rather semi-conceptual fashion. For example, an a**phasia patient** looking at a clock would describe it not as a clock, but rather as something round with two hands, one larger and one shorter and not with an hour and a minute hand, which would then imply the existence of higher order concepts and ideas. This is the only instance in which we may indeed be able to justify a stage of conceptual development with a near zero level of abstraction in contrast to the interpretation given at 75th Street Elementary School (Los Angeles Unified School District) on July 20, 2004 in relation to factual knowledge in accordance to Bloom´s Taxonomy.

Interpreting data from a "factual knowledge" point of view in accordance to Bloom´s Taxonomy may represent a rather materialistic format of interpreting human development. Teachers should become aware of the very specific stage in which the children exist, because no matter what we do we cannot, for example, teach reading at first grade level to a six-year-old child who speaks like a three-year old:

Essentially, learning how to read and write is a direct function of oral language development: Learning how to read and write at first grade level implies the existence of certain oral language and cognitive competencies expected of a six-year-old (i.e., the ability to speak fluently syntactically, morphologically and semantically). On the one extreme, a three- or four-year old child who speaks like a six-year-old has the potential of learning how to read texts written and developed for the kindergarten or first grade curriculum[10] and on the other extreme, a six-year-old child who speaks like a three- or four-year old "generally speaking" (with some exceptions) will not be able to learn how to read at age expected levels no matter what initial reading method may be applied.

In general terms, a child has the potential of learning how to read and write **only** that which he or she is already able to speak. Thus, instead of focusing our attention on developing and implementing isolated "initial reading methods" (without integrating for example a dialectical approach to the art of teaching as postulated by Gestalt-Dialektik: www.gestaltdialektik.com), we should first and foremost make sure that the child learns how to **speak** at age-appropriate levels. Music (especially singing) and the arts (i.e., drama, poetry, children´s ability to retell stories in their own words and other dialogic strategies) are believed to be the best therapy via which children´s oral language development may be successfully promoted.

Thus, instead of focusing our attention as to the different hierarchical levels of conceptual development, we should emphasize the different levels of conceptual development in which the children interpret their own reality. At the very early stages of child development, we may indeed be dealing with relative low levels of abstraction, but not in the context as presented in the current research on Bloom´s taxonomy, but rather within the context of a **dialectic understanding of human development in accordance to Vygotskyan psychology.**

At the very early stages according to Vygotsky, that is, at the preschool age or younger, the names of objects may represent concrete attributes. For a very young child to name an object means to give it a specific attribute. A door is just there to be closed or to be opened as a square or rectangular looking thing through which one comes in and out of a place and a house is that particular box-like place where he and his parents live in. That is far from the most abstract interpretation of a door as an entrance to paradise or as an exit into freedom within a poetic or philosophical point of view. Nevertheless, even at the concrete level, the names of objects may be considered to represent "semi-concepts" with a certain level of abstraction, or even "true concepts" depending on the stage of development of a particular child. A round thing with two hands is now viewed as a clock and no longer as something round. **Semantic mediation replaces the pure and direct perceptual experience and meaning becomes a part of our consciousness.** At this level of semantic mediation, we are now dealing with a generalization of a generalization according to Vygotsky. In this case, the clock becomes a generalized concept out of a particular optical frame of reference. In other words, the clock has been generalized out of a particular perceptual physical surrounding, which is also considered to be a generalization at a more concrete level. Thus, we are dealing with a first and a second level of generalization or a generalization of a generalization.

However, a concept may also result out of "a generalization of a generalization of a generalization." This would be the case of a multiplication fact such as 5 x 5, which according to the interpretation given at 75th Street Elementary School on July 20, 2004 represents a low level of abstraction. **That is**

[10] For example, classical children´s books such as "Where´s Spot?" by Eric Hill, "Goodnight Moon" by Margaret Wise Brown and "Are You My Mother?" by P. D. Eastman, etc.

not the case!!! Yes, there are instances in which we may speak of a particular phenomenon with a relative low level of abstraction, but such a phenomenon is viewed differently in Vygotskyan theory. Here we are dealing with a truly cognitive based experience and not with a theory dealing with “factual knowledge” in accordance to Bloom´s Taxonomy. Knowledge of particular arithmetic facts implies the existence of a concept; otherwise, we could not be dealing with “arithmetic knowledge” per se. **Thus, factual knowledge in some cases, such as “5 X 5,” may exist at a relatively higher level of abstraction.** Thus, the case of the multiplication facts should make this postulate clear: at the lowest level of abstraction, we may indeed be dealing with “some kind of factual understanding or knowledge,” but not necessarily in accordance to Bloom´s taxonomy, but rather with a very perceptual experience or schema in Piagetian terms. A child, for example, may see three groups of oranges. Here, we are dealing with a generalization of the first order because the child, although he is not able to conceptualize the three groups in any abstract form, is still able to generalize or single out the three groups out of their physical surroundings. For such a child, the three groups of oranges are just there in a particular visual field and do not represent any kind of mathematical relationship. The three groups of oranges are just there next to each other as objects to be eaten or to play with.

However, for the elementary school pupil, the three groups may indeed represent a conceptual idea, such as **a fact of addition**: “4 oranges, plus 4 oranges and plus 4 oranges are 12 oranges put together.” In this case, we may indeed postulate a generalization of a generalization. The first generalization represents the optical field in which a child has been able to single out the three groups of oranges out of their more general physical surrounding. The second generalization may be represented by its arithmetic addition, “4 + 4 + 4,” which can be further generalized into the more abstract level of a multiplication fact such as 4 + 4 + 4 = 4 x 3. This multiplication fact may be further generalized into the next higher level of abstraction such as in some algebraic relationship ($4 \times 3 = 3 \times 4 \longrightarrow ab = ba$). Thus, algebraic operations may represent a generalization of a multiplication. Likewise, a multiplication is a generalization of an addition and an addition is a generalization of the more concrete and perceptually based physical phenomena. This means that by the time that we deal with algebraic operations, we may in effect be dealing with generalizations of a fourth order, that is, **“with a generalization of a generalization of a generalization of a generalization.”**

This theoretical perspective should prove the point that the interpretation behind Bloom´s Taxonomy, as given by the teaching staff at 75th Street Elementary School on July 20, 2004 in regards to factual knowledge, is misguided and fails to explain human knowledge of a more factual and/or perceptual nature.

2.1.4 The Vygotskyan Predicate: Unitary vs. Elementaristic Analysis

The principles of gestalt psychology, especially those related to the Berlin school of thought (i.e., Köhler, W. (1971) at the turn of the twentieth century brought about a new era of psychological research. According to Wolfgang Köhler, the whole is not just greater, but also different from the sum of its parts. The whole does not exist isolated from its parts and vice versa and the parts not only have to be considered within the context of the other parts in question, but also in accordance to the entire phenomenon in which they appear. Yet, in Bloom´s Taxonomy we still read statements that tend to emphasize the parts in an isolated format. The handout on Bloom´s Taxonomy given to the teachers on July 20, 2004 indicates **"that Factual Knowledge can be isolated as elements or bits of information that are believed to have some value in and of themselves."** A typical example of this atomistic approach is the tendency to give each level of Bloom´s taxonomy a corresponding set of verbs:

1. ***Knowledge***: arrange, define, duplicate, label, list, memorize, name, order, recognize, relate, recall, repeat, reproduce, state.
2. ***Comprehension***: classify, describe, discuss, explain, express, identify, indicate, locate, recognize, report, restate, review, select, translate,
3. ***Application***: apply, choose, demonstrate, dramatize, employ, illustrate, interpret, operate, practice, schedule, sketch, solve, use, write.
4. ***Analysis***: analyze, appraise, calculate, categorize, compare, contrast, criticize, differentiate, discriminate, distinguish, examine, experiment, question, test.
5. ***Synthesis***: arrange, assemble, collect, compose, construct, create, design, develop, formulate, manage, organize, plan, prepare, propose, set up, write.
6. ***Evaluation***: appraise, argue, assess, attach, choose compare, defend estimate, judge, predict, rate, core, select, support, value, evaluate.

With such lists, it is hoped that teachers and curricular developers may create a better pedagogy. According to this approach, the cognitive levels may be viewed from the perspective of particular sets of actions **(i.e., certain specific lists of verbs that correspond to a particular level of Bloom´s taxonomy)**. Such an analysis in accordance to Vygotskyan theory may be ill conceived. Vygotsky´s main method is not to focus on the elements (i.e., lists of verbs), but rather on the units relevant to a particular whole. He warns the researcher that an investigation based on the elements is incomplete. Rather than conforming to the laws and characteristics of the elements, Vygotsky exhorts the researcher to look for units in terms of molecular movements that may also be able to reflect the characteristics of their corresponding whole. In his book, "Thought and Language" (1999, 2002) Vygotsky declares for example that an analysis of water into its chemical elements (oxygen and hydrogen) will not explain the characteristics of water in its ability to extinguish fire because oxygen stimulates the combustion of fire and hydrogen is an inflammable element. So, why does water have the characteristics of extinguishing fire? Well, for sure we will not be able to find out if we were to analyze water into its basic elements, oxygen and hydrogen!

According to the interpretation of Bloom´s taxonomy on factual knowledge as given by the teaching staff at 75th Street Elementary School, the basic elements represent a tremendous wealth and because of it, a student could not learn them all relevant to a particular subject matter. That may be true, but more important than such a statement would be to find out how the elements interact with one another in terms of molecular units of movements. What´s important for example is not to investigate brain research or develop pedagogical principles, or even design a battery of standardized tests in terms of the amounts of words[11] a child has learned or heard up to a particular age, but rather how a child is able to express a concept relevant to a particular social context. Social, cognitive and affective meaning for example is more important than an additive value of linguistic and pedagogical research. It is not the one-to-one correspondence between letters and sounds that is important, but rather the relationship between phonemes within the context of a word, words within the context of a sentence, sentences within the context of a phrase, topic or overarching idea and such an idea within the context of the entire human nature of a child, be it in the affective, social, cognitive or psycholinguistic domain.

However, and because Bloom´s Taxonomy on factual knowledge may be interpreted as being imbedded into a psychology based on elements and bits of pieces of information (i.e., the targeted list of verbs), its entire frame of reference may under such atomistic interpretations become "anti-holistic" in nature. In the aforementioned handout, we read for example that even experts in the social sciences, sciences, and humanities have difficulty keeping up with all the new elements. Such a statement is worth mentioning, but it nevertheless fails to mention any holistic principles of investigation. If the researchers do happen to have problems keeping up with the tremendous amounts of information, then it would be wiser to focus on the molecular or unitary movements of the corresponding facts. To claim that we as humans live in a world with a tremendous wealth of information is not enough. To claim that a scholar has to deal with a million bits of pieces of information represents a view that reflects an additive piece-meal philosophy which is completely out of focus. Our worldview becomes too limited, which may have rather negative consequences:

> As our knowledge increases in the social sciences, sciences, and humanities, even experts in these fields have difficulty keeping up with all the new elements. **Consequently, some selection for educational purposes is almost always required.**
>
> **Source:** A Handout given to the teaching staff at 75th Street Elementary School on July 20, 2004 based on a document titled "A Taxonomy for Learning, Teaching, and Assessing- A Revision of Bloom´s Taxonomy of Educational Objectives" (name of author or year of publication was not given to the teachers).

We end up conforming to an information age delineated into different fields of expertise with a rather negative and fatalistic consequence: instead of transcending a field of knowledge, some of our scholars may end up as one-dimensional specialists. As such, they may become individuals unable to bring all the parts together into a philosophical approach to life. Worst of all, most modern students are no longer capable of mastering and transcending a particular field of knowledge and as such, they are no longer **masters of philosophy, but rather experts or specialists of a particular field of knowledge**. In essence, the elements of this materialistic world have managed to master and enslave their **very own souls and spirits**. In our vanity, we have ceased to view life from the broadest perspective possible, and as such have become captives or our own materialistic paradigms.

[11] For example, list of verbs that correspond to the different levels of Bloom´s taxonomy.

3.0 The Pentagonal System and the GD-Structures according to Gestalt-Dialektik

The “Pentagonal System” as a GD method is not a “mathematical base system” such as “base 10” for the decimal system or “base 2” for the binary system. Thus, it´s not a “base 5” system in mathematical terms, but rather a “five-based system” in psychological terms, especially as it relates to groups of five items in the optical field. Thus, what I propose is a theoretical and practical approach to replace the decimal systems in the optical-spatial field, but not in the mathematical sense. It is still a decimal system, except that it´s reflected in terms of groups of five items up to the number 50. Once the children have mastered “number sense” from 1 to 50 as per the “pentagonal exercises”, then they enter into the decimal system in all its dimensions, including the optical-spatial field.

In the proposed Pentagonal System, no item is repeated more than five times. Even when we count, the number 5 and 6, 10 and 11, 15 and 16, etc., are separated:

1,2,3,4,5 6,7,8,9,10, 11,12,13,14,15 etc.

The Pentagonal System is an approach in order to introduce the number 1 to 20 as groups of 5 in their optical perception. Children will learn to perceive them in the optical field as well as in their cognitive understanding of what the numbers represent, mathematically speaking. Thus, instead of writing $2 + 6 = 8$, they will write $2 + (1 + 5) = 8$ because the 6 has to be represented by 5 and 1. If for example, they want to add $8 + 4$, they need to convert the first number into $(5 + 3)$:

$$8 + 4 = (5 + 3) + 4 = 12$$

Likewise: $14 + 6 = (5 + 5 + 4) + (5 + 1) = 5 + 5 + 5 + 4 + 1 = 5 + 5 + 5 + 5 = 20$

Thus, the Pentagonal System is a special Ansatz (approach) in the field of perception, i.e., as a number continuum in the optical field so that the children can follow the numbers with their eyes. This optical perspective will allow them to eventually generalize the numbers as cognitive tools that they can manipulate at will. The main objective is for the children to grasp any particular number from 1 to 20 as a concrete whole and as a bridge towards the numbers from 20 to 30, and from there to 50; once this stage has been mastered, then it becomes a bridge towards the numbers 1 to 100. It´s all about a “constructivist approach”, building “conceptual bridges in Piagetian terms. (Griffin & Case, 1997). In order to build such a bridge, the teacher must determine the cognitive level of the child; that is, the teacher must know if the child is capable or not of accomplishing a corresponding skill. The teacher has to decide, what should the child learn and how is he/she going to go from point A to point B in order to develop a didactic strategy so that the child can actually performed the desired outcome.

According to Griffin and Case, teaching is about the determination of certain factors:

- The readiness level: what the child is capable of achieving (the child´s cognitive condition)
- What should be learned so that a bridge can be built from point A to point B
- The teaching units with all the methods and support strategies in order optimize learning.

There are certainly other factors to be considered, depending on the hypothesis of the educator, but these three factors are foundational. Within the three factors the Zone of Proximal Development according to Vigotsky will play a central role. In accordance to GD, it is assumed that certain cognitive structures must exist, which have to be internalized so that the child can make any progress. One can postulate certain **Central Conceptual Structures** according to Griffin and Case. Furthermore, it is assumed that a **Central Conceptual Structure** is present within the entire cognitive structure of a child; it is the very essence of thinking, grasping and understanding at a particular level of cognition. The objective of any lesson is to develop strategies in order to reinforce cognition within its corresponding conceptual structure so that a child can progress into the higher level of cognition within a continuum.

The First Gestalt-Dialektik-Structures or First GD-Structure

In terms of the natural numbers, the numbers 1 to 3 are hereby theorized as the First GD-Structure. In accordance to the Law of Proximity in gestalt psychology, it´s hypothesized that three objects close to one another represent a natural "gestalt" in the field of perception; **that is an optic quality, which is self-evident and *a priori***. This means that three items are perceived as "three" without having any previous experience. Thus, a child is capable of perceiving three objects simultaneously if they are close to one another relative to their physical environment and so, they perceive them as "three" and not as "two" or as "one." This experience takes place simultaneously *a priori*. Afterwards, the numbers 1 to 5 are considered to be the Second GD-Structure, but not as a natural human phenomenon a priori. In theory, a set of numbers 1 to 5 could be postulated as a "semi-natural phenomenon", i.e., as "semi *a priori*" so to speak. This means that in order have mastered this structure, the child must have some experience or some training. Yes, this child may be said to grasp a set of three oranges as "three" without any prior experience, but in order to grasp "five oranges", the child would need some "world experience" in order to understand that he or she is dealing with five items and not four. With appropriate experiences the child will be able to eventually perceive the whole set of four or five items simultaneously, which is the main objective of GD as per its Pentagonal System: to be able to know instantly if a set has three, four or five items. If the child has to count a set of five items one by one, then it means that it has not yet mastered the Second GD-Structure.

Also, in the early stages of number sense, **mastery of the one-to-one correspondence is essential:**

> Children love to count. They count everything from the steps they take to get from their bedroom to the kitchen, to how many friends are in school each day. Counting helps them make sense of the world and to find out how many of something. With time and practice, children develop an understanding of the "rules" or principles of counting.
>
> One such principle is known as one-to-one correspondence. It's the idea that numbers correspond to specific quantities. For example, in playing a game, a child counts 1, 2, 3, 4, 5 dots on the die and jumps 1, 2, 3, 4, 5 spaces on the board because 5 dots correspond in quantity to 5 jumps. The number "five" always corresponds to that precise quantity, no matter what it is you are counting.
>
> A hallmark of accurate counting, then, is when preschoolers begin to assign one number, and only one number, to each object as they count. We see this achievement when a child touches or tags each object in time with saying the counting words. And, this is no small achievement as it requires coordinating motor movement and speech with exact synchrony.
>
> But even when children tag every object one-to-one with a counting word, they may not yet have full understanding of one-to-one correspondence. Understanding the correspondence between a quantity and its number name (and numeral) is more than the action of tagging or keeping track while counting.

> Children often first develop a sense of one-to-one correspondence by playing with toys that require matching one object to one space, such as putting plastic eggs in an egg carton or fitting shapes into a shape puzzle. Eventually children can put objects into one-to-one correspondence themselves, such as setting the table with one plate and one napkin for every seat. But children can do this without fully understanding that the corresponding number of plates, napkins, and seats is the same.
>
> It's important to discuss correspondences that occur naturally, and meaningfully, in the life of young children. When putting on winter gloves, does every finger find an opening? Are there enough glue sticks for everyone at the table? How many garages do we need to park all the toy trucks?
>
> This student has a completely different question to answer. He needs to figure out how to fairly share cookies between two friends
>
> Sapoznick, J. and Brownell, J. (October 29, 2019). https://earlymath.erikson.edu/why-one-to-one-correspondence-math-matters/

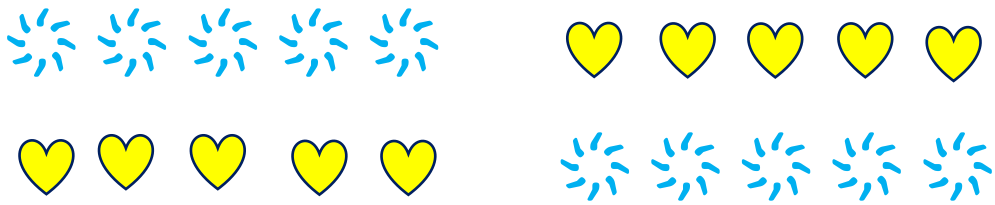
The corresponding **GD-Structure** is unconditional because it has to be mastered so that the child can climb up to the next step, which in itself also implies the mastery of the one-to-one correspondence: the knowledge for example that three bananas correspond to three children (Elena, Peter and Eva). That means that each child can only eat one banana. It´s amazing that the mapping in the cardinal numbers (https://www.splashlearn.com/math-vocabulary/number-sense/cardinal-numbers) 1 to 5 can be achieved with relative ease (Bryant 1974).

In essence, the first three numbers, 1, 2, 3 represent the **First GD-Structure** as a natural *gestalt a priori*. The **Second GD-Structure** can only be achieved if the first one has been mastered. Here are the steps of the hypothetical GD-Structures:

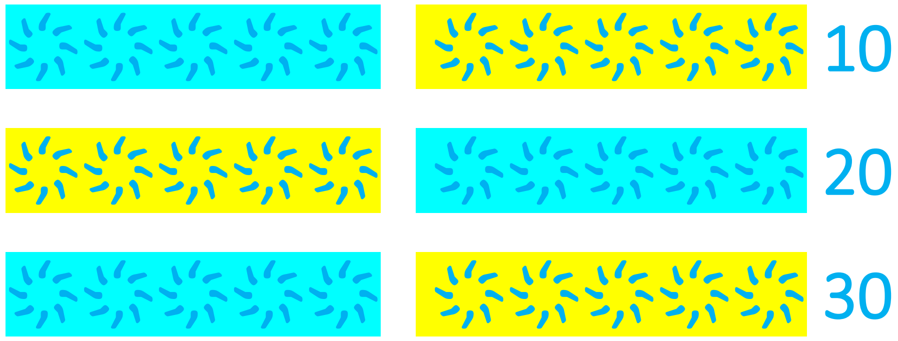
The five GD-Structures

- The First GD-Structure is believed to corresponds to the cardinal numbers 1,2,3.
- The Second GD-Structure corresponds to the cardinal numbers 1 to 5 in context of the GD Pentagonal System
- The Third GD-Structure (or 1-10-GDS) corresponds to the cardinal numbers 1 through 10 within the GD Pentagonal System such as:

- The Fourth GD-Structure (or 1-20-GDS) corresponds to the cardinal numbers 1 through 20 within the GD Pentagonal System. Here is an example:



Thus, 30 could be represented as follows:



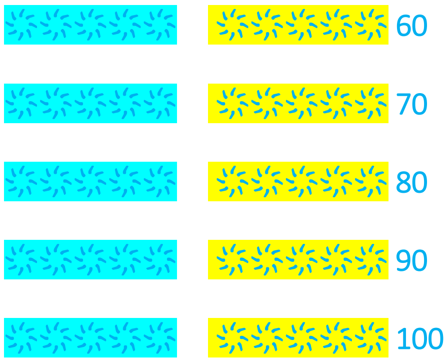
● The Fifth GD-Structure (or 1-50-GDS) corresponds to the cardinal numbers 1 through 50 are also postulated to be promoted within the GD Pentagonal System. Here is a practical example:

$$31 = 5 + 5 + 5 + 5 + 5 + 5 + 1$$

However, here is where the Pentagonal System becomes a subset of the GD Decimal System whereby the number 10 as a group slowly takes center stage:

$$31 = 5 + 5 + 5 + 5 + 5 + 1 = 10 + 10 + 10 + 1$$

● The Sixth GD-Structure (or 1-100-GDS) corresponds to the cardinal numbers 1 through 100, whereby the GD Decimal System takes center stage in context of the Pentagonal System. Here is a practical example where the GD Decimal System is reflected in the Power of the Five:



● The seventh hypothetical GD-Structure (or 1-1000-GDS) represents the natural numbers 1 to 1000 as a pure decimal system without any direct connection with the GD Pentagonal System. From there, the numbers are analyzed into infinity via the decimal system, but also taking into consideration the binary and the Mayan vigesimal system (a base-20 numeral system).

The First GD-Structure is very interesting because of its natural *a priori* phenomenon. It is believed that all children are born with this ability as part of the Sensorimotor Stage according to Piaget:

> The sensorimotor stage is the period of development from birth through age two. During this initial phase of development, children utilize skills and abilities they were born with (such as looking, sucking, grasping, and listening) to learn more about the environment.
>
> In other words, infants and young children experience the world and gain knowledge through their senses and motor movements. Through trial and error, children discover more about the world around them.
>
> Cherry, K. (February 28, 2023). https://www.verywellmind.com/sensorimotor-stage-of-cognitive-development-2795462

In GD, children are capable of perceiving three or two objects simultaneously as naturally as if they were one and this happens even if the objects are hidden. Objects continue to exist even if they are not seen or heard according to the theory of object permanence by Piaget:

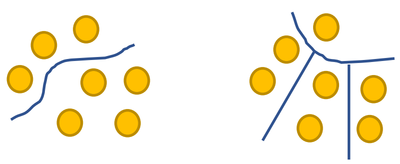
> The main development during the sensorimotor stage is the understanding that objects exist and events occur in the world independently of one's own actions ("the object concept", or "object permanence").
>
> > Object permanence means knowing that an object still exists, even if it is hidden. It requires the ability to form a mental representation (i.e., a schema) of the object.
>
> For example, if you place a toy under a blanket, the child who has achieved object permanence knows it is there and can actively seek it. At the beginning of this stage, the child behaves as if the toy had simply disappeared.
>
> The attainment of object permanence generally signals the transition from the sensorimotor stage to the preoperational stage of development.
>
> Mcleod, S. 2023, March 8. (https://simplypsychology.org/Object-Permanence.html)

In essence, “cognitive development occurs through the interaction of innate capacities (nature) and environmental events (nurture), and children pass through a series of stages” (Mcleod, S. 2023, https://simplypsychology.org/piaget.html). As per GD, this theory should encompass not just one object, but also groups of two or three objects as per the First GD-Structure. In other words, a group of three or two objects be perceived *a priori* just as one object is perceived and that is simultaneously, *a priori*, without prior experience because children are born with that human ability. This means for instance that three objects are not only perceived as three elements within a group, but also as a *Gestaltqualität* (quality of gestalt), as if the three objects had the equivalence of being captured as one entity vis-à-vis other objects in the physical environment. Children are therefore can see and conceptualize three objects as if they were one as per GD theory.

On the other hand, mastery of the cardinal value of objects within a group assumes that the child can count them forwards and backwards. Mastery of any GD-Structure presupposes the cardinal reversibility within a set of numbers. For example, if the child has mastered the Third GD-Structure, that is, the numbers 1 to 10, then it also means that the child can count them backwards: 10, 9, 8, 7, 6. 5. 4. 3, 2, 1. If they cannot do it, then it means that mastery has not been achieved.

However, for the First GD-Structure, the simultaneous perception is innate, *a priori*. According to Kant space, geometry and time may be perceived *a priori* and, in this sense, any given set of three objects are perceived in space as if they´re one entity as “pure necessity” in Kantian terms. When something is perceived out of “pure necessity” it doesn´t depend on any experience. In effect, a quantity of three items is self-evident, but not a quantity of 5 to 10 items.

> Kinder können Anzahlen bis vier durchaus sofort erfassen und benennen, und sie greifen dabei nur selten auf das Abzählen zurück. Zwar fußt diese Fähigkeit auch auf vorausgegangenen Zählerfahrungen (Gelman/Gallistel 1978), ermöglicht aber bereits im frühen Alter eine gleichsam automatische Antwort.
>
> Krauthausen, 1995, S.94
>
> [Children can readily perceive a quantity up to four and be able to name it and very seldom do they need to count them one by one. This may be based on previous counting experiences (Gelman/Gallistel 1978), but it enables them at a very early age an automatic answer so to speak]
>
> Während die Anzahl der Elemente bei kleineren Mengen (bis zu vier) in der Regel „auf einen Blick“, d.h. ohne zu zählen, bestimmt werden kann, gelingt eine sog. quasi-simultane Anzahlerfassung bei einer größeren Menge nur dann, wenn diese zugleich in ihrer Gesamtheit und als Zusammensetzung unterschiedlicher Teile bzw. Gruppen gesehen werden kann (vgl. Ruwisch, 2015; Wittmann & Müller, 2009).



> [While the quantity of elements in small groups (up to four) can be ascertained at “at a glance” as a general rule, i.e., without counting, a quasi-simultaneous quantity-assessment with a bigger group is possible if in its totality the composition can be perceived as a result of the corresponding parts and groups] (Source: https://pikas-mi.dzlm.de/node/122).

According to *Deutsches Zentrum für Lehrkräftebildung Mathematik*[12]:

https://pikas.dzlm.de/schumas/arithmetik-12/modul-1/mini-module/13-z%C3%A4hlen

at age 5½ to 6 children are able to build structures within quantities from objects. For example, they can count forwards two-by-two and backwards a number at a time. I don´t think that most children at that age have achieved this ability as a general rule, but certainly with appropriate training, they can count two-by-two. However, counting backwards is more challenging and requires special training, which are given within the GD methods.

Complex requirements are needed according to this German research center. It is noted that by the time children enter elementary school they have already acquired diverse mathematical abilities and counting experiences. Thereby the learning preconditions for school beginners are very heterogenous. (see Krauthausen & Scherer, 2007; Schipper, 1998). While some children can count up to 100, there are other children with the numbers up to 10 and cannot yet, given a number, count forwards or backwards (see Gasteiger, 2011).

What´s important according to *Pikas compakt*[13] (2023, April 6) is the partitioning of numbers (their decomposition), which is essential in the mastery of number sense. Children for example need to learn how to decompose the numbers 2, 3, 4, 5, 6, 7, etc. into all pairs of summands as follows:

2	3	4	5	6	7	8
2 + 0	3 + 0	4 + 0	5 + 0	6 + 0	7 + 0	8 + 0
1 + 1	2 + 1	3 + 1	4 + 1	5 + 1	6 + 1	7 + 1
0 + 2	1 + 2	2 + 2	3 + 2	4 + 2	5 + 2	6 + 2
	0 + 3	1 + 3	2 + 3	3 + 3	4 + 3	5 + 3
		0 + 4	1 + 4	2 + 4	3 + 4	4 + 4
			0 + 5	1 + 5	2 + 5	3 + 5
				0 + 6	1 + 6	2 + 6
					0 + 7	1 + 7
						0 + 8

The objective is for children to grasp the whole-to-part and part-to-whole relationships as a number concept. Thus, quantities can be conceptualized as entities composed of variant parts within specific patterns. For example, within the odd numbers all the parts have a, "interchangeable pair" such as 5 + 1 and 1 + 5, but in the even number there is pair that is only given once. Also, for each number there are n + 1 pairs, whatever n may be. So, if the number is 24 for example, the children will eventually be able to predict that it has 25 (n + 1) pairs with one, 12 + 12, being the one that is unique because it´s only given once and each summand is half of its corresponding total. In the uneven numbers all pairs have a corresponding interchangeable pair. For example, for the number 7, we have the following interchanging pairs:

[12] Deutsches Zentrum für Lehrkräftebildung Mathematik (German Center for Teacher Education in Mathematics) https://pikas.dzlm.de/

[13] Pikas compakt (2023, April 6). ZAHLVERTÄNDNIS ANFANGSUNTERRICHT – Zahlzerlegung [Understanding of numbers at beginning/elementary instruction – partitioning of numbers].
Retrieved from https://pikas-kompakt.dzlm.de/sites/pikaskp/files/uploads/08-ZahlvorstellungenAnfangsunterricht/zahlvanf_zahlzerlegung.pdf

7= 7 + 0 / 0 + 7 6 + 1 / 1 + 6 5 + 2 / 2 + 5 4 + 3 / 3 + 4

Furthermore, we can also conclude that all even numbers have the same amount of pairs as the corresponding number, plus one pair that has no interchange functions as it stands by itself in the middle of the pattern.

The Pentagonal System as a GD theoretical framework

In terms of number sense, a pentagonal system is much more obvious and reasonable than a decimal system. In essence the numbers 1,2,3,4,5,6,7,8,9,10 do not constitute a clear and obvious transparency in the field of perception. In other words, the decimal system with its numbers up to 10 represent a very awkward optical organization.

The four-year old child can readily perceive "1,2,3" as a clear and transparent whole because they appear simultaneously in the vision of a child (Krauthausen 1995, Bryant 1974), but that is not the case with the numbers 1 through 10. According to Piaget the numbers 1 to 6 are readily visible in a clear and transparent way because "they are bound to certain perceptual structures" (Piaget 2000, p. 162).

In Piagetian terms concepts of quantities are constructed by the child in different stages, beginning with the sensorimotor stage up to age 2; from there the preoperational stage appears with the use of language and symbolic though, but with a lack of logic and rational thought. "The child links together unrelated events, sees objects as possessing life, does not understand point of view and cannot reverse operations. For example, a child at this stage who understands that adding four to five yields nine cannot yet perform the reverse operation of taking four from nine" (Ojose, B. 2008, p.27).

One has to deal with the number concept. According to Piaget´s "genetic epistemology", the concept of numbers has to be built by the child according to his or her corresponding cognitive stage (Griffin / Case, 1997). Under this constructivist perspective, the adults need to help the child to assimilate insights through the process of adaptation: assimilation and accommodation:

> Assimilation involves the interpretation of events in terms of existing cognitive structure whereas accommodation refers to changing the cognitive structure to make sense of the environment. Cognitive development consists of a constant effort to adapt to the environment in terms of assimilation and accommodation. In this sense, Piaget's theory is similar in nature to other constructivist perspectives of learning (e.g., constructivism, social development theory).
>
> Source: https://www.instructionaldesign.org/theories/genetic-epistemology/

One assumes that conceptual development takes place genetically and naturally, which can be researched with empirical data. Thereby a contradiction arises between gestalt psychology of the Berlin school of thought and constructivist theory. In gestalt psychology phenomena are studied for their natural appearance *a priori* without much concerned as to its genetic development, where as in Piagetian psychology, it´s all about how the learning develops *a posteriori* in ontogenetic terms. Gestalt psychology deals with how the minds sees things in the visual world, without much concerned as to how the corresponding mental ability was developed (Köhler 1971, Katz 1944). GD will take into consideration a constructivist-genetic perspective along with the basic principles of gestalt psychology, plus most of the Vygotskyan psychology. The objective is to integrate those methods that offer the best possibilities and results.

GD´s gestalt principles dealing with number sense

A baseline would be to find certain numerical qualities that deal with specific visual patterns not only in terms of number sense, but also as to how numbers influence lists of words in terms of initial reading and writing. The patterns have been natural from the ontogenetic point of view, that is, they have to be appealing to the human being as "quasi simultaneous images". In this sense, GD postulates that a set dealing with more than 7 objects is not natural to the human vision; the human mind cannot simultaneously see the cardinal value of an x number bigger than 7 such as a row of 10 dots:

●●●●●●●●●●

It is not clear and transparent if we´re dealing with 10 because without counting them one by one, we´re not sure of as how many there are. However, it´s certainly clear and transparent that we see three or two items simultaneously:

3 2

The First GD-Structure represents the first generalization as a Vygotskyan principle

In essence, the first GD-Structure represents the first arithmetic generalization as explained before:

> **The first generalization represents the optical field in which a child has been able to single out the three groups of oranges out of their more general physical surrounding**. The second generalization may be represented by its arithmetic addition, "4 + 4 + 4," which can be further generalized into the more abstract level of a multiplication fact such as 4 + 4 + 4 = 4 x 3. This multiplication fact may be further generalized into the next higher level of abstraction such as in some algebraic relationship **(4 x 3 = 3 x 4 → ab = ba).**

In other words, children are able to generalize the objects in their surroundings as "schematic figures" in Piagetian terms, which happens naturally *a priori* according to GD. However, in order to generalize the Second GD-Structure, that of a group of items up to 5, its assimilation implies an authentic social interaction, especially at home. If the interaction is socially meaningful, then generalizing the Second GD-Structure will be optimal, which may be possible by age three, especially for linguistically advanced children.

For a one-year-old boy one can assume he has zero level of generalization in cognitive terms, but in the visual area, he´s nevertheless able to generalize three objects *a priori*. One can assume in Vygotskyan terms (Vygotsky 1978) that he sees a clock not as a clock, but rather as something round with two hands, one larger and one shorter and not with an hour and a minute hand, which would then imply the existence of higher order concepts and ideas. The First GD-Structure is the only instance in which we may indeed be able to justify a stage of conceptual development with a near zero level of abstraction.

Likewise, for the child who has not mastered the Second GD-Structure, the numbers 1 to 10 look like isolated figures, perhaps with a "gestalt", but they have no optical or semantic meaning that can be simultaneously perceived. In other words, a row of ten items does not have a dynamic *Gestaltqualität* because the child has not been able to assimilate the items in any meaningful way.

The perception of three objects within the First GD-Structure does not mean that the children can generalize them as a group, but rather as a natural perception *a priori*. Mastery of the Second or Third GD-Structures implies their corresponding reversibility. This means that the children, once they have mastered the numbers 1 to 10, can also count them backwards and in any order at will, but this is not the case within the First GD-Structure stage.

Contrary to Piagetian theory, GD postulates that a group of three objects [* * *] are perceived within the same perceptual structure as one [*]. This means that they both belong to the same perceptual schema in Piagetian terms. According to this GD hypothesis, children up to age two can perceive a group of three objects the same way as they perceive two or one. Thus, according to this hypothesis, three things as a group belong to the same perceptual field.

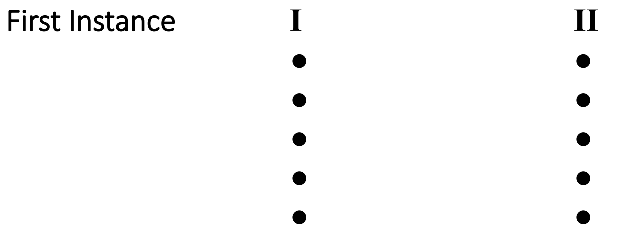
In this sense, GD represents a philosophical perspective and not an empirical science because it´s irrelevant if the children can see a group of three objects *a priori* or not. What´s relevant is the understanding that the First GD-Structure is a precondition of the Second GD-Structure. Research shows that children can perceive up to four items simultaneously and that is significant. With a group of four items, one is not sure if the perception takes place *a priori or a posteriori*, but for a group of five, some level of experience and cognitive maturity are required.

As a Second GD-Structure the objective is to train the children to perceive a group of five items as a simultaneous entity. With a group of 6 or 7 items, GD assumes that for most children, concrete experiences are necessary in order to perceive them and to assimilate them, but in context of the Second GD-Structure, in context of the Pentagonal System:

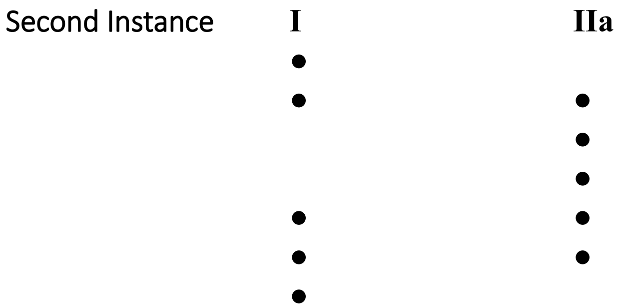
6

7

The Second GD-Structure should be analyzed as the precondition in order to assimilate a set of items up to 10. The Power of the Number 5 is the precondition to master number sense first up to 10 for the Third GD-Structure, which then becomes the precondition for the next higher GD-Structure ad infinitum. For example, as per the Second GD-Structure, children may not believe in the invariance of two groups of five items once the second group is presented in a row that is longer. In order to conduct an experiment, two columns of items with the same size can be presented to children:

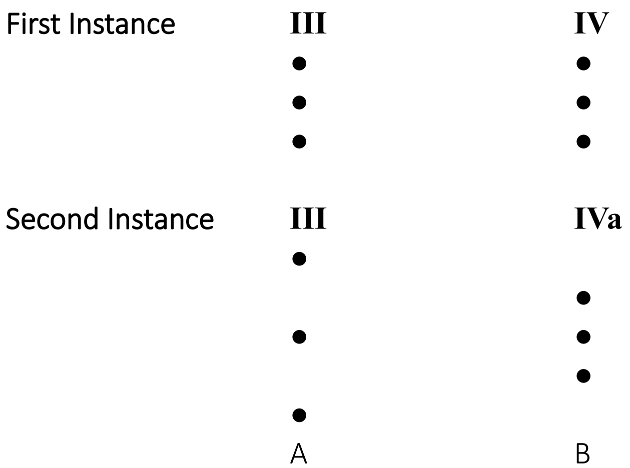


In the first instance children believe in the invariance of both sets.



In the first instance the children are confronted with two columns, each have five items and children have no doubt as to their invariance: they both have five items. There are two factors to consider: first the one-to-one correspondence as the items are presented in pairs from left to right and second, the length of the two columns. Because of these two factors six- and seven-year-old children say that both columns, **I** and **II**, have the same number of items. Thus, they believe in their invariance. However, in the second instance, the invariance disappears. Children at that age believe that **I** has more than **IIa** (Bryant 1974).

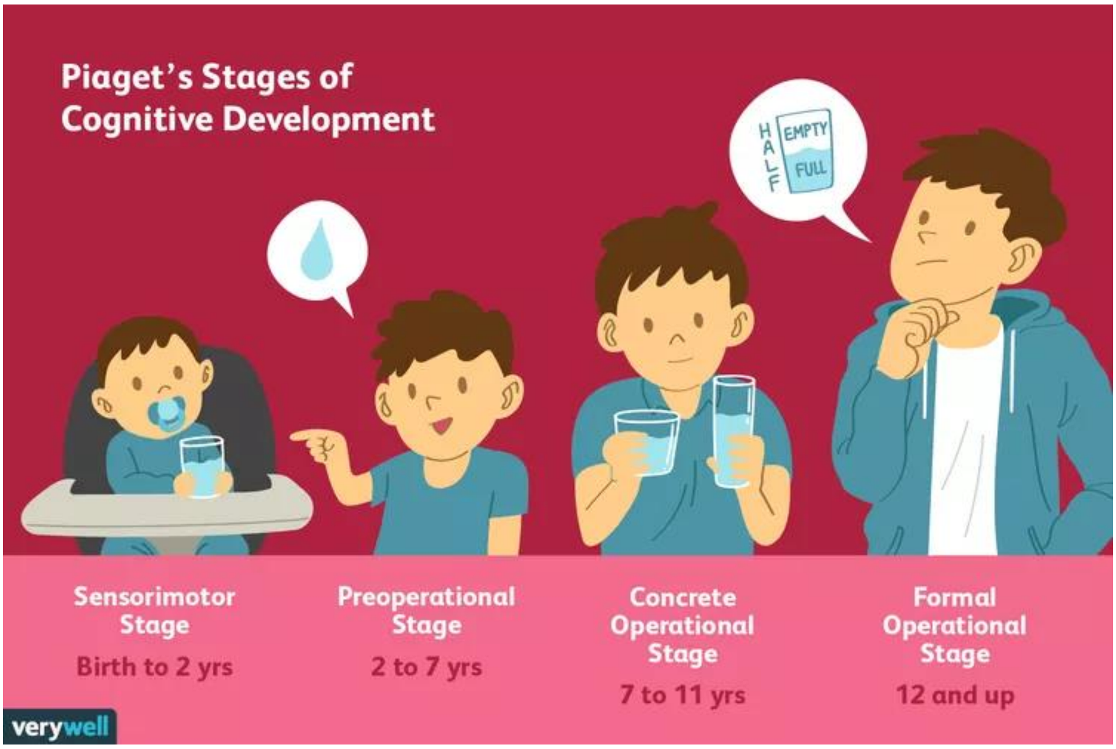
According to Bryant the children have difficulties dealing with the conflict of the one-to-one correspondence and the length of the columns. They need to slowly become conscious that the length of the column is not the decisive factor, but the one-to-one correspondence. However, according to GD, the reason that the children fail to ascertain the invariance at the level is because they have not yet mastered the Second GD-Structure. If the First GD-Structure thesis is correct, the same children will not fail to ascertain the corresponding invariance if each column had two or three elements regardless of the length of any column.



In the case of **III** vs. **IVa**, it´s predicted, according to GD-Structure´s theory, that children will be able to ascertain the invariance of both columns. Certainly, an experiment is necessary in order to evaluate the significance of the First GD-Structure, but there are plenty of anthropological references that this hypothesis is valid. Thus, depending on their psycholinguistic maturity, as the children ascertain the invariance with columns of up to five elements, it also means that they are on their way to leave the preoperational stage and enter the next one, the concrete stage:

Conservation in the Concrete Operational Stage

> Another key development at this stage is the understanding that when something changes in shape or appearance it is still the same, a concept known as conservation.
>
> Kids at this stage understand that if you break a candy bar up into smaller pieces it is still the same amount as when the candy was whole. This is a contrast to younger children who often believe that pouring the same amount of liquid into two cups means that there is more.
>
> For example, imagine that you have two candy bars of the exact same size. You break one candy bar up into two equally sized pieces and the other candy bar up into four smaller but equally sized sections.
>
> A child who is in the concrete operational stage will understand that both candy bars are still the same amount, whereas a younger child will believe that the candy bar that has more pieces is larger than the one with only two pieces.
>
> One of the key characteristics of the concrete-operational stage is the ability to focus on many parts of a problem.
>
> While kids in the preoperational stage of development tend to focus on just one aspect of a situation or problem, those in the concrete operational stage are able to engage in what is known as "decentration." They are able to concentrate on many aspects of a situation at the same time, which plays a critical role in the understanding of conservation.
>
> Source: Cherry, K. (2023, March 01). https://www.verywellmind.com/concrete-operational-stage-of-cognitive-development-2795458



Source: Cherry, K. (2022, December 16) https://www.verywellmind.com/piagets-stages-of-cognitive-development-2795457

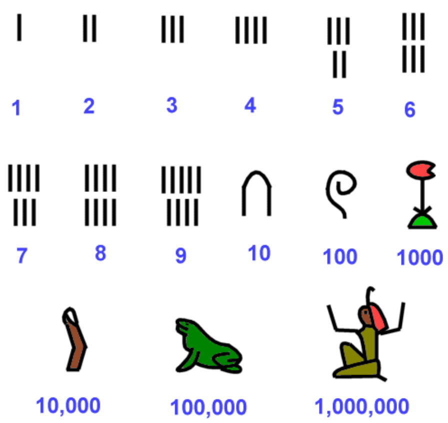
By age 6.6 to 7.8 children enter the concrete operational stage according to Piaget (2000) whereby the invariance is perceived on both columns. From the metaphysical point of view, a decentralization process takes place from the Preoperational Stage into the Concrete Operational Stage. The child is no longer bound by its physical environment similar to the experiments by Köhler (1925), who was able to show that Chimpanzees could reconfigure their perception in order to solve a problem, that is, in order to gain an "insight" into the order of things. In gestalt psychologists believe that the whole brings meaning to the parts in a top-to-bottom process.

> For example; a triangle is more meaningful than the three line segments. Gestalt theory main study area is perception, according to them through perception, we are able to acquire knowledge of the world, interact with it and connect with others. The Gestalt psychologists rejected the theory of 'trial and error theory'. **Learning by insight** is the sudden grasping of the solution to a problem, or a flash of understanding without any trial and error process. It is Eureka or a sudden understanding of a problem or concept. It is a moment of 'ahaa' or 'I have found it' etc. Insight learning is not the result of trial and error, responding to an environment stimulus. It is a cognitive experience to visualize a problem, think and finds a solution to solve the problem.
>
> Source: Neise Kehie, (2020, August 6): https://www.youtube.com/watch?v=0OK0qt_sxKk

Here is an excellent link about the gestalt principles (Colver, J. 2023, April 10):
https://www.youtube.com/watch?v=RWJSC1HU32c

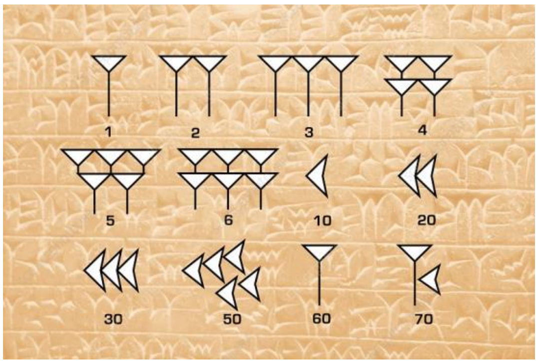
In essence, the chimpanzees may be able to gain an insight into a solution within the Second GD-Structure, contradicting the trial-and-error S-R reactions that would be expected of an animal, but nevertheless, they are still bound to the visible items in their environment.

The validity of the Pentagonal GD-Structure can be analyzed in the anthropological accounts as to how different civilizations reacted the numbers one through five. For example, in the Egyptians had a base 10 number system, but the corresponding hieroglyphs for 1, 10, 1000, etc. could not be repeated more than five times:



Source (2023, April 10): https://www.sutori.com/en/item/3200-b-c-egyptian-numerals-hieroglyphics-the-ancient-egyptian-number-system

Likewise with the Babylonian cuneiforms to represent the numbers.

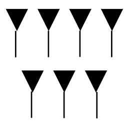


Source: Terry, S. (2023, April 10): https://www.sciencephoto.com/media/82989/view/babylonian-cuneiform-numerals

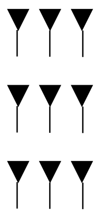
It would have been very chaotic if the Babylonians had to repeat any of their symbols more than 7 times:

How many?

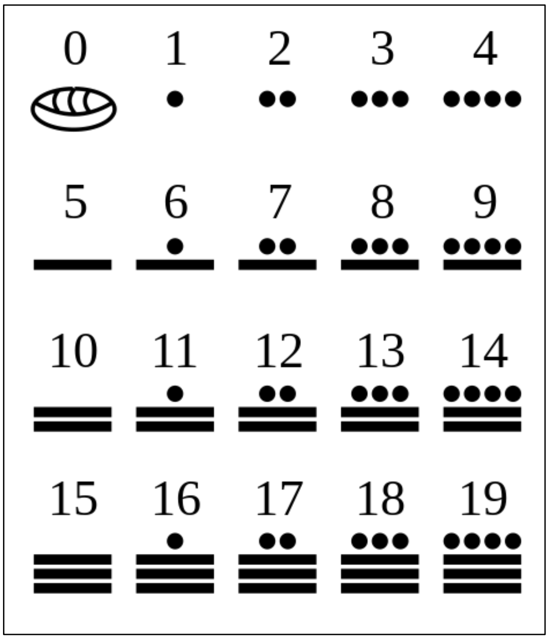
In essence, the Babylonian system represented the power of the number three in its optical configuration. In other words, symbols were group together mostly in groups of three in combination with groups of four. The number 7 for example, could be represented as follows:



The number 9 would be represented by a repetition of 3 plus 3 plus 3 symbols:



On the other hand, the GD structures up to the number 20 represent a pure “pentagonal system” similar to the Mayan Number System in the “optical-spatial field.” Here is for example the Mayan numbers 1 to 19, whereby no item is repeated more than four times and where the 5 is represented by a bar segment:



https://de.wikipedia.org/wiki/Maya-Zahlschrift
Source: Wikipedia, Die freie Enzyklopädie

The power of the number three is self-evident in many practical applications. For example, a telephone number is usually given in sets of two or three digits. In the USA the area code has three digits; after that the telephone number is written in sets of three or two digits. In order to call a telephone number from Germany it would something like +1 702 308 7943. As per GD, the power of the number five is the guiding force into the mastery of the numbers up to 50 and will serve as a bridge into the decimal system because 5 is half of 10 and because children can readily sense the number five as represented in the five fingers. The objective is to use the power of the number five in order to perceive the cardinal value of any set of number up to 50. Organizing numbers accordance to the decimal system is not clear and transparent:

A ●●●●●●●●●● B ●●●●●●●●

How many we see in A? Is it 10? Even in B, we need to count them one by one in order to be sure. Once we divide them into groups of either five or four, then we can perceive their cardinal value right away:

A ●●●●● ●●●●● B ●●●● ●●●●

Likewise in an election of candidates X and Y for a political office, we can easily hand count their votes in groups of five:

Candidate X: ~~////~~ ~~////~~ ~~////~~ /// = 18 votes

Candidate Y: ~~////~~ ~~////~~ //// = 14 votes

Up to the number 30, a pentagonal system is the most efficient way to hand-count votes in a way that the result can be perceived almost simultaneously.

Iconic representation of the GD Pentagonal System

On a daily basis, the power of the number five should become a motif in the first illustrations of children.

(2013, December 22): https://draw-a-city.tumblr.com/post/73432328703/282-5-windowed-house

We want to encourage children to detect the number 5 in nature as in the number of petals in a flower. Once they do find certain flowers with five petals, we want them to illustrate them so that the five becomes an essential motif within GD Pentagonal System:

(2023, April 10): free.com
https://www.freepik.com/free-vector/colourful-flowers-leaves-isolated-pink-background-flat-design_6748483.htm#page=4&query=flower%20illustration&position=14&from_view=keyword&track=ais

This also means that the children will be able to systematically index all the flowers with five petals so that they can contrast them with flowers with different amounts of petals. In other words, the children will learn about the names of those flowers with five petals and be able to name them with their special characteristics. Here is for example and illustration of a blue Forget-me-nots flower:

pixabay (2023, April 10): https://pixabay.com/illustrations/drawing-forget-me-nots-blue-bloom-2993282/

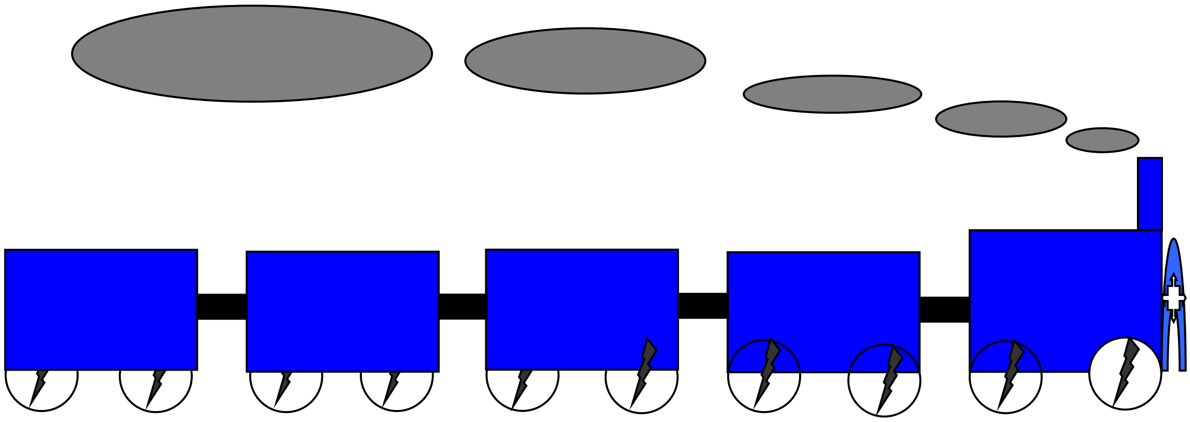
Essentially, the five becomes the baseline of their analysis of flowers in nature with five petals vs. all other flowers. Here is for example, a link to the 20 most beautiful flowers with five petals:

BalconyGardenWeb (2023, April 10): https://balconygardenweb.com/flowers-with-5-petals/

And here is close-up shot of a Buttercup Flower, which is one of the most salient five-petal flowers:

Tigiser, E. (2023, April 10). A Close-Up Shot of Buttercup Flowers
https://www.pexels.com/photo/a-close-up-shot-of-buttercup-flowers-7749004/

In terms of geometrical figures, one could start with a very simple train in which the number five becomes rather conspicuous:

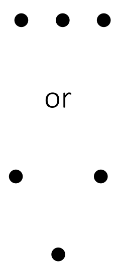


A set of five elements should become motif in as many experiences as possible; a motif to reflect on and to internalize as the basis for the addition and into the multiplication. They will be encouraged to draw as many phenomena as they can perceive in their environment, even with rhymes and songs such as Five Little Ducks Went Out One Day by Super Simple. Songs:

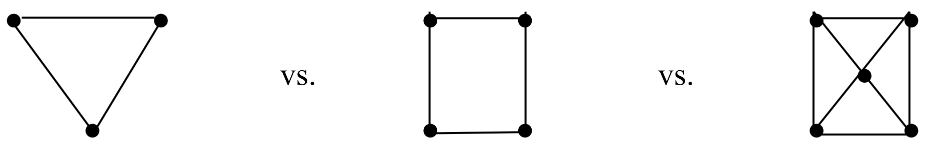
https://www.youtube.com/watch?v=pZw9veQ76fo

The question is, can the five-year old children draw a train with five wagons? That is the first objective: the ability to draw any object with five parts. That implies mastery of the first two GD structures, which can be depicted as follows:

First GD-Structure



Children will be taught how to contrast groups of three, four and five, be it with different types of flowers and their number of petals, or by drawing geometric figures such as trains and houses with their corresponding number of wagons, windows, or for example an apple tree with x amount of apples. We can also emphasize different point-constellations such as the following:



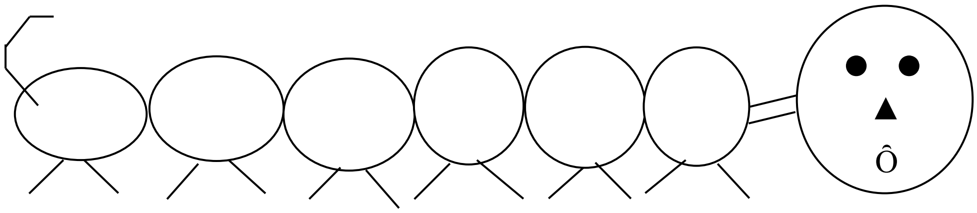
Geometric constellations

Linear constellations

If the child can do it, then he or she is ready to enter into the Third GD-Structure, that of mastering the numbers 1 to 10.

Most five-year old children while trying to draw a figure with five parts may end up drawing it with more than five

If a five-year old child is asked to draw a worm with five-parts may instead draw it with more than five parts, not paying attention to the one-to-one corresponding:



This may be because the one-to-one correspondence is not salient in their consciousness, especially with a counting process (Ansatz), which I call "back-coupling": A process by which a child adds or subtracts items from a set in order to arrive at a specific set. Here is an example:

Teacher:	Ok Maria, you´ve drawn a nice worm-looking animal, but let´s count them: one, two, three, four, five, six and seven. How many body parts it should have?
Maria:	Five!
Teacher:	So, what should we do so that it only has five?
Maria:	Take one away!
Teacher:	Ok, let´s now count them again: one, two, three, four, five, six. So, what should we do?
Maria:	Take another one away!
Teacher:	(After erasing one body part). Yes, now it has five; four body parts and one head. Excellent!

Thus, the power of the number five as a general motif must be incarnated into their perception, into their daily activities. It´s something that children need to "learn by doing" and in this case by drawing geometric figures as well as organic forms such as flowers with five petals. That way they can contrast those flowers with those with different sets of petals. In essence, we want to find out if nature favors those flowers, plants and fruits related to the Fibonacci sequence and for that we could start with a catalogue in terms of the number of petals in a flower such as the following:

3 petals: lily, iris

Mark Taylor (Australia), a grower of Hemerocallis and Liliums (lilies) points out that although these appear to have 6 petals as shown above, 3 are in fact sepals and 3 are petals. Sepals form the outer protection of the flower when in bud. Mark's Barossa Daylilies web site (opens in a new window) contains many flower pictures where the difference between sepals and petals is clearly visible.

4 petals: Very few plants show 4 petals (or sepals) but some, such as the fuchsia above, do. 4 is not a Fibonacci number! We return to this point near the bottom of this page.

5 petals: buttercup, wild rose, larkspur, columbine (aquilegia), pinks (shown above) The humble buttercup has been bred into a multi-petalled form.

8 petals: delphiniums

13 petals: ragwort, corn marigold, cineraria, some daisies

21 petals: aster, black-eyed susan, chicory

34 petals: plantain, pyrethrum

55, 89 petals: michaelmas daisies, the asteraceae family.
Some species are very precise about the number of petals they have- e.g. buttercups, but others have petals that are very near those above, with the average being a Fibonacci number.

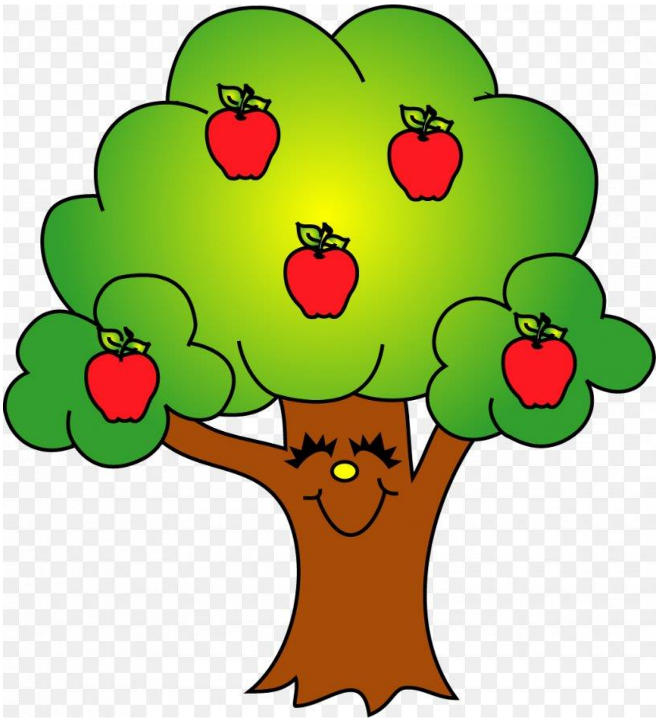
Knott, R. (2023, April 11): https://r-knott.surrey.ac.uk/Fibonacci/fibnat.html

For more advanced students, say beginning at the second-grade level, we could make a deeper analysis of the Fibonacci sequence such as the one made by Dr. Ron Knott:

https://www.youtube.com/watch?v=_GkxCIW46to

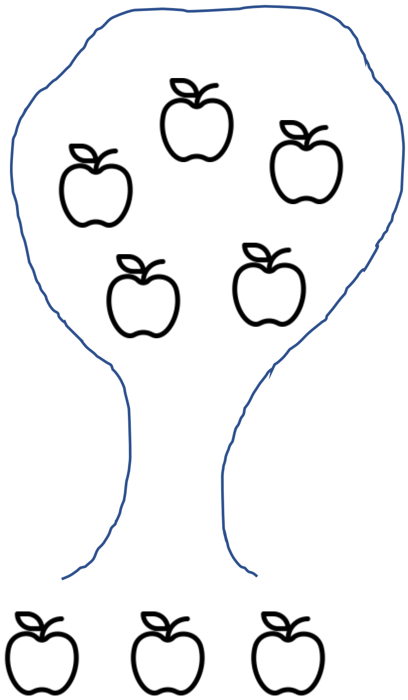
The "back-coupling of numbers" within the one-to-one correspondence

The coupling back of numbers into an intended amount starts with a child trying to find such an amount by making an illustration. It could be done with geometric or organic figures. If the intended figure was to have 7 parts, then the teacher helps the child to make the appropriate illustration. For example, the child may be asked to draw an apple tree in order to portray 7 apples in context of the Pentagonal Systems. So, the child may draw a tree with too many apples, say with 8 instead of 7. In this case the teacher asks the child to count them one by one. Once the child knows that the tree has 8 apples instead of seven, the teacher asks the child for a solution and the solution is to subtract one apple. In order to contextualize the number 7 within the Pentagonal System, the teacher for example first encourages the child to draw a tree with five apples as the following illustration:



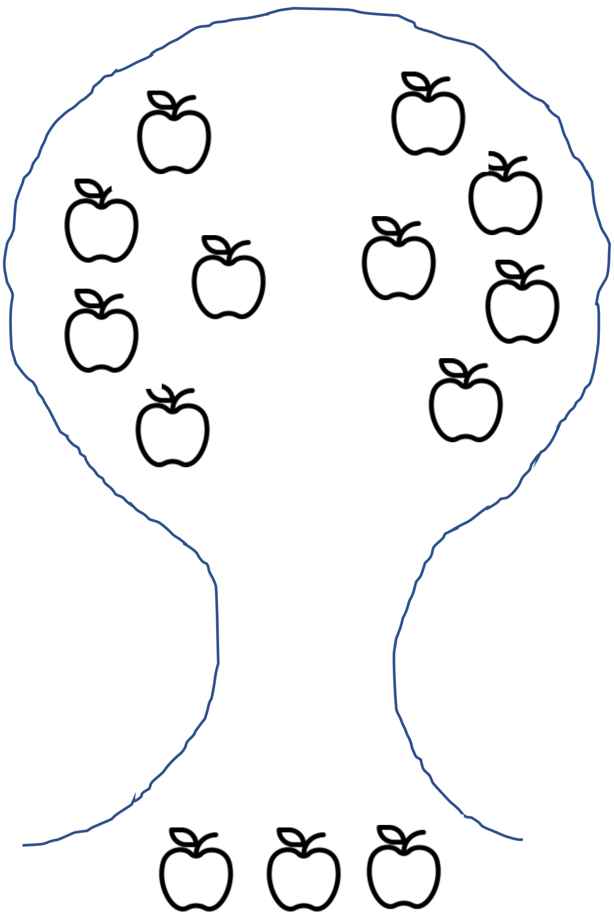
(2023, April 11) https://clipground.com/pics/get

Then the child would draw two apples on the ground, and perhaps with a worm nearby. Likewise, the child could draw 8 apples, five in a tree and three on the ground:



(2023, April 11): https://www.freepik.com/free-icon/apple_15481468.htm#query=apple&position=43&from_view=keyword&track=sph

The teacher explains to the child that the three apples on the ground fell off the tree because they got too ripe. The point is now to challenge the child in illustrating the sets of numbers within the Third GD-Structure, that is the numbers between 5 and 10. Once they can do this, then the illustrations would target the numbers between 10 and 20 within the Fourth GD-Structure. For example, the child may be encouraged to draw an apple tree with 10 apples in the tree and three on the floor:



Besides the iconic representations in different illustrations dealing with flowers, fruits, geometric figures within the Third, Fourth and even the Fifth and Sixth GD-Structure (the numbers between 30 and 50 and from there up 100), there is a melodic strategy in which the numbers are spoken either very loud or very softly. For example, the teacher may say the numbers "one, two, three, four" very slowly, and very loud for number five as she counts with the child the number of apples in the tree. After saying **FIVE!!!,** there is a small pause and again the numbers "six, seven, eight, nine" are spoken with a very low voice, and then very explosive for **TEN!!!** The same strategy is done with the numbers up to 20 and from there to 30 and then to 50:

1-2-3-4-5 6-7-8-9-10 11-12-13-14-15 16-17-18-19-20 Etc.

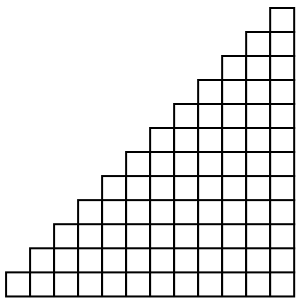
However, the numbers up to 20 need to be completely mastered first, especially in terms of their reversibility. The drawing, plus the melodic and the reversibility strategies will help the child to in incarnate into their minds the first GD structures into clear and transparent schemas. With these and other strategies, mastery of the numbers up to 100 will be possible by the end of second grade.

Another strategy for example is the awareness that arises when children are given an individualized number according the alphabet. For example, within a group of twenty children, we find out whose last name comes first according to the alphabet and thus Maria could be given number 1 if her last names starts with "a" and George could be given number 2 if his last name starts with "b" etc. Thus, a group of 20 first graders know that they each pupil has a particular number corresponding to the alphabet. That way, the numbers become tools in order to find out who sits where. Maria having number 1 will sit on chair number 1 and the first five chairs are grouped together where the corresponding children with their numbers 1,2,3,4,5 will sit and work together as a group. Thus, if

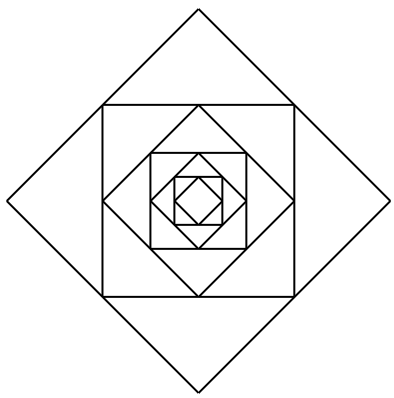
George were to be absent the next day, then one chair would be empty that day with the group only having four children. We could even play “moving chairs” because then the kid who had number 3 would at least for one day adopt number 2; so, each kid beginning with number 3 would slide down the “number line” and adopt the lower number as long as George is absent. So, if George were to be present the next day, then he would reclaim number 2 and everybody would go back to their “real number” as if such a number were to be an identity characteristic of their last name. In essence, such a number becomes a tool to finding out who´s present and absent on a daily basis and it helps children to understand the value of numbers and how they can help redefine the daily activities. Numbers now become part of their social upbringing and thus part of their socialization.

When numbers become part of their consciousness children are able to understand their surroundings much better. Number sense becomes a reality in order to perceive and organize their environment with numbers, especially through patterns and number configurations like having a number attached to their last name according to the alphabet. Slowly new number patterns are analyzed in order to encourage the children to create their own patterns:

Ascending Stairs 1 to 12



Diagonal Inner Squares

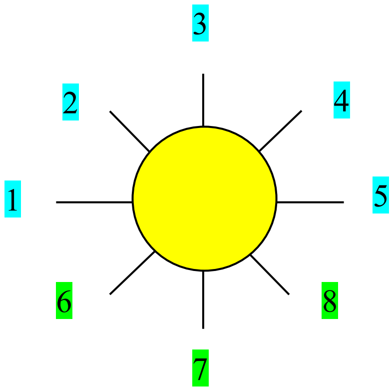


In the second diagram with the diagonal inner squares, the first square could be 8 inches in length. That means that the second square has 4 inches in length. So, we help the children to “build” the squares with materials of any kind, whereby later on we can cut the squares out of wood and/or make them out of clay. Now the child can make a Tower of Babel with such word or clay squares and they could further experiment to finding out how many smaller and smaller squares they can make in order to make the tallest Babel Tower. Perhaps a square is not the ideal geometric figure in order to create the tallest Tower of Babel; perhaps it´s a circle the ideal figure. So, the children can experiment with various materials and geometric figures in order to make the tallest Tower of Babel and they can also experiment with other types of patterns dealing with groups of three, four, five, etc. whereby those patterns with groups of five will always enjoy the highest esteem.

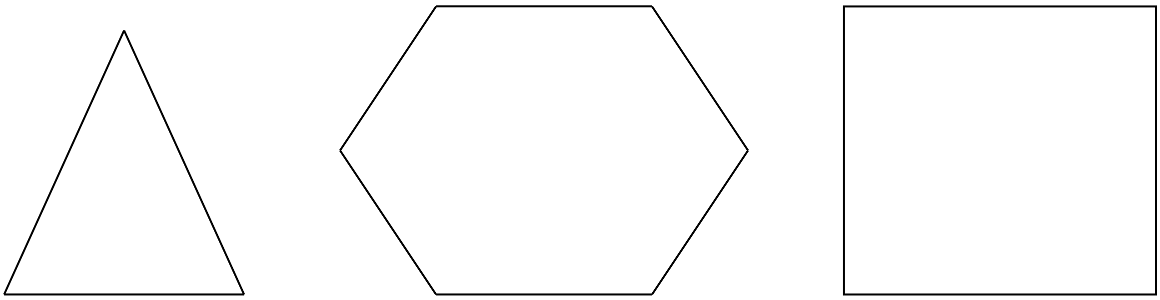
Sometimes there may **not** be a visual configuration of the number 5 like with the raindrops:

Freepik.com
https://www.freepik.com/free-vector/dark-clouds-with-rainfall-thunder-flash-background_15244408.htm#query=rain%20cloud&position=9&from_view=keyword&track=ais

Neither do the sunrays around a sun have a natural number-5 configuration. So, the children would need to come up with a creative way to count them in terms of 5 + 3:



The point is to show the children how we can "abstract" a number-5 constellation and contrast it with those figures that have a different constellation, whereby the power of the number 5 becomes the baseline, that contrasting entity against the number 3, 4, 6, etc.



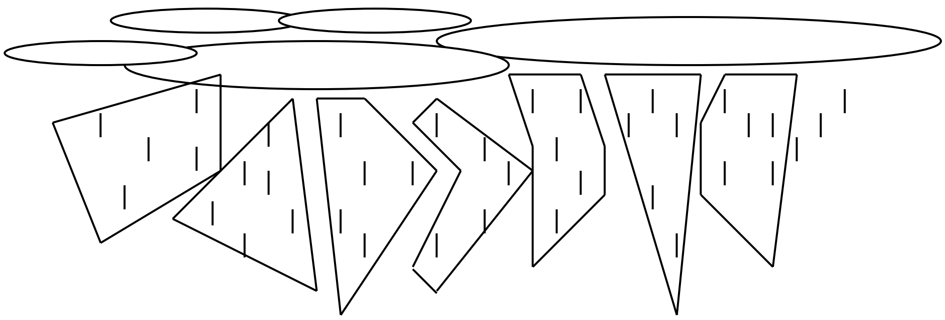
And in terms of the raindrops, we could even reconfigure them in a way that the number 5 becomes the focal point for example with five clouds with two clouds having each a group of 5 raindrops, which is very clear and transparent, while another cloud has 6 raindrops. The children would then analyze the situation in which the group of 6 raindrops becomes like in a world of itself, contrasting with its visual chaos against a background with two groups of raindrops, each have a clear and transparent set of five raindrops:

5 clouds with sets of 5 + 6 + 5 raindrops and 2 bolts of lightening
Source: Freepik.com
https://www.freepik.com/premium-vector/rain-clouds-lightening-bolt-paper-cut-weather-storm-time-rain-drops-thunder_34670056.htm#query=rain%20cloud&position=18&from_view=keyword&track=ais

The question is, how many raindrops do we have? The mentor would advise the children to "divide" that group of 6 into 5 + 1 and so the children could perceive the whole number configuration as:

$$5 + 6 + 5 = 5 + 5 + 6 = 5 + 5 + (5 + 1) = 16$$

The children may be encouraged to create their own raindrop configurations in which sets of five raindrops may be encapsulated as in the following example:



In this case, the addition now becomes a little bit more challenging:

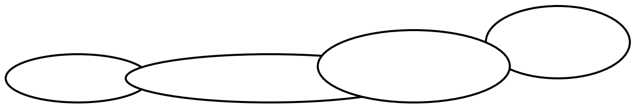
$$5 + 5 + 5 + 5 + 5 + 5 + 5 + 3 = 38$$

The back-coupling *Ansatz* and the Pentagonal System

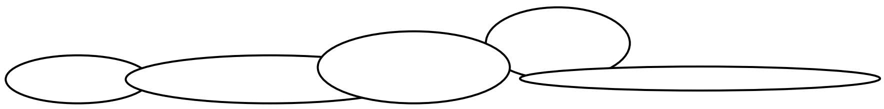
With the back-coupling *Ansatz* the main objective is for the child to understand that 6 = 5 + 1 and likewise 7 = 5 + 2, 8 = 5 + 3; 9 = 5 + 4 and 10 = 5 + 5 with their corresponding reversibility. Back-coupling means to reverse the action of a practical *Ansatz* (approach). The goal is for the children to acquire number sense by being able to divide all numbers up to 50 into sets of 5 whereby the 5 becomes a "quasi common numerator." For example, for 43 we may have 8 sets of 5 with a residual of 3:

$$5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 3 = 43$$

It´s a long process in order to acquire mastery of all number up 50 including their reversibility. It all starts with simple sets dealing with the Third GD Structure, i.e., the mastery of numbers up to 10. Children at age 4 or even 5 may not be able for example to draw five clouds. Instead of five clouds, the child may draw only four.

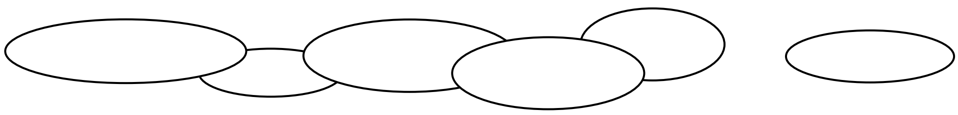


With the "back-coupling" we then ask her to count them and once the child realizes that it´s only four, he or she will add a fifth cloud.

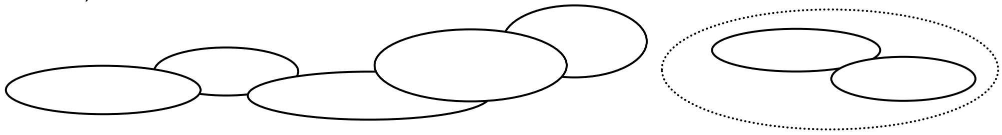


The next challenge would be to keep on drawing clouds up to 7:

Teacher: How many clouds have you drawn so far?
Child: Five!
Teacher: How many more clouds do you need to draw in order to have 7?
Child: One more!
Teacher: Go ahead and let´s find out!

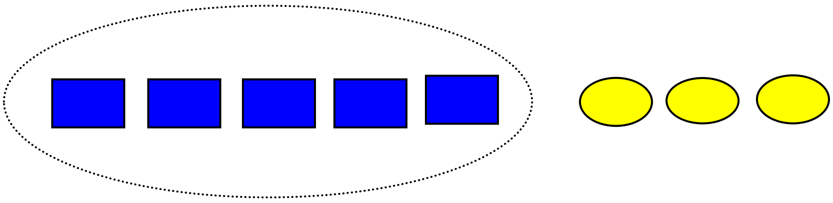


Teacher: Let´s count them.
Child: They´re six. So, we need one more to have 7.
Teacher: Yes, that´s great thinking. You came up with the right solution! But now that we have 7 clouds, imagine that it stops raining and two clouds go away. How many would we then have left? What do you think? What will happen if we erase two clouds. How many will be left?
Child: Five! Yes, five clouds!

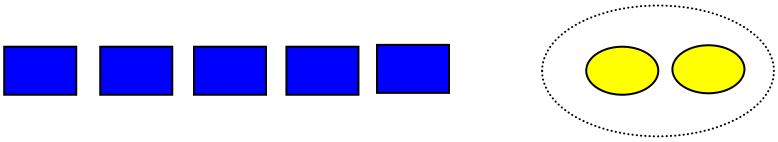


Teacher: That is perfect! How did you know? Excellent thinking!

With any materials in the environment such as books, pencils, notebooks, chairs, tables, etc. such **"back-coupling**" activities should be encouraged, especially if they can be contextualized into an authentic life situation, such as in the case of the clouds. Everything depends on the creativity of the teacher. **"Back-coupling"** is about a perceptual awareness that numbers can be reconfigured into set of 5 and that these sets can be manipulated at will. Slowly this perceptual ability becomes incarnated into the consciousness of the child. Children will be able to internalize the Second GD Structure, the one dealing with the power of the number five as a set, and once that happens, the numbers up to 10 become readily available. In essence, the Second GD Structure become the foundation towards number sense because it becomes internalized with a special *Gestaltqualität* (gestalt-quality):

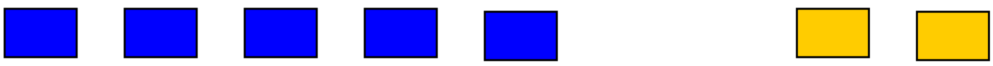


The essential aspect of this concept is for the child to become aware as what happens if one, two or three elements are taken away or added to the set five. With a "five-plus-two" constellation, it becomes self-evident that the answer is 7 and likewise, once this is ascertained, the child then is asked the inverse, that is, what happens if two from the seven are taken away:



Once the children can perceive the five and the seven as autonomous configurations independent from one another then the answer becomes automatic in either directions, adding two to seven or subtracting two from seven in order to perceive five. Once this happens, the reversibility slowly becomes more challenging depending on their Zone of Proximal Development:

Teacher: Ok children, here we have seven books, five blue and two orange. Now, however, I would like to lend three books for a day to three children. How many books are left?



Children: Three!
Teacher: Let´s find out. We take three away, one by one. How many are left then?
Children: Ok, four are left!
Teacher: That´s right. Now, the next day the three children return the books. How many would we then have by then? Ok, let´s put them back one by one. How many? Here we have five, then six and seven!
Children: Seven!
Teacher: Yes, we come back to the same number of books that we had at the beginning. Then the next day, I go to the bookstore and buy three more books. How many books would I then have? Who can tell me?
Child X: Then we have nine!
Teacher: Let´s count them starting with seven. After seven is eight, then nine. That´s

only two more books: seven books, then eight and nine books, but I bought three books and not two. How many then?

Children: Ten, ten books!

Teacher: That´s right. 7 plus 3 is 10. Repeat after me: "seven plus three..., seven, eight, nine, then."

Children. "Seven plus three..., seven, eight, nine, then."

Teacher: And so, we now have ten books, five blue and five orange:

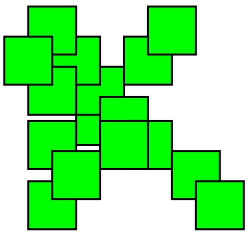
However, the very next day, I will then lend to Maria and Paula each one book. How many books are left?

Etc.

Each time that the children give the right answer they should be praised. That way, they will start valuing the power of the number five with some reverence, which will help them organize their counting abilities into more and more complex forms within the GD constellations.

One effective strategy is simply to ask the question about a pile of objects as to its approximate amount. If it is a very large pile, children should be given a star as a price if their calculation is given within a range of five from the right answer. Children collect stars as a valuable asset. Once they collect 100 stars, they get a special treat like going to a farm to ride a horse or anything that is exiting for them, like a special trip to the mountains, etc.

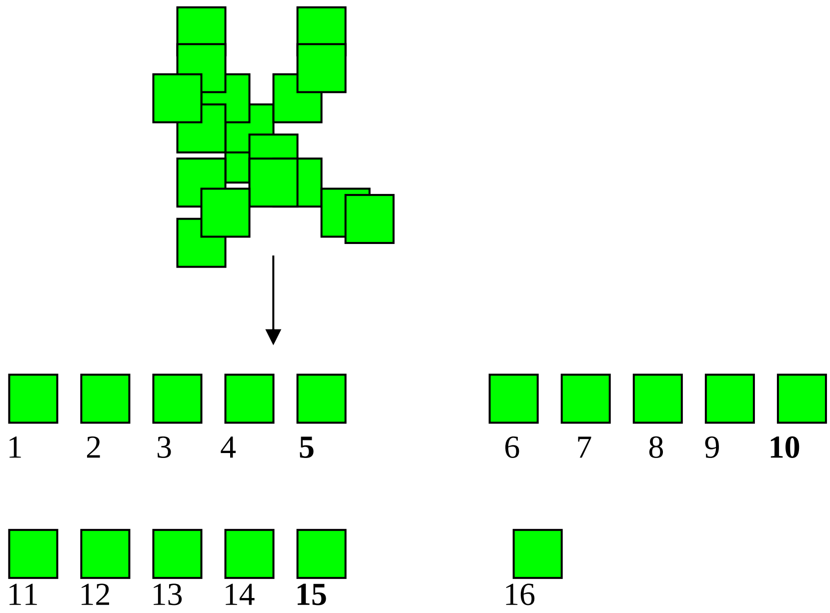
Teacher: So, children who can guess how many books we have on the floor?



Whoever comes close to the right amount gets a star. Your answer would need to be within a range of five in order to get a star for your collection. Maria, how many stars have you collected already?

Maria: I have 27 stars. I´d say, there are 20 books on the floor. Will I get another star? Once the activity is done in the style of a riddle, it becomes an interesting play and the children enjoy it. The objective is for them to become curious and actively involved in solving the riddle. After each child estimates an approximate amount of books in the pile, they will want to find out who came up within a range of 5 with their respective answers and whoever did gets one more star to the collection:

Teacher: Yes of course. Let´s count the five by five:



After a couple of weeks of such calculating experiences, the children come closer and closer to the right answer given a similar amount of books or pile of objects to be counted. By then, only the child who gives the right answer would be given a star for his or her collection, and there will be a lot of excitement if any child comes up with exactly the right amount asked. Once a child reaches 100 stars there should be a celebration; all children must somehow benefit if only one child reaches 100 stars so that all rejoice in the success of one.

Children will become more and more self-conscious of their newly acquired abilities as they progress in getting their price: reaching 100 stars. Through the “back-coupling” and other activities, children will eventually master the next GD-Structure, that of the numbers up to 20. In order to do this, it´s important that they can perceived what is 9 – 2 or 7 + 3, which implied the mastery of the Third GD-Structure. Each successive GD-Structure depends on the mastery of the previous one, especially in reference to its corresponding reversibility. The objective of the Third GD-Structure is to be able to manipulate two sets of five within a perceptual field with the numbers 1 through 10:

* * * * *   * * * * *

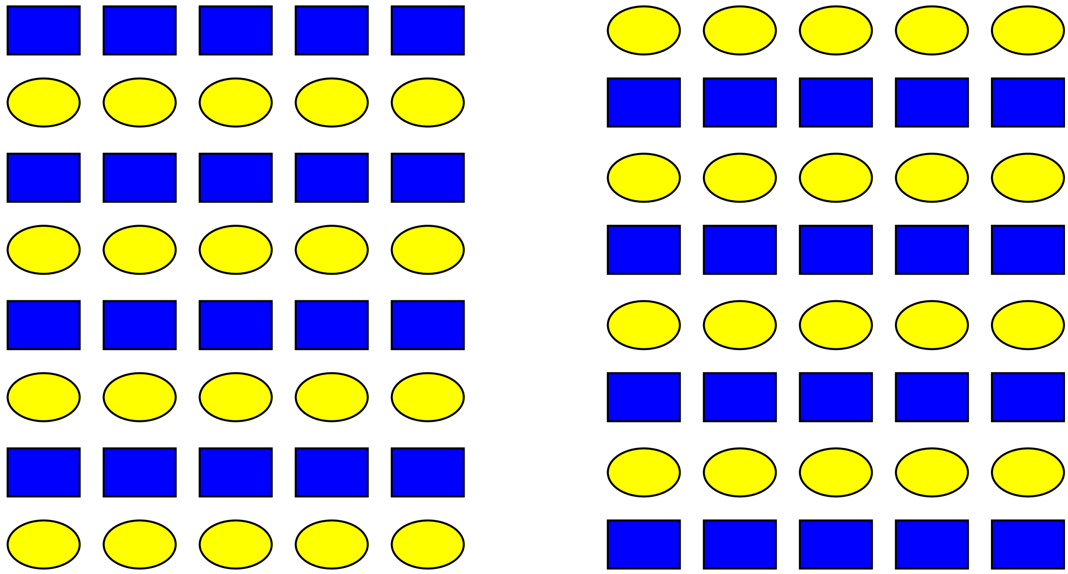
Mastery implies the complete manipulation forwards and backwards. They may be able to perceive right away, what is 7 – 2 or 10 – 5, but 7 – 3 or 9 – 3 is still challenging. The children are still “prisoners” of the optical constellation, and have not internalized the complete reversibility as an abstraction. Before this is completed, experience shows that the introduction of higher numbers, those of the Fourth GD-Structures (1-20) will help in the internalization process within the Third GD-Structure. It´s a type of spiraling; although the numbers 1 to 10 have not been completely internalized forwards and backwards, the introduction of the 1-to-20 constellation adds more color; it makes the whole interplay more interesting and as experience takes places, the eventual mastery of the numbers 1 to 10 will become reality in context of the Fourth GD-Structure in its optical constellation, which the children will start to manipulate:

* * * * *   o o o o o

o o o o o   * * * * *

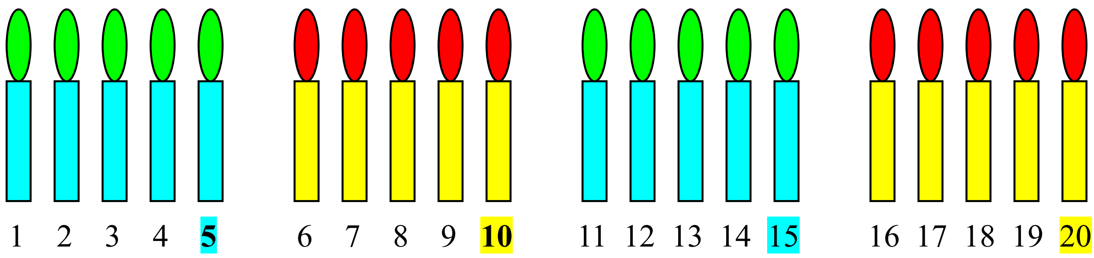
This 20-Field of Perception (Fourth GDS or 1-20-GDS) will be obviously processed in context of the GD Pentagonal System, especially within an optical zig-zag strategy as follows:

Zig-zag constellation within the GD Pentagonal System



At first one could simply illustrate a linear constellation as follows:

Linear constellation within the GD Pentagonal System

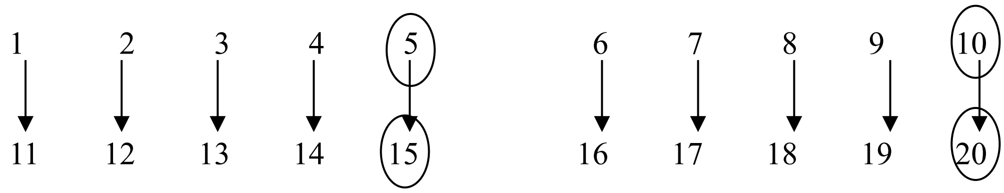


The pentagonal structures are also illustrated with some peculiar color constellations: red goes with green and blue goes with either blue or purple.

The GD mapping place value constellation

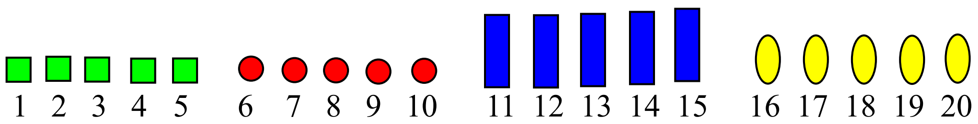
At this point a new numerical constellation is introduced as a new GD strategy in which the ones digits in the first row are aligned with the same digits in the second row as follows:

The GD place value mapping constellation of the numbers 1 to 20



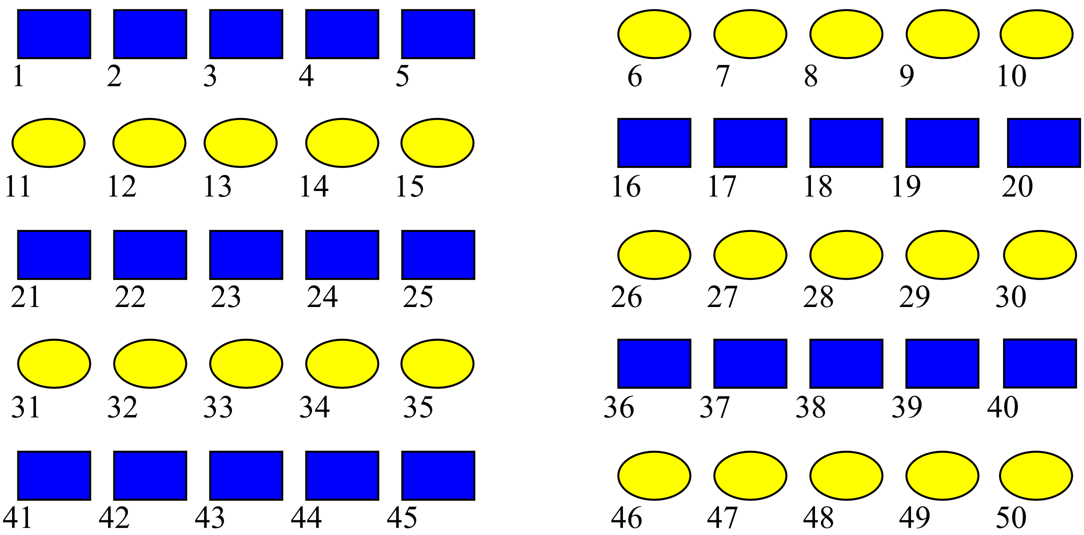
That´s a difficult assignment that can lasts month at the K-level. At first, the groups are colored within a pentagonal constellation as follows:

Color Constellation



Children choose what color they prefer for the first group of five items. However, they are encouraged to place a group of green items next to a group of red and likewise five would be placed next to five blue ones. This is a challenging task for five-year old kids and as they enter first grade, the color constellation could increase in its complexity up to the 1-50-GD-Structure:

The mapping constellation with two colors and two contrasting figures

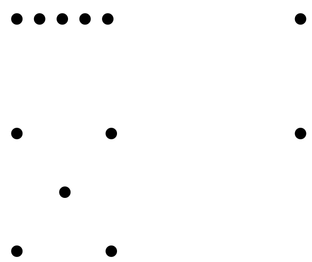


However, the objective is for the children to 4th the Fourth GDS (#s 1-20) even as the children enter into the territory of the Fifth GDS (numbers 1-50). In other words, in order to master the numbers 1-20 with all their back-coupling activities, they must interact within the 1-50-GDS. This allows the children to gain insights into the fifth GDS (1-50) as they master the fourth GDS (1-20). The activities would become too static and monotonous if the children were to deal exclusively with the number 1-20. The children must reach to the moon in order to reach the mountains and eventually, they must reach to the stars in order to reach the moon and as they do, they experiment with different stimuli, different objects in a real-world situation. As they do, the children will reach their upmost potential.

3.1 The empty space and the graphic system

In gestalt psychology all visual aspects are integrated into the whole, even the background. For example, in illustrating six dots on a piece of paper, the space between the dots and the whole space from the piece of paper are all integrated into a graphic representation of the numbers 1 to 6. In this case, what´s important is not only the one-to-one correspondence, but also the relationship of the first group of five dots separated as its own group from the sixth dot, which is placed rather far away from the constellation of the five groups to represent 5 + 1. The space between the group of five and the sixth dot plays an important role and eventually the whole space of the piece of paper will also play an important visual representation.

Space between a group of five dots and the sixth dot



In other words, the conscious manipulation of the space relations becomes an important factor in the pentagonal system according to GD. Little by little, children learn how to represent any number in groups of five in which the spacing between the groups becomes an essential factors of mastering the numbers from the addition to the multiplication. Everything that is countable is placed within the pentagonal system such as the number of characters of the first cartoon story. One objective of the story is for children to number the characters in a way that each character can identified with a corresponding number as follows:

The syllabic chart according to the GD pentagonal system

The objective is for the children to become aware of the number patterns relative to the space within the numbers, between the groups as well as the spacing outside the group, so that the entire space of the piece of paper becomes integrated into the optical field. In this case, the numbers and their optical constellation takes into account how much space there is between the groups and how the space is configured outside the groups. Let´s say for example that the children want to write x amount of words on a piece of paper. Well, that paper has to be folded in the middle so that numbers 1 to 5 and 6 to 10 are allocated on the left half side of the paper and likewise 11 to 15 and 16 to 20 on the right half side.

This pentagonal system can also be used to promote initial reading and writing as the children organize the number of words they´re learning to write in groups of five, following the number pattern. Thus, if they want to write 10 words, then the words are subdivided into two groups of five on the left-half side of the piece of paper. In this sense, literacy and arithmetic skills become integrated in the learning process; they are no longer viewed as two completely separated domains, but instead become intertwined and interlocked in the same lesson. Children start to read and write words with the aid of the pentagonal system. Children may then decide not only what words they want to write, but also how they want to present them in an arithmetic chart, following the pentagonal system. Before they even start writing the words, they learn how to fold the piece of paper twice as follows.

A piece of paper is folded twice in the middle and write 20 words as per the pentagonal system

1.	barbie	11.	apple
2	cat	12.	tree
3	flower	13.	moon
4	house	14.	sun
5	mouse	15.	car
6	hammer	16.	sister
7	jaguar	17.	baby
8	lamp	18.	school
9	mom	1.9.	window
10	dad	20.	door

The children begin to appreciate how to use the pentagonal system as a tool in order to organize the words they write in groups of five within a Graph-of-20. By doing so, they subdivide the space of a piece of paper into a graph of 20 words, subdivided into groups of five. In this sense, the children experience the power of the number five. First of all, children must have an experience and, in this case, they experience the pentagonal system in a way that helps them organize the words they write. Slowly the children adapt and internalize the pentagonal system within the *Law of Prägnanz*[14] a principle of gestalt psychology:

> The Law of Prägnanz suggests that when people are presented with complex shapes or a set of ambiguous elements, their brains choose to interpret them in the easiest manner possible.
>
> Source: https://adamfard.com/blog/pragnanz-ux

Accordance to the *Law of Prägnanz*, there is a tendency to simplify the perceptual field into regular and symmetric patterns. In the GD *Ansatz*, it´s important that all stimuli are considered and regulated into the field of perception. As the children perceive and consciously manipulate the number of words they want to write, there is a greater sense of the whole, which is no longer just a set of words, but a graphic chart of words organized into patterns of five, which are eventually integrated into the decimal system.

The numbers may be organized across the space by learning how to fold the piece of paper in half, which allows the numbers to be written in groups of five as follows:

Folding a piece of paper into two halves in order to master the space within the paper

1	2	3	4	**5**		6	7	8	9	**10**
11	12	13	14	**15**		16	17	18	19	**20**
etc.										

In her chapter of Visual Routines[15] Jessica F. Shumway (2011) and Douglas Clements (1999)[16] write about the ability of children to recognize numbers in two ways. At first perceptually and eventually, as they gain experience, conceptually. In Shunway´s book (2011), she builds her number sense activities in groups of five by asking relevant questions:

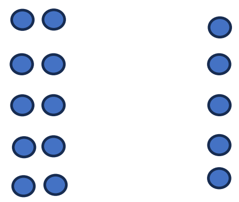
[14] Ali, N & Peebles, D. (2013) The effect of Gestalt laws of perceptual organization on the comprehension of three-variable bar and line graphs. Hum Factors. 55(1):183-203. doi:10.1177/0018720812452592

[15] Shumway, J. (2011). NUMBER SENSE ROUTINES: Building Numerical Literacy Every Day in Grades K-3. Stenhouse Publishers. ISBN 978-1-57110-790-9 www.stenhouse.com

[16] Clements, D. (1999). “Subitizing: What Is It? Why Teach It?” Teaching Children Mathematics. 5(7): 401.

How many dots do you see?

You probably did not need to count the dots by ones, but instead saw the amounts in groups. Did you see a group of five and a group of two, then combine them to make seven? Try this one:

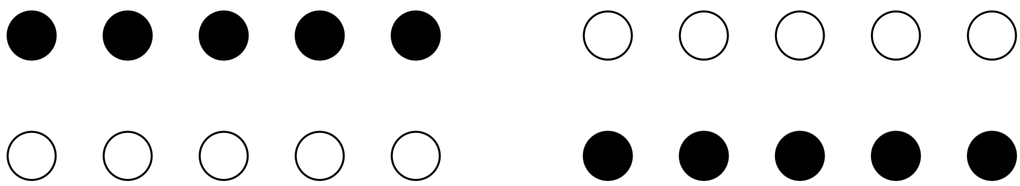


Maybe you say ten and five or three groups of five in this second illustration. Again, you probably did not count each dot one by one. You were able to recognize small amounts without counting—you were conceptually subitizing. Subitizing was something you were able to do early in your path to number sense.

Source: Shumway, J. (2011) page 33

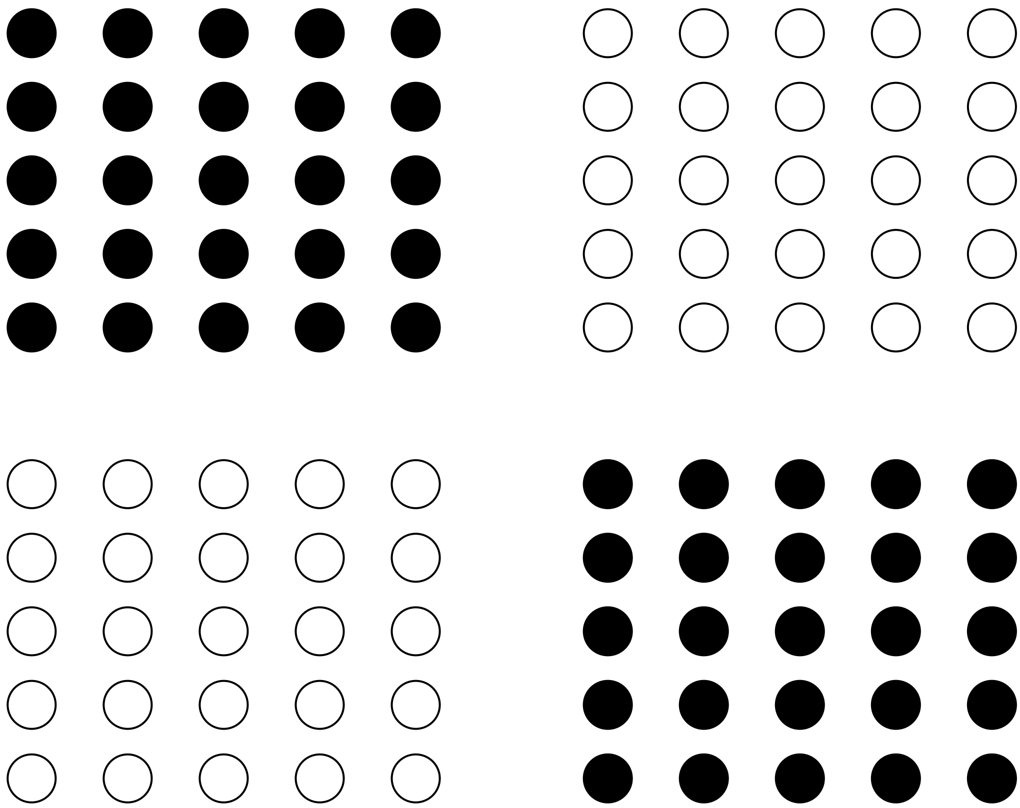
In other words, the power of the number five within a field of 20 items has been recognized by several researchers of number sense (Krauthausen 1995, Wittmann / Müller, 2000). What is new in the GD *Ansatz* is the integration of the space as background of the number configurations in which the power of the number 5 is represented with its color-effect as a contrast between white vs. Black, green vs red or yellow vs. blue. Here is an example of black vs. white within the 20-Field Configuration:

Color-Contrast Effect within a field of 20



The Field-of-20 should be taught in order to internalize the *Law of Prägnanz* so that regularities and symmetry may be experienced. What´s unique in the GD Ansatz is the integration of the space inside and outside of any number configuration as the children master the GD structures. By folding the paper in two halves, children become aware of the importance of space relations. Here is an example with the Field-of-100 configuration:

Field-of-100 configuration



This 100-field is very practical in its space configuration as it allows space to be an integral aspect of the learning process. In this case, a piece of paper is subdivided into four quarters and each quarter is reconfigured with its corresponding groups of five. This perceptual field is valuable when children are dealing with addition, subtraction, multiplication and division because it allows the power of the number five to be used as a practical tool. This field allows the children to build a foundation in order to reach the next GD-Structure, the of the field of 50.

The objective of GD is to assimilate the experiences from one GD-Structure in order to reach the next one, in this case from the field of 20 to the field of 50, the so-called Fifth GD-Structure, taking into consideration the Zone of Proximal Development according to Vygotsky. As the children master the field of 20, then the question is, what´s next? That will be the field of 50, which can then be analyzed in terms of the decimal system as opposed to the pentagonal system. As the children reach the Fifth GD-Structure, they no longer need the power of the number five as they enter the domain of the decimal system.

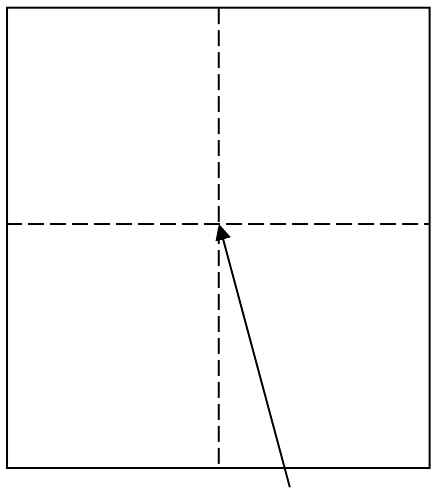
However, in order to reach the Fifth GD-Structure, it´s imperative to master the Fourth GD-Structure not as a perceptual field, but as a conceptual experience. This means that the children must be able to perceive and conceptualize all number up to 20 forwards and backwards. If we show children 8 items such as these, *****   *** and they´re still counting them one by one, or as one group of five, plus six, seven and eight as opposed to 8 as a whole number, then it means that they have not yet master the Third GD-Structure, that of a field of 10. These children are still lingering within the Second GDS, that of the number up to 5 in a *semi a priori* domain. Up to the numbers 1, 2 and 3, children experience them *a priori*, that is, automatically and without the need of an experience because they are born with that ability. In order for the children to master the numbers one to five, they may need actual experiences, but because they are easily acquired with few experiences, they may master them in a ***semi a priori fashion*** vs. the first three numbers (one-two-tree), which takes place ***a priori***, that is, without any experience.

In this sense, the First GD-Structure, that of the numbers one, two, three, is ***a priori*** and the second one, that of the numbers up to five, is ***semi a priori***.

At any rate, the teacher must determine the activities she wants to teach in order to help the children reach the next GDS. If the children are able to conceptualize the numbers up to five simultaneously, as a whole, then they are ready to for the next GDS, that is the numbers up to number 10 and likewise, if they can grasp it as a whole and be able to count the numbers forwards and backwards, then they are ready to conquer the next GDS, that of the field of 20 (the numbers 1 to 20) forwards and backwards and simultaneously as a whole.

What are the activities that can help children to master the field of 20? Such activities are based on different domains and depend on the creativity of the teacher. For example, as the children fold the paper twice the objective is to perceive the center out of which several graphic configurations can be designed:

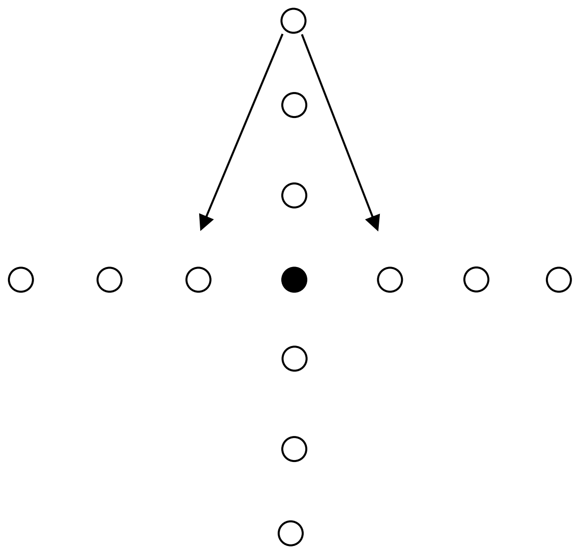
Folding a piece of pater twice in order to determine its center



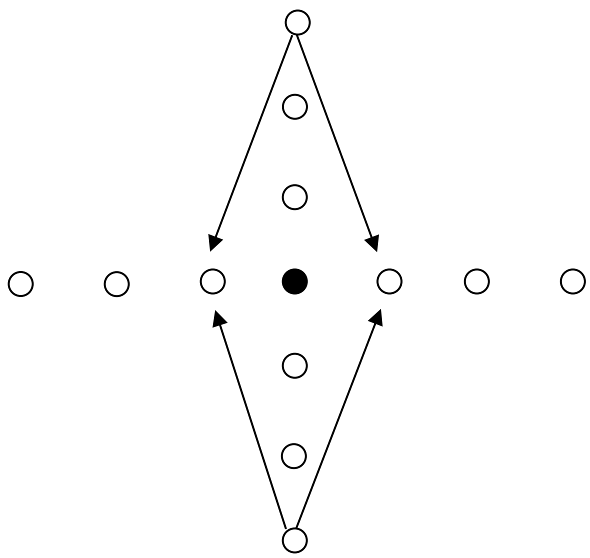
From the center a symmetric design can be illustrated so that dots are configured within a particular distance from the center:

A) First the piece of paper is folded in half twice, half from its length and half from its width as shown above.
B) A zero, a dot or a small circle is placed in the center.
C) In equal distances three dots are placed symmetrically above, below, left and right from the center.
D) From the third vertical dot above the center a line segment is drawn to the first dot left and right from the center on the x axis.
E) Then the same process is repeated, but this time from the third dot below the center onto the first dot left and right from the center on the x axis.

Symmetry I

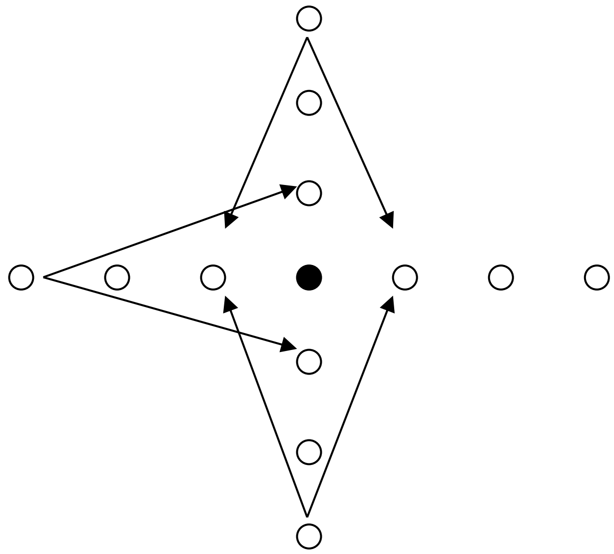


Symmetry II

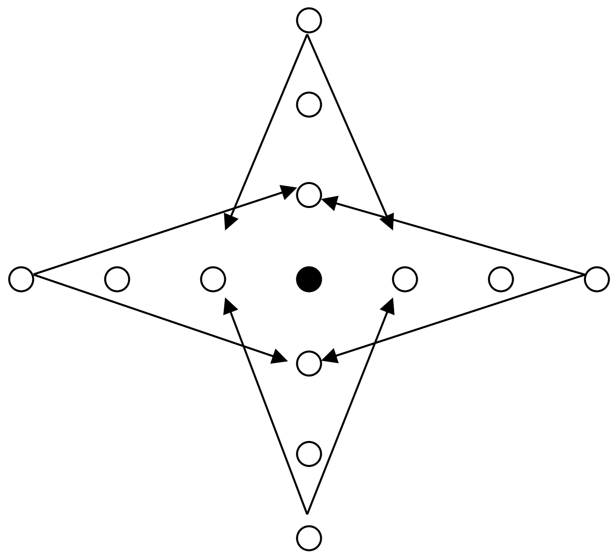


The next step is to repeat the same process, but this time from the x axis to the y axis, from the third dot, left and right, from the center up and down one dot from the y axis:

Symmetry III

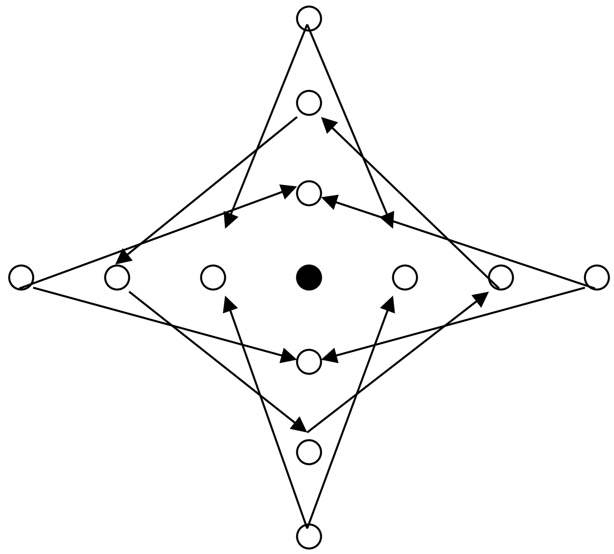


Symmetry IV



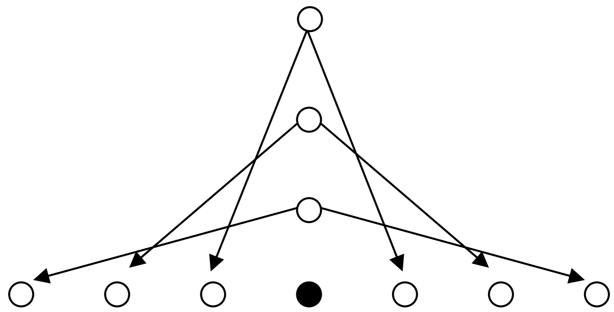
Then we follow the same logic with the other four line segments, connecting the second dots on the x and y axis:

Symmetry V

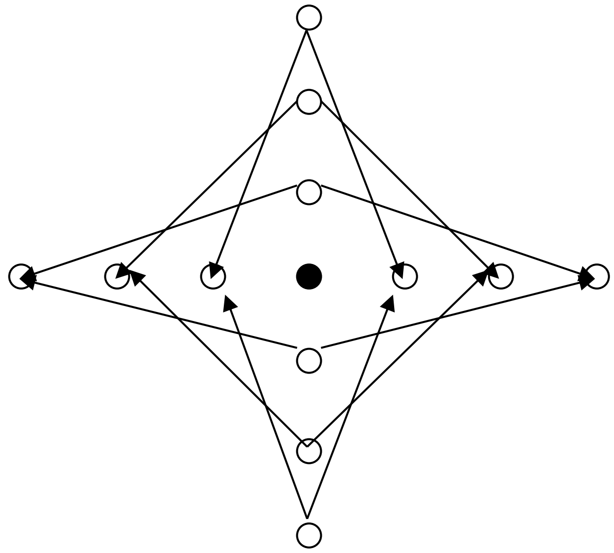


Afterwards, a line segments connects the first dot above and below the y axis with the third dot of the x axis as follows:

Symmetry VI

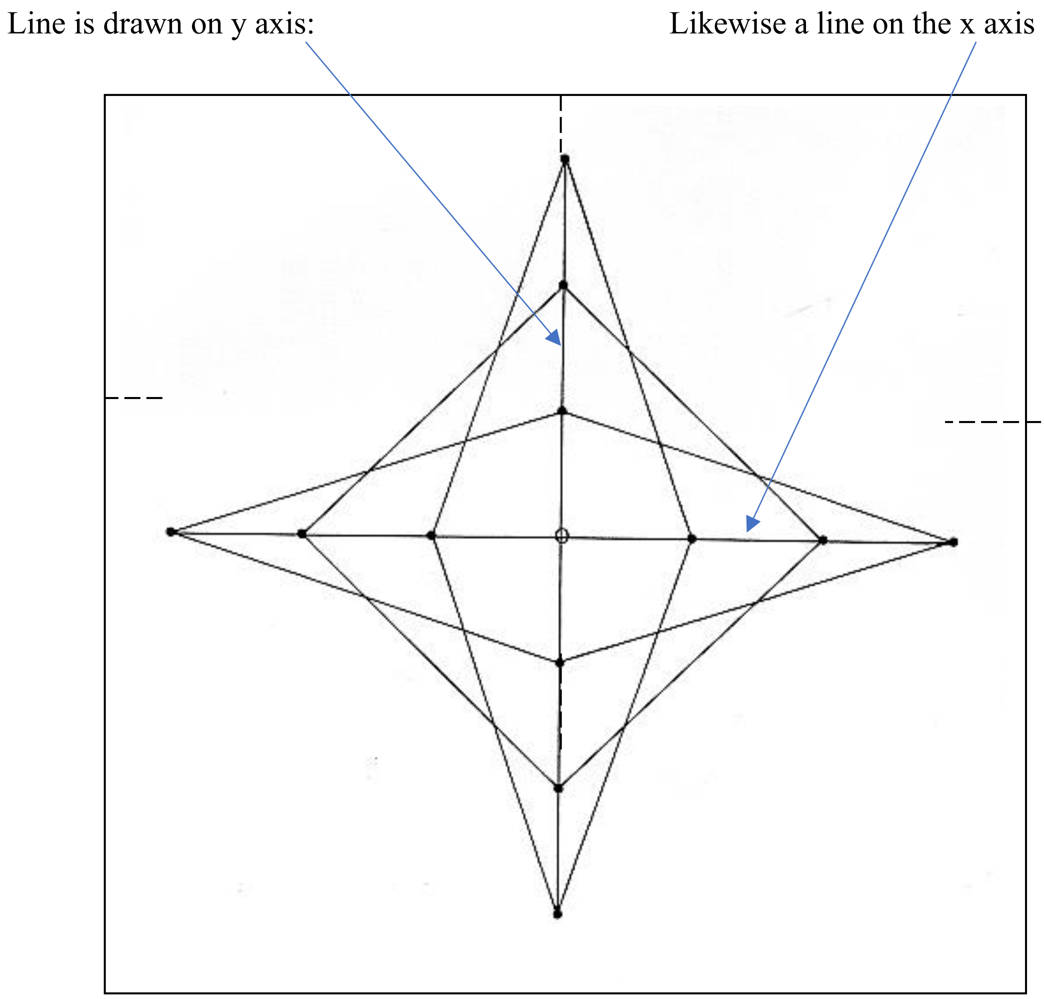


Symmetry VII



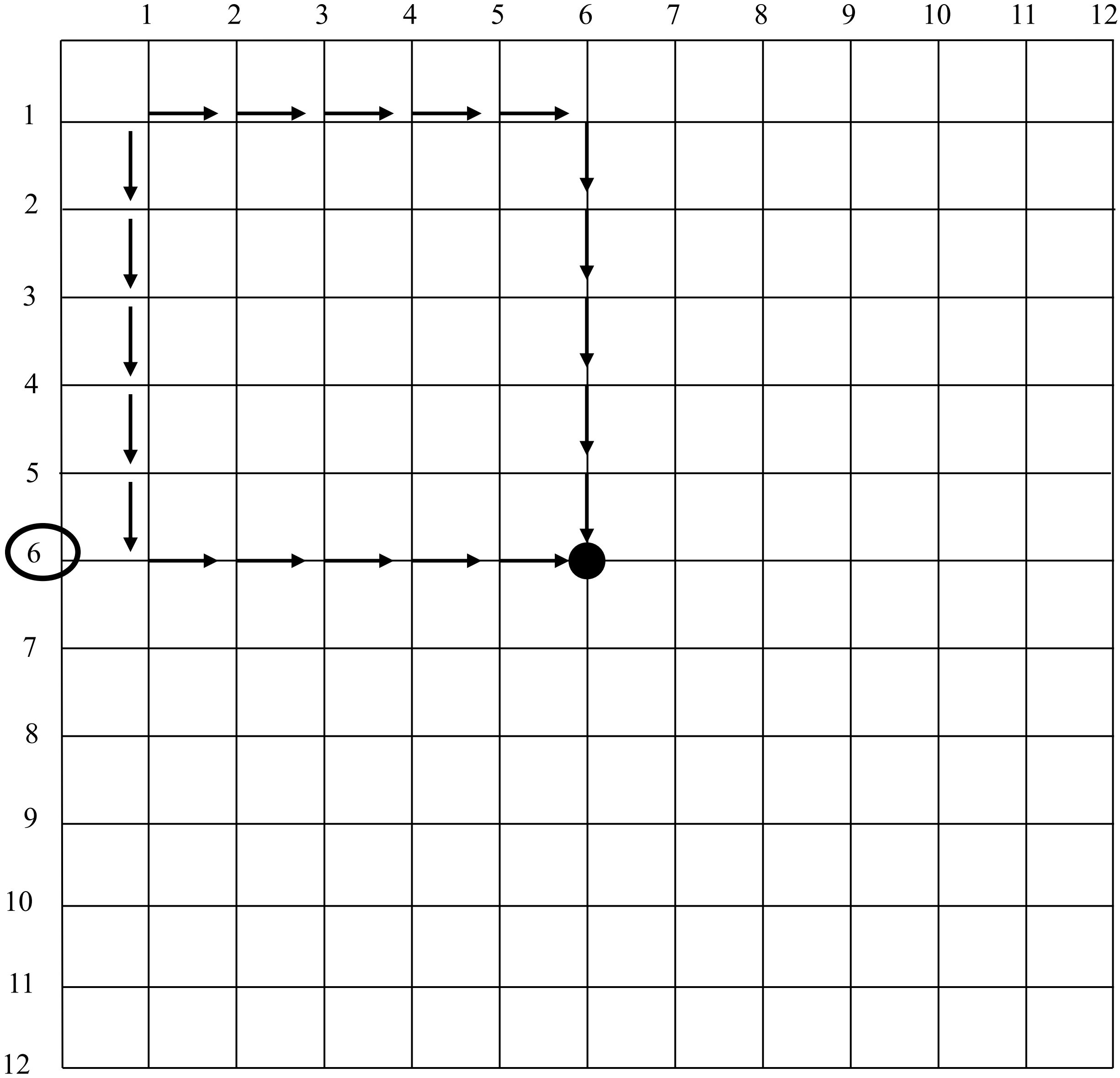
In the final step, a line segment is drawn on the y and x axis, thereby creating a star-looking figure as follows:

Symmetry VIII



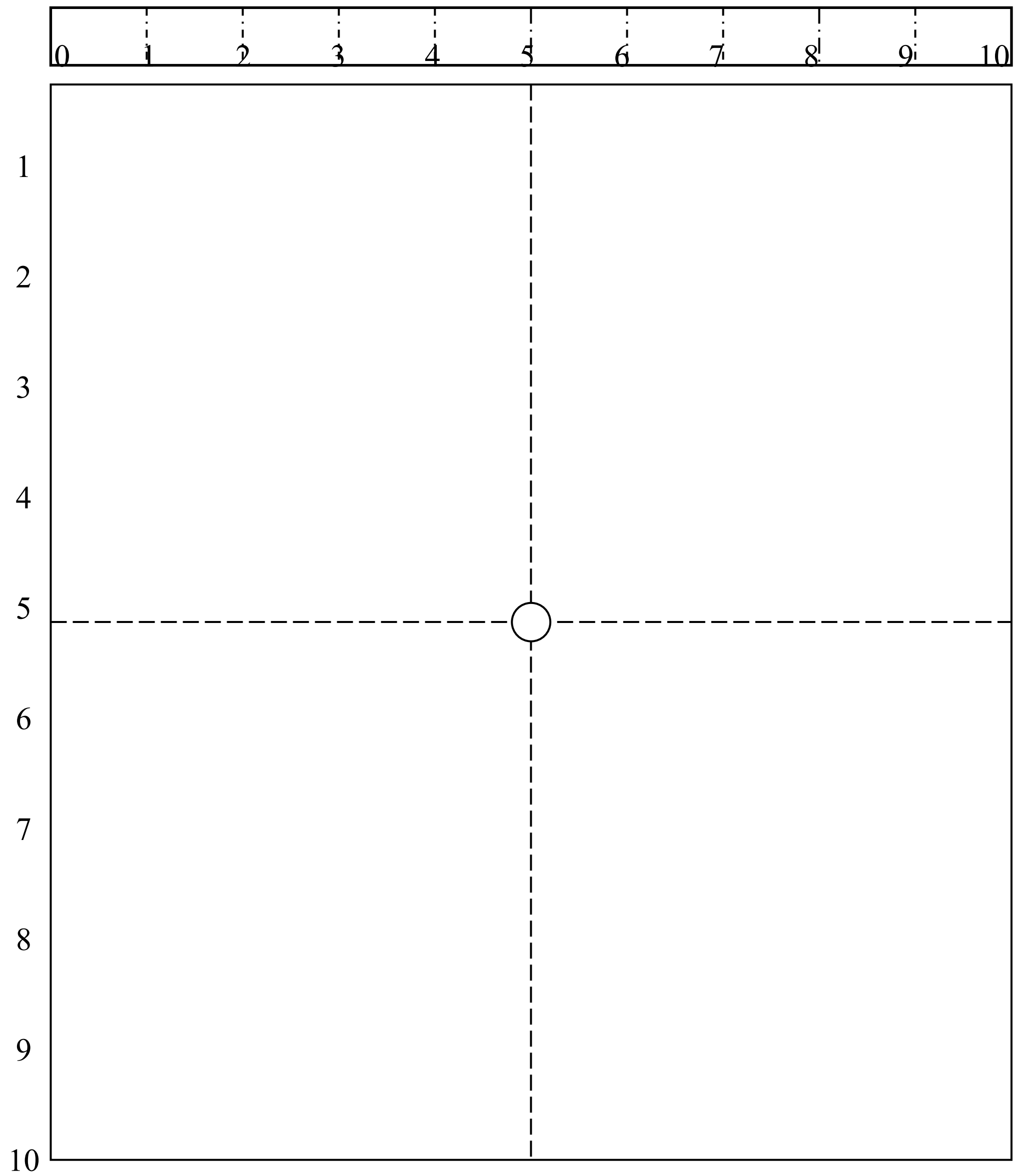
If grid paper is available, then the children instead of folding the paper twice, they may determine where the center is by counting the lines. A child could for example count six line segments downwards and six horizontally in order to find the center. The objective is to locate the sixth line on the x and y axis and how they intersect in the center of the grid as follows:

Grid paper



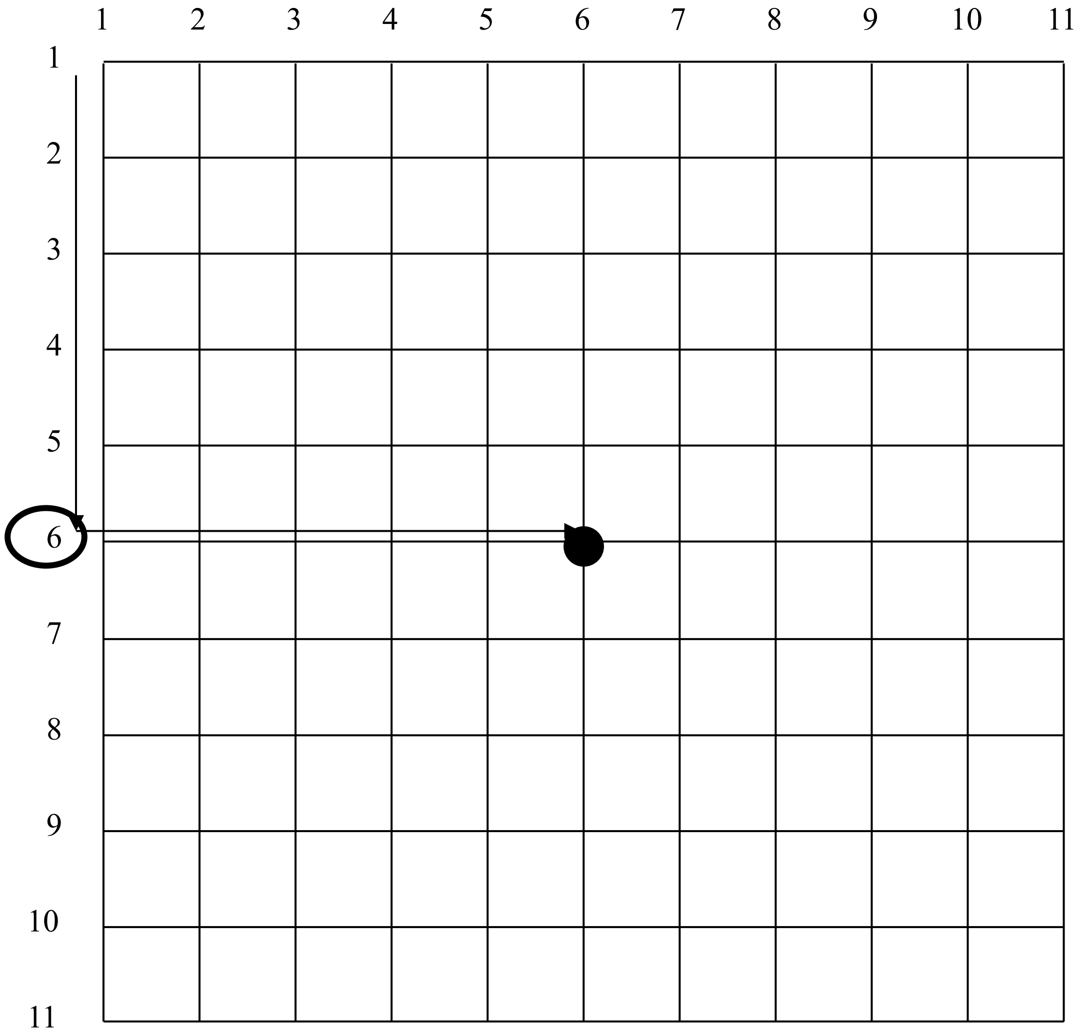
Afterwards, the objective is for the children to be able to find the center with a rule. They will need plenty of practice to determine where the y and x axis intersect in the center of a piece of paper. This challenge requires a lot of concentration and patience, but once they succeed, they surely feel a lot of satisfaction and motivation, which is needed in order to master more challenging graphic designs with the ruler. And as they do, they no longer depend on folding the paper twice or on using grid paper. All they need from now on, is a blank piece of paper, which they can manipulate in order to make any design that they so desire.

The ruler



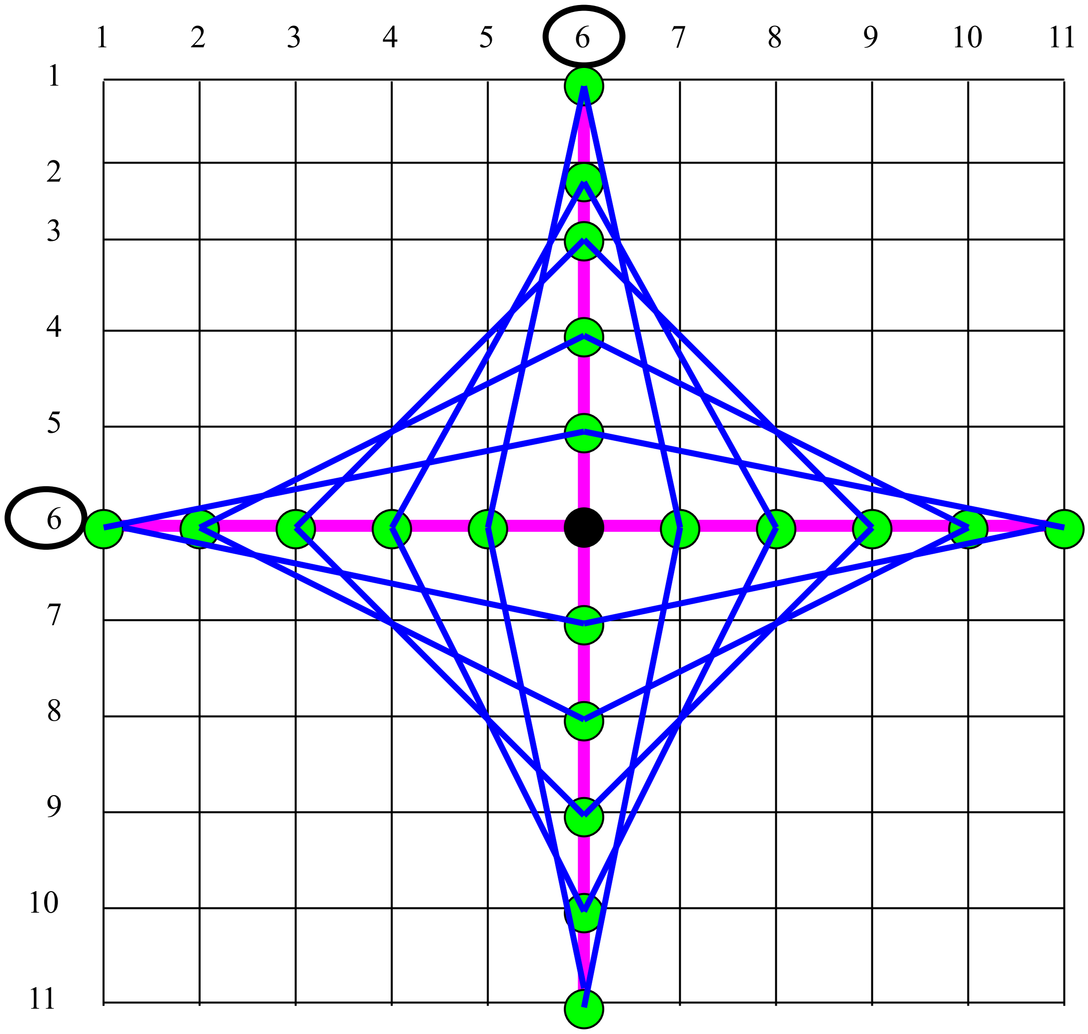
Another variation would be for the children to design their own grid on a blank piece of paper. As they number the line segments, one of the objectives would be to determine the center as follows:

Design of a self-created grid and determining its center



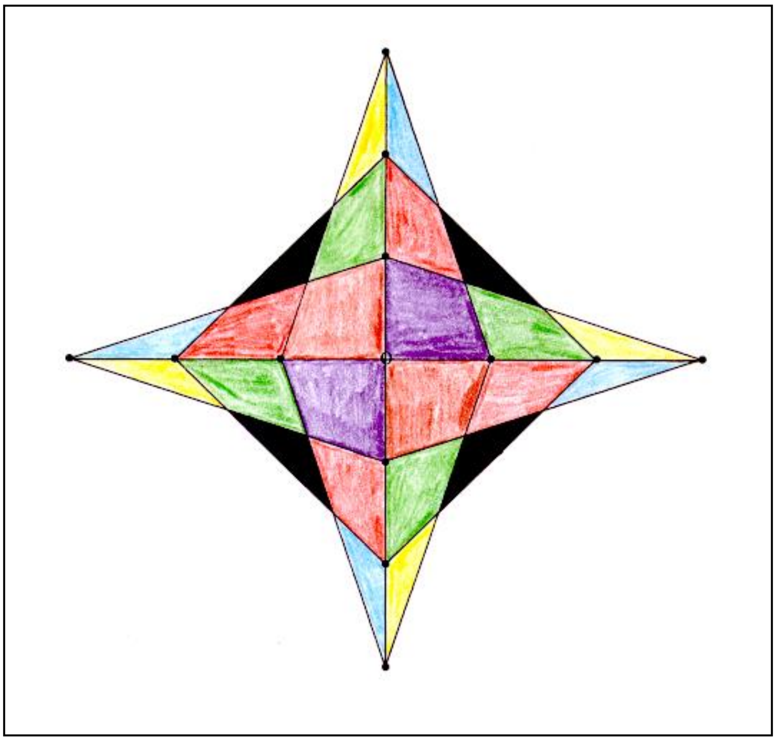
Once they´ve mastered the use of the ruler as to how to find the center, most children are ready to illustrate a rather complex star as follows:

Star as a graphic design on a grid



The ability to design a star based on five dots or more implies a great degree of precision. By the end of second grade, several children should be able to design a star with a least five dots from the center. However, most children, even in first grade should be able to master a star with three dots from the center as follows:

Colorful Star



The star itself is a design with patterns, dealing not only with line segments, but also with contrasting colors. Blue and yellow are placed next to each other and so are red and green. For each of the four tips of the star, the children can choose either blue or yellow. If they were to choose say yellow on the left side, then it´s yellow on the left side of each of the four tips and the space on the right side must be colored in blue. Thus, on each of the four tips yellow is placed next to blue. Then we ask the children to choose either red or green on the space below yellow. If they choose green, then red must be on the adjacent right side because red is contrasted with green. On the four outside spaces, diagonal to the center, they must choose black. For all other spaces, the children can choose whatever color they have not used. However, the spaces diagonal to the center are colored with the same color.

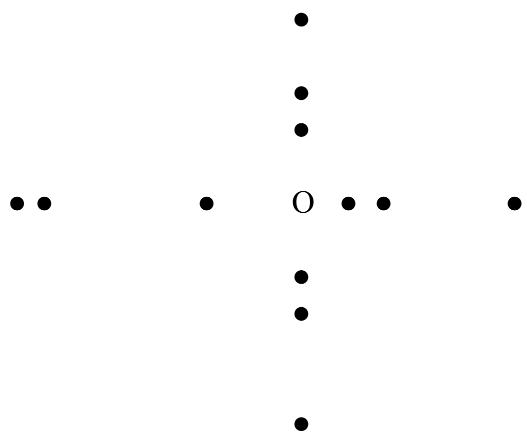
The objective is not to color the star with any colors, but with the four contrasting colors on each of the four tips of the star: yellow vs blue and green vs red, and with black on those four outside spaces diagonal to the center. Around the four diagonal spaces adjacent to the center, the same color is chosen, but that color cannot be a repeated color; thus, it cannot be yellow, blue, red, green or black.

For the color constellation there is a precise pattern of colors that children need to follow. **In essence, the teacher is the one making most of the decisions and the children just follow the directions.** At the beginning of the GD pattern, that´s how it works. The teacher is the one guiding the children in discovering the line and color patterns, **but slowly the teacher will give them more and more freedom to design their own graphic patterns with whatever colors they prefer:**

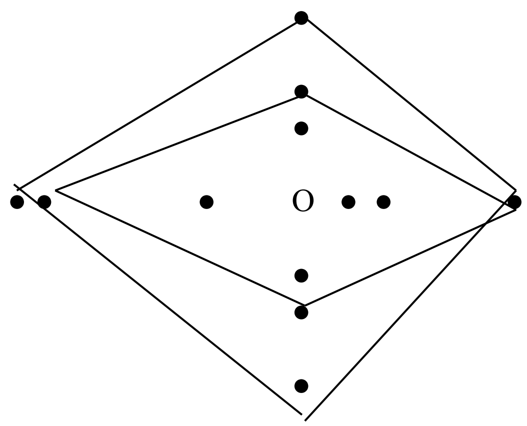
Teacher: Children, what color do we choose on the upper left side of the star?
Children: Brown!
Teacher: I recommend red, green, yellow or blue, and I highly recommend yellow. If we color the space yellow, what color should be on the right side?
Children: Red.
Teacher: If we choose yellow on the left side, then blue must be on the right side because blue is the contrasting color to yellow.

At the beginning, the children are instructed to follow a precise color constellation in which yellow is place next to blue, red next to green, but eventually, the children are given more and more freedom to make their own decisions. They must also be able to place the dots equidistant to the center, where the x and y axis intersect and that is not easy. At the beginning, the dot constellation may look very asymmetrical. They may place the dots too close or too far from each other:

Asymmetrical arrangement of dots

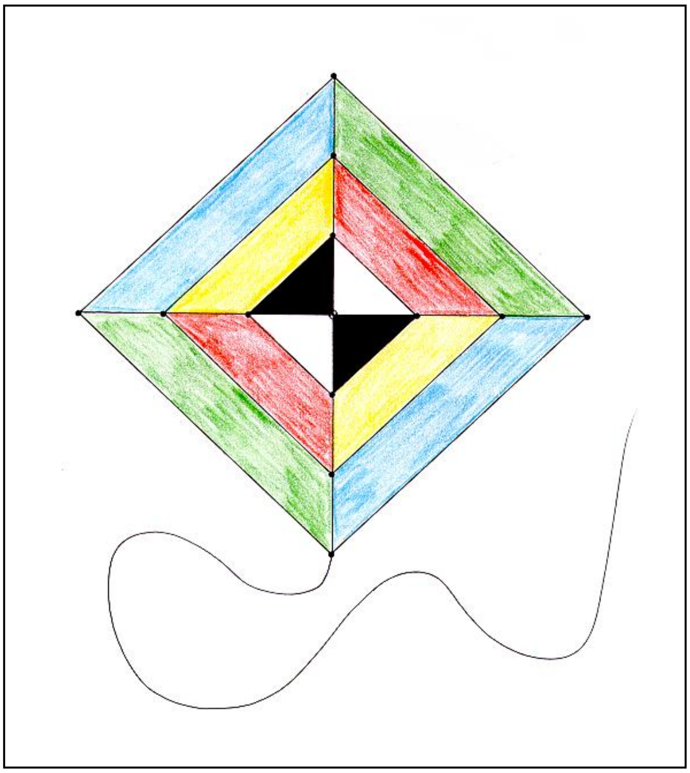


Many children also have problems with the line segments. For example, instead of making a line from the third on the y axis to the first dot left and right of the centers in the x axis, they connect it with the third dot left and right:



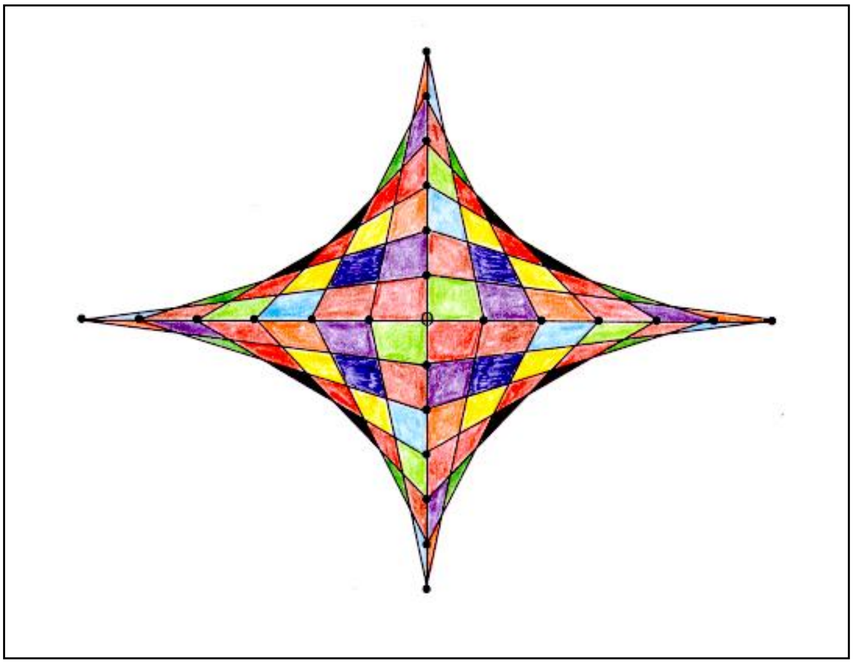
The majority of first grade children will need a lot of help in order to create acceptable symmetric line patterns with this lesson. Eventually, as we create more graphic patterns, better and better symmetries are created such as that of a kite:

The kite as variation of the star



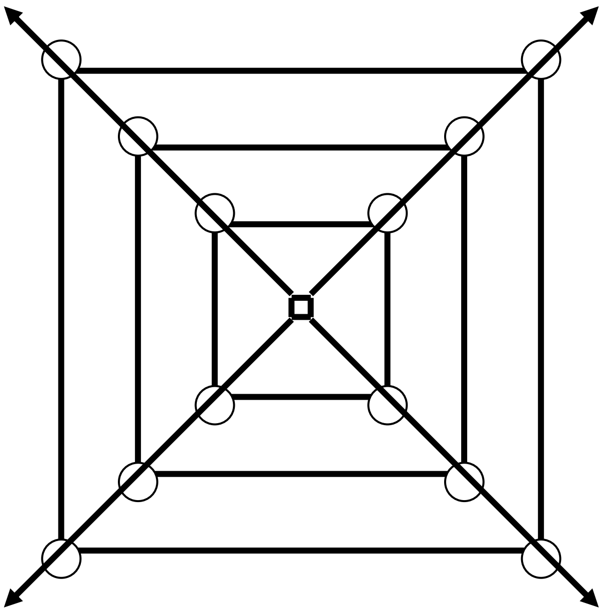
One can object, that the teacher controls everything, that they´re only following instructions, imitating the designs as the teacher wants them to be. These objections are valid. The GD Ansatz allows for the teacher to be in total control of the instructions, but through imitation, the children eventually discover their own designs, their own pathways. However, an advantage is that as the children master the specific patterns as instructed, they can also work on their own pace, according to their own understanding. The children are learning at their own individualized ability. Slowly, the patterns allow them to master very challenging designs as per the ability of each child because they discover that the pattern could be endless. The children begin to discover that the sky is the limit. The more advance children may even design a star with six dots from the center, which requires a lot of precision and ability and if they can do a star with six dots, then they know of the possibility of a star with 6, 7 or 8 dots.

A complex star with its patterns of lines and colors for the more advance children



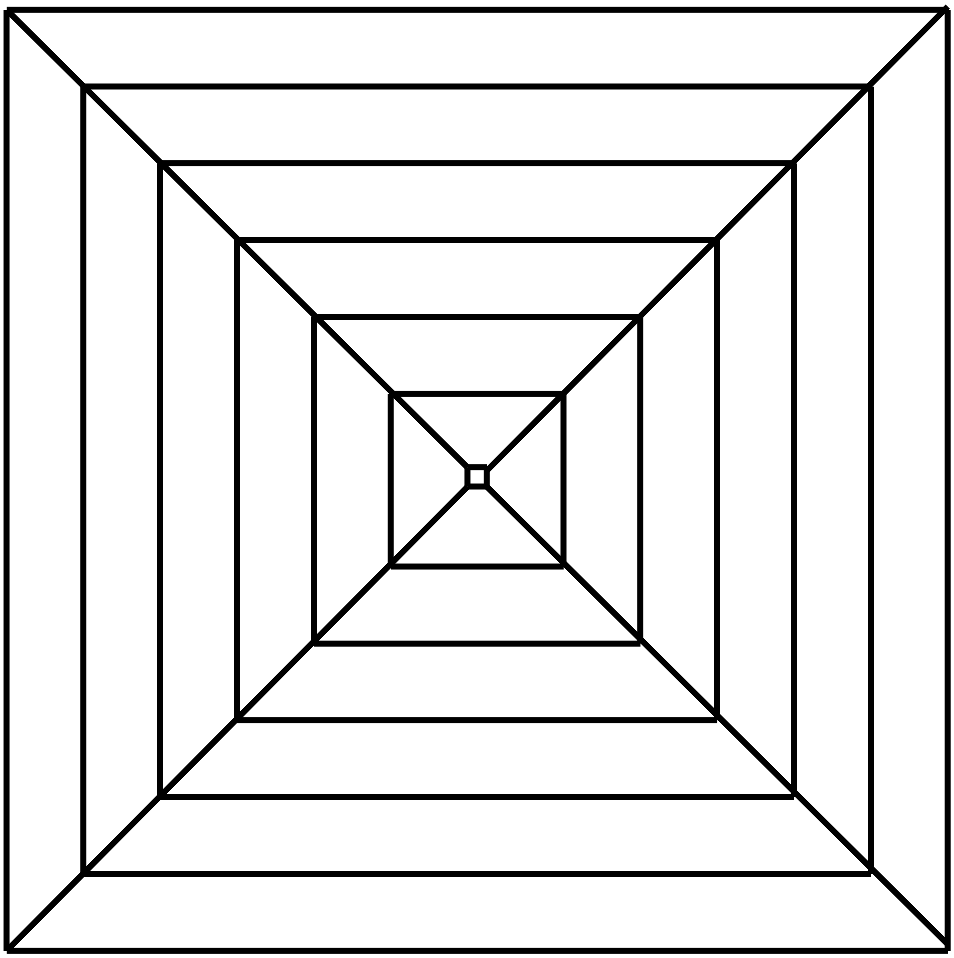
As for the gifted children, there are not limits as to the complexity of the patterns, not just with the graphic star and kite, but also with other designs such as that of a pyramid. A limitless continuum with several gestalt qualities is being created, starting with very simple patterns, leading up to more and more complex ones. With the design of a pyramid the children are exposed to endless possibilities, which they begin to discover. Just like with the star, the children are taught how to create a pyramid with three dots, but eventually the design becomes more complex as the children master a greater number of equidistant dots from the center of the pyramid:

The three-dot pyramid



A gifted child would be able to increase the complexity of the pyramid up to six dots, diagonally from the center of the paper:

The more complex pyramid

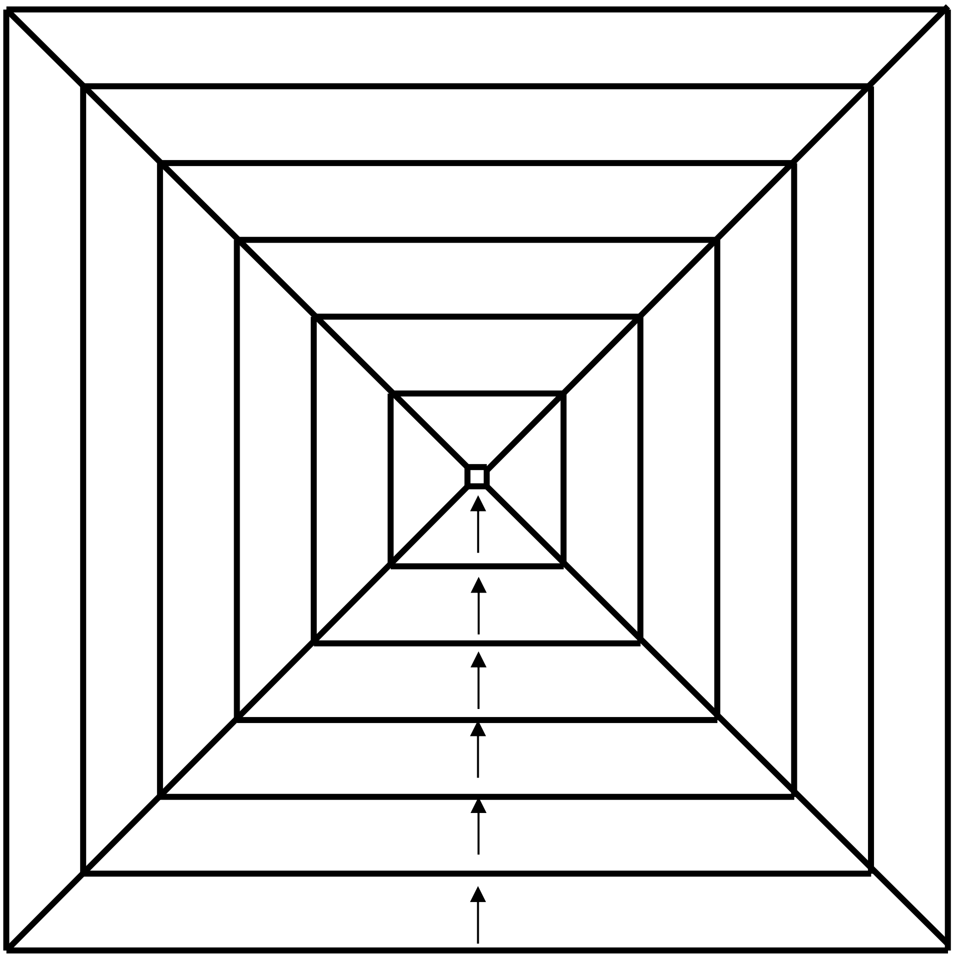


If the line segments are precise, the children will be able to interpret the design either as a pyramid or tunnel. With certain designs, the observer can switch the figure for the background and the background with the figure. These figure-ground relations are known in gestalt psychology:

> The figure-ground concept is a cognitive function that allows a person to direct their attention to a figure rather than its background. For example, black font grabs our attention when placed on a white page, and we pay little attention to the texture of the white paper. But white words on a black page, with the same two colors, achieve the opposite effect.
>
> **Why is the figure-ground relationship important?** The figure-ground relationship is an important cognitive function, and it is one of the first cognitive abilities to develop in young babies. The human brain uses color, shape, and motion cues to perceive an object of focus vs. background objects. This has important evolutionary benefits, including finding food, recognizing the faces of familiar humans, and even detecting threats such as snakes.
>
> **What is an example of figure-ground perception?** The Rubin vase faces (a drawing by Danish psychologist Edgar Rubin) is a classic example of figure-ground perception. The image depicts a black vase in between two profiles of white faces. Depending on the observer's focus, one can perceive a black vase on a white background or they can perceive two white faces on a black background.
>
> Source: **Figure-Ground Perception | Definition, Principles & Types**
> https://study.com/learn/lesson/figure-ground-perception-relationship-examples.html

For children it becomes an interesting phenomenon as they are able to perceive either a pyramid or a tunnel, depending on how precise the design might be. If they can perceive a tunnel, then they may wonder about the "light at the end of the tunnel". The children may be able to switch back and forth from a tunnel to a pyramid and be motivated to create their own designs.

The Tunnel



Here is the entrance to the center of the tunnel

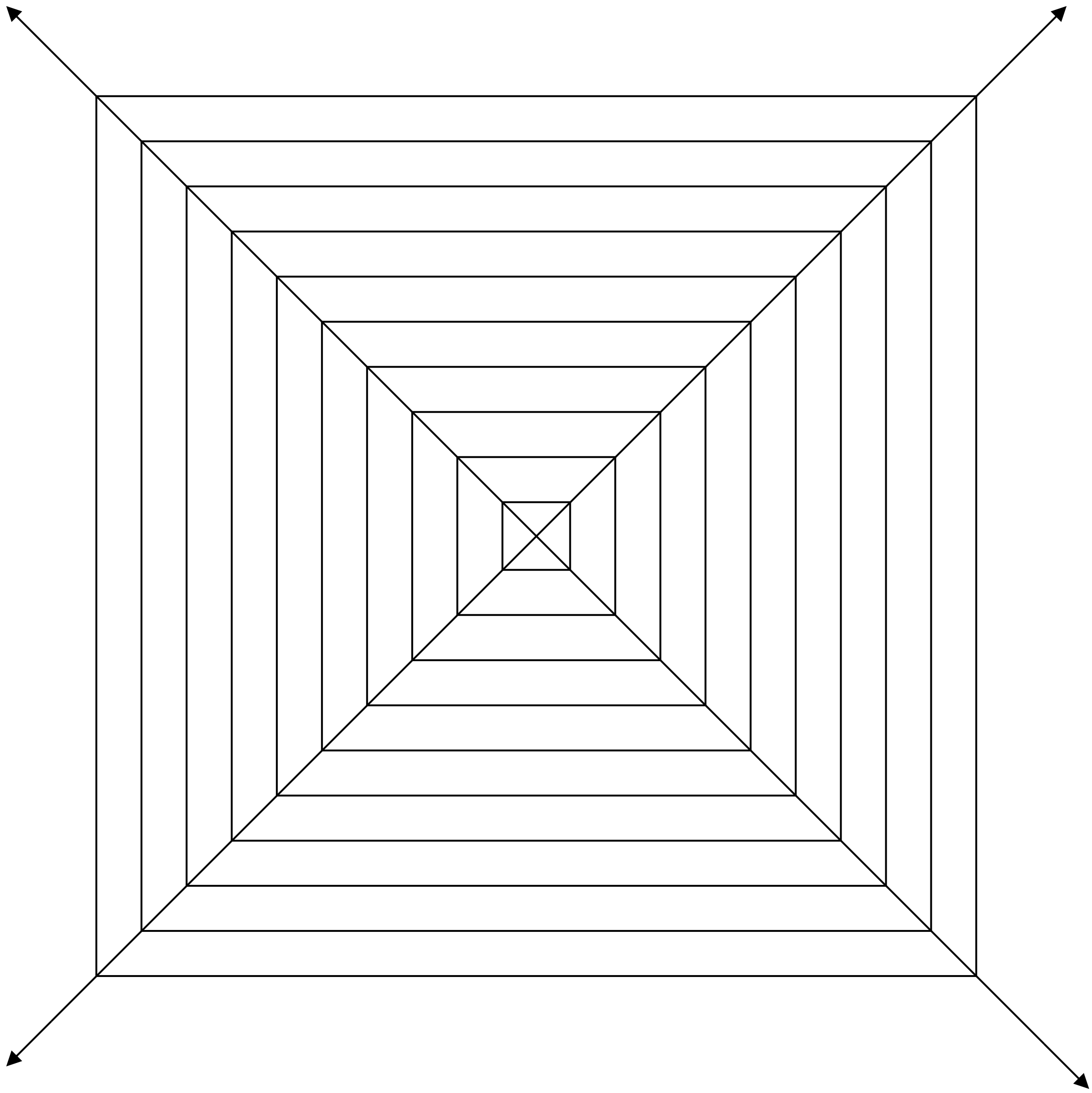
The children should be able to perceive not only *the light at the end of the tunnel*, but also the center as the peak of the pyramid as seen from above. Let´s suppose that children can illustrate a pyramid with six dots. This implies that they can measure the intervals in the space very carefully, all equidistant from the center. The best strategy would be to teach them how to draw squares around the center with each square increasing with equal distance to the center as the previous square.

Slowly the children realize that the pyramid design can become infinite, that, for example, they can add more and more steps. The infinite nature of the design is an essential principle of the GD Ansatz. It allows children to master the space of a piece of paper with increasing complexity ***ad infinitum***.

In essence, this *Ansatz* promotes the so-called spatial intelligence, which is "the ability to generate, retain, retrieve, and transform well-structured visual images" (Lohman 1996). According to Melissa Kelly (updated on May 30, 2019), "those with spatial intelligence have the ability to think in three-dimensions. They excel at mentally manipulating objects, enjoy drawing or art, like to design or build things, enjoy puzzles and excel at mazes" (https://www.thoughtco.com/spatial-intelligence-profile-8096). She recommends for children to practice visualizing techniques, including artwork, learning how to draw, do puzzles and give students step-by-step instructions, etc.

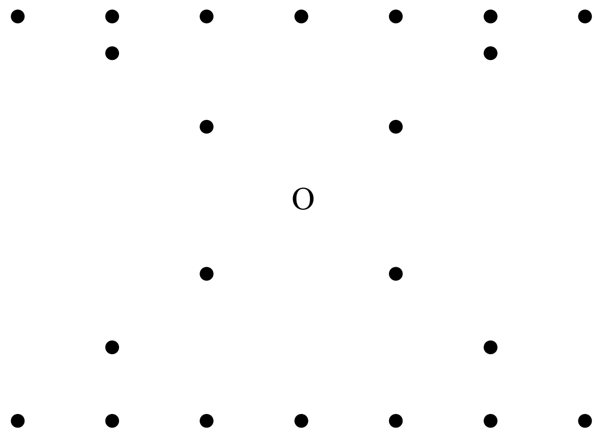
In the GD Ansatz, one of the objectives is to promote spatial intelligence through the mastery of space relations on a blank piece of paper. This type of spatial intelligence will promote individual creativity and concentration. The spatial designs will help them to discover infinite possibilities with the line segments, which will uplift their self-esteem. The graphic designs help the children to highly concentrate on the spatial details, thereby allowing them to discover their own patterns. The main factor is that the patterns can become infinite by design.

An expanding pyramid ad infinitum

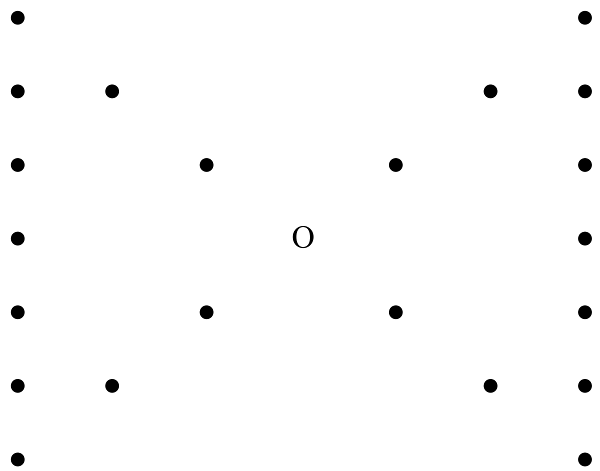


Because of their experiences with the color and line constellations with the star and the kite the children will eventually star creating their own graphic designs. Slowly they become independent from the teacher as they discover other line and dot designs such as that of a necktie or a sandtimer:

Sandtimer

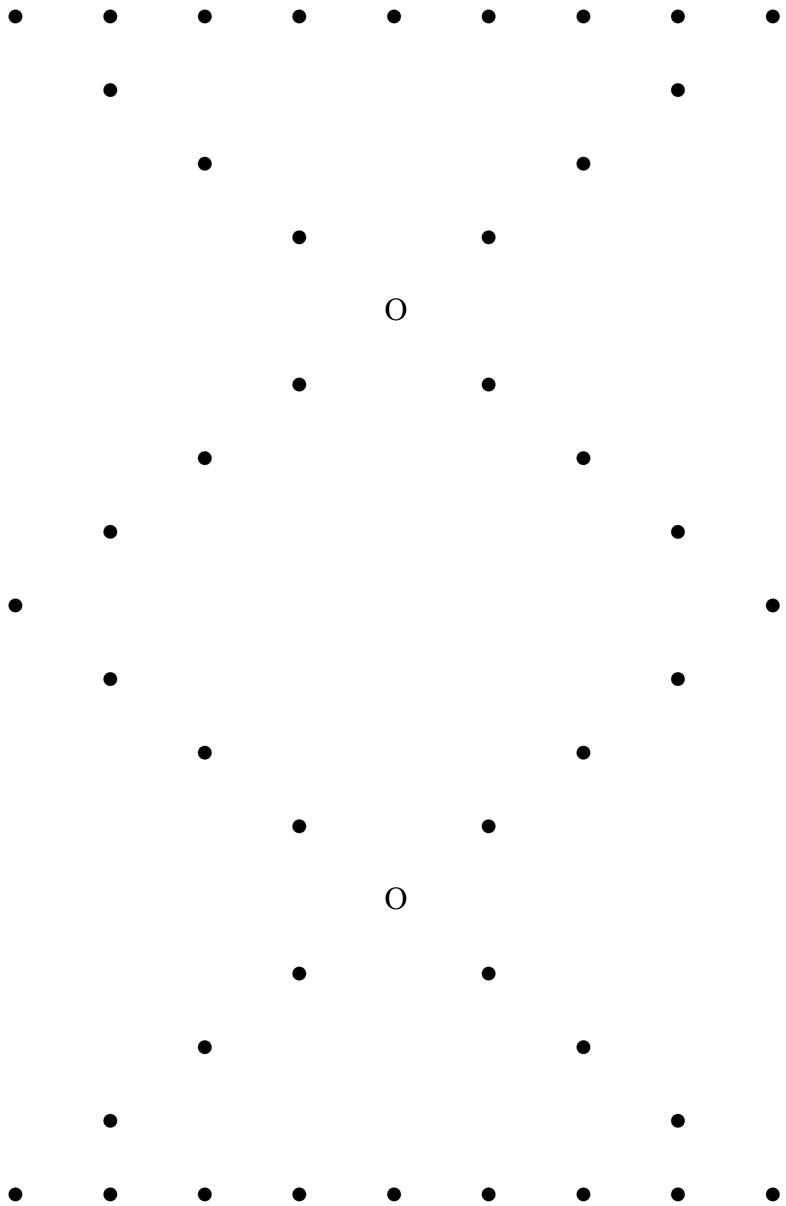


Necktie



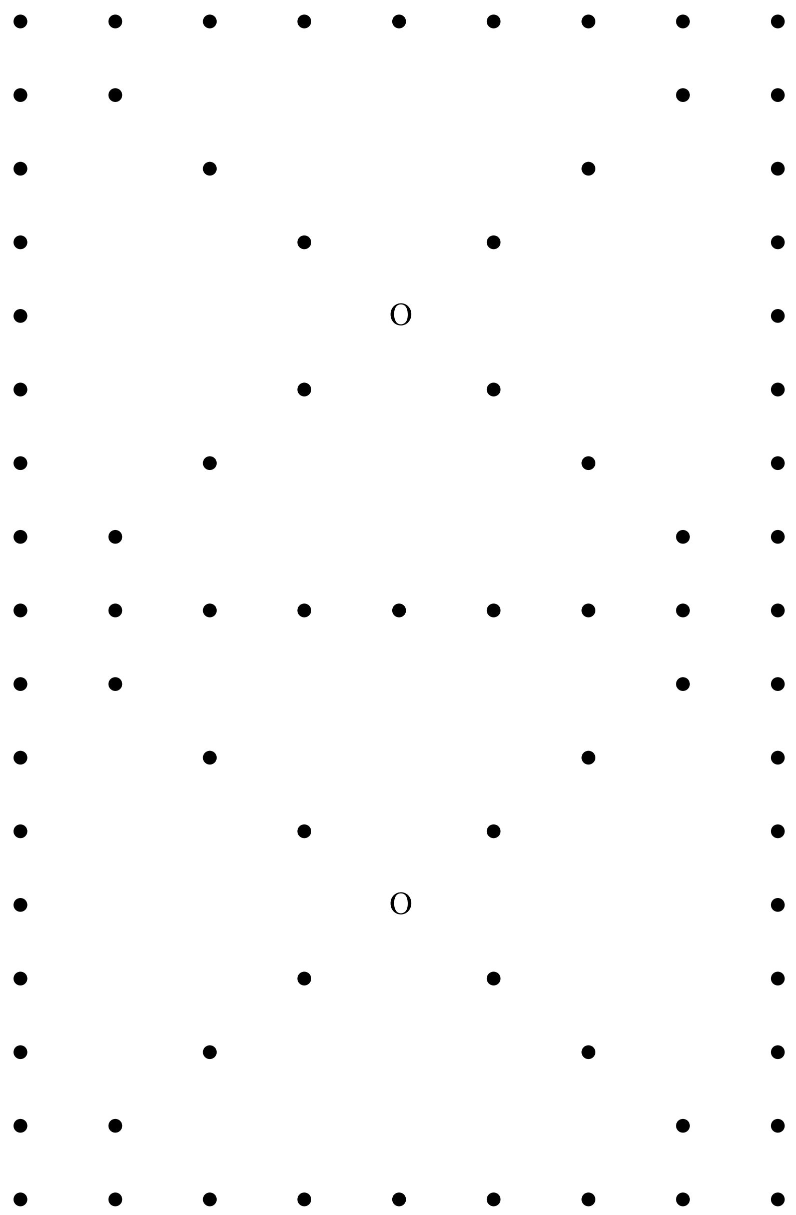
The possibilities become endless. For example, a type of zigzag configuration can be created by drawing a sandtimer on top of another sandtimer:

Zig-zag figure



The teacher can show them the visual effects of drawing two pyramids, one on top of the other one, so that the children can experience new creations based on those they have already experienced. In this case, the children can see, for example, either a vertical rectangle or 8 triangles, or they can also view the two pyramids, one on top of the other one:

Double Pyramid



Gradually the children decide what and how they want to draw and as they do, they start experimenting with their own line and color constellations. Number sense becomes an inner motive, a way to define certain patterns. With the help of geometric figures, the children eventually discover how to make curvilinear relationships between the straight lines, especially with the graphic design of stars. Children begin to experiment with graphic curvatures, how they can bend the line making more organic designs.

A summary of the graphic constellations

The dot constellations do not play a direct role in the pentagonal system. However, as the children master the spatial relations, they gradually experience the numbers with their corresponding geometric forms in a more aesthetic and creative way. Beginning with simple geo designs they gradually discover infinite patterns of lines and dots, manipulating them into very interesting and creative curvatures. In other words, out of the initial simple constellations, the children eventually discover infinite possibilities in geometric and artistic forms.

In essence, out of the center of a page emerges a linear movement into infinity, thereby allowing the children to master spatial relations according to their own individual experiences and potential. At first, the teacher is in control of the patterns as per the Zone of Proximal Development (ZPD) of each child, but gradually they become more and more independent, until they eventually discover their own designs in infinite ways. According to their ZPD, what´s important is not only their current development, but what they will be able to accomplish independently in the near future.

3.2 First formulations of the multiplication tables and their self-control mechanisms

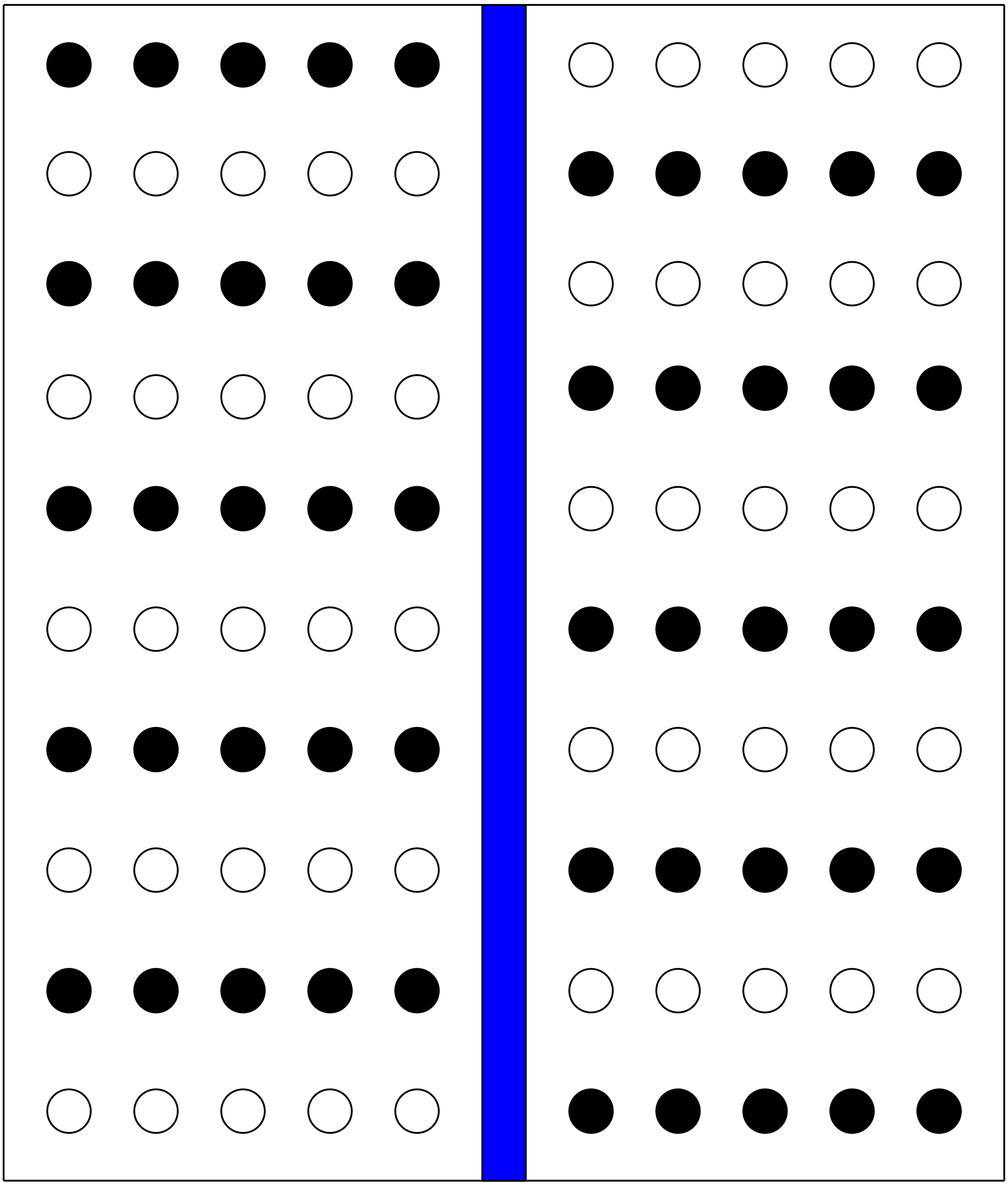
With the increasing mastery of the space, it will be possible to reach the sixth GD-Structure, that of the field of 100. The children are required to fold a piece of paper into two equal halves lengthwise and write the sequence in multiples of 10 up to 100 as follows:

The 1-to-100 Multiplication Table

1	2	3	4	**5**		6	7	8	9	**10**
11	12	13	14	**15**		16	17	18	19	**20**
21	22	23	24	**25**		26	27	28	29	**30**
31	32	33	34	**35**		36	37	38	39	**40**
41	42	43	44	**45**		46	47	48	49	**50**
51	52	53	54	**55**		56	57	58	59	**60**
61	62	62	64	**65**		66	67	68	69	**70**
71	72	73	74	**75**		76	77	78	79	**80**
81	82	83	84	**85**		86	87	88	89	**90**
91	92	93	94	95		96	97	98	99	**100**

Likewise, they can fill out the field with contrasting dots as per a zigzag formation, in this case white vs black in context of the pentagonal system, but it could also be blue vs yellow or green vs red.

Zig-zag constellation contrasting black vs white up to 100



The same pattern is used with the time table of two up to 200 in multiples of 10, whereby the corresponding operations of the addition and subtractions become apparent. An essential factor is the reversibility of the numbers. We start with 1 to 10 forwards and backwards and once they master it in about 5 to 6 seconds, the children start practicing 2 to 20, 3 to 30, 4 to 40, 5 to 50, etc. until they can do them in seconds:

T__ First forwards: One, two, three, four, FIVE, ___ six, seven, eight, nine, TEN
And then, backwards: Ten, nine, eight, seven, six, ___ five, four, three, two, one, Zero.

The objective is to recites the numbers very fast. For 1 to 10 and backwards, children should be able to recite them in about 5 seconds with a lot of practice, which may take days to accomplish. Likewise with 2 to 20 forwards and backwards, thereby integrating certain physical movements, as if they were frogs or kangaroos, jumping with both feet at the same time, forwards up to 10, pausing a little bit, and from there up to 20. From 20, the children immediately jump backwards to 10 and from there to zero, kangaroo style.

T___ Well children, lets jump like a kangaroo reciting the numbers two-by-two up to 10. When we get to 10, we jump higher and say TEN louder and make a small pause. After that, we recite the numbers up to 20 and backwards back to 10 and zero. Ready:

Two, four, six, eight, TEN twelve, fourteen, sixteen, eighteen, TWENTY
eighteen, sixteen, fourteen, twelve, TEN eight, six, four, two, ZERO!

What´s challenging is not counting the numbers two-by-two up to twenty, but counting them backwards. Once they get to 20, most children have the tendency of saying twenty-two afterwards, instead of eighteen. What´s practical is to introduce a graphic representation of the sequence as follows:

T___ Here children we have 20 boxes.

■■ ■■ ■■ ■■ ■■ ■■ ■■ ■■ ■■ ■■
2 4 6 8 10 12 14 16 18 20

What number do we see before 20?

Another strategy is to recite the odd numbers with a very low voices, almost whispering, and the even numbers very loud in order to train the sequence in a memorable way:

T___ Children, we recite the odd numbers with a low voice and the even ones very loud like this:

one-TWO, three-FOUR, five-SIX, seven-EIGHT, nine-TEN … (Pause)
eleven-TWELVE, thirteen-FOURTEEN, fifteen-SIXTEEN, seventeen-EIGHTEEN [nineteen-TWENTY]

(Pause)

nineteen-EIGHTEEN, seventeen-SIXTEEN, fifteen-FOURTEEN, thirteen-TWELVE, eleven-TEN, (Pause) nine-EIGHT, seven-SIX, five-FOUR, three-TWO, ZERO.

With this spoken-word strategy the corresponding sequence will have a memorable effect, allowing the children to internalize the numbers with more ease. The objective is to generate a contrasting effect so that the even numbers become the focus, and the odd numbers the background. The strategy becomes a mental interplay of add-vs-even numbers, whereby we ask the children to only recite the even numbers of the sequence forwards and backwards against a time trajectory. At first, the children may be able to recite the sequence in 40 seconds, but gradually, with practice, they recite it faster and faster so that in a couple of weeks, they can do it in about 5 to 6 seconds. The strategy is to mentally supersede or displace the odd numbers, leaving them aside, so that the children only recite the even numbers against a time schedule. As the contrasting effect become internalized, the odd numbers become the background in a very unique mental ability in order to interpret the time tables in a rather dynamic and memorable fashion.

In essence, the objective is for the children to generate the numbers 2 to 20 forwards and backwards as fast as possible so that in due time, they can recite them in 5 to 6 seconds. Reciting them forwards allows the children to relate the multiplication with the addition and by reversing them the children gain insights in their ability to divide and subtract them in a clear and logical pathway. It will take a lot of practice and patience in order to recite the numbers forwards and backwards in about 5 to 6 seconds:

T___ Ok children, when we say *two, four, six, eight, ten (pause), twelve, fourteen, sixteen, eighteen and twenty*, what number do we have before twenty? Do we have 22?
C___ No, we have 18!
L___ And before 18?
C___ Then 16!
T___ And then?
C___ Then 14, 12, 10 (Pause), 8, 6, 4, 2 and ZERO!
T___ Who can recite the number 2 to 20 and back to zero?
C___ I!
T___ Go ahead please!
C___ 2,4,6,8,10, (Pause), 12,14,16,18,20 (Pause), 20, 18,16,14,12,10 (Pause), 8,6,4,2, ZERO!
T___ BRAVO. EXCELLENT Jasmin! That´s a great accomplishment, but don´t tell your parents[17] how well you did the numbers, 2 to 20 and backwards because they would be too surprised from your intelligence. And besides, they may not even believe you.
C___ What? I´ll certainly let them know how well I did!
T___ You recited the numbers in 40 seconds. You see, I measured them with my iPhone. They were exactly 40 seconds. Now, who else would like to recite the numbers up to 20, two-by-two, and backwards back to zero?

On the other hand, it would be better if the children had a clock with a second hand. That way, the children would wait until the second hand comes around and when it passes by the 12, then a volunteer child would start counting the numbers 2 to 20 and back to 2 or zero as fast as possible. We would use the second hand to find out in how many seconds they did it, instead of using an iPhone.

By going forwards and backwards, we´re not only dealing with number sense, but also with a time continuum in which time and number sense merge with one another. Hopefully this will help a child to perceive time in a very concrete way. It all begins with a general feeling of what a day and a night is, that there is a sunrise and sunset, that time is marked in days and nights. But, how do seconds and minutes as concepts become a psychological reality to them? The best way I believe is integrate activities that can be measured in seconds or minutes such as counting numbers up to 20 and backwards or finding out how long it takes for an egg to boil. In this case, we want to find out how long does child x can count the numbers 2 to 20 and backwards and for that we can wait until the second hand of a wall clock passes by the 12. The child would start counting the numbers as soon as the second hand passes by the 12 on a wall clock and as soon as the child counts the numbers back to zero, we make a notation as to where the second hand is on the clock.

Slowly they come to an awakening in their consciousness of what time means, not just from the perspective of a day and night, but of what hours, minutes and seconds represent. They now become aware that seconds, as measured by the second hand, can even help them master the number sequences forwards and backwards in a time continuum. The apperception of time in the GD *Ansatz* is an important factor in mastering the time tables. On a daily basis, they now have a device that tells them if they are making progress or not. At first, they may take 40 or 50 seconds to recite the numbers 2 to 20 forwards and backwards, but gradually they can do it faster and faster until days or weeks later,

[17] Asking them not to tell parents has the opposite effect. Children would laugh at the request, knowing that indeed they want to tell their parents how well they did.

they can do it in 5 to 6 seconds.

Teacher___ OK Jasmin! Did you see the second hand as you said *zero*? When you finished counting back to zero, the second hand was passing by the number 8 and this means that it took you 40 seconds for you to say the numbers 2 to 20 and back to zero. Do you think you can do it faster if you try again? Look at the second hand! It´s halfways by the 6. Be ready when it passes right by 12… Ready?

As the second hand comes close to 12, we can even dramatize the exact timing by counting the seconds as a scale around the clock. In this case, we also need to teach the children that the space between 12 to 1 represents 5 seconds as marked by the little dots underneath the numbers. Then, from 1 to 2 we have another 5 seconds and each number represents 5 seconds. Thus, when the second hand passes by the number 10, this really means 50 seconds. At this point, we can warn to child to be ready because very soon, the second hand will pass by the 12, which represents 60 seconds around the clock, but it´s also the point at which the child should start counting the sequence of 2 to 20 and back to zero. Then, as the **second hand** passes by the 12, the **minute hand** also moves simultaneously to indicate that a minute has passed and that another minute begins as indicated by 60 seconds around the clock. In other words, the children will slowly realize that a minute has 60 seconds. In this way, the seconds and minutes are correlated.

Teacher__ Are you ready Jasmin? Wait for the second hand to pass by the number 12, which represents 60 seconds. Look at it. The second hand is passing by the number 11 and this number means 55 seconds and see the second hand moving fast to 12, this means that we now have 56 seconds right after the number 11 and the next little dot means 57 seconds and watch out, it will be now 58 seconds with the third little dot, and now 59 and 60 seconds with the number 12. At the number 12, you need to start counting 2, 4, 6, 8, 10---- 12,14,16,18,20; 18,16,14,12,10----8,6,4,2,0. Are you ready? Let´s wait again until the second hand goes around the clock one more time and when it passes by the 12, you start counting. Are you ready?

Jasmin__ 2-4-6-8-10 12-14-16-16-20 20-22-19-22, I mean 18....

As the children begin to perceive time as a concrete measure, their sense of reality becomes more complex. In this case, Jasmin knows that all children are watching how she manages to recite the sequence in a specific number of seconds, which may lead to a deeper understanding of time because she now has a first-hand experience of how the seconds and minutes move time forward. Thus, Jasmin should be developing a deeper understanding and a personal appreciation of those assignments dealing with time in seconds, minutes or hours. In the case of the next recess, the waiting time may be a couple of hours and in terms of going to the restroom, we may ask the child to wait until the minute hand goes to the next number in the clock so that we can count the minutes in intervals of five. The child may for example go to the restroom as the minute second moves to the number 5, which is 25 minutes after the hour and as the student returns to class, we all analyze the clock to see where the minute hand is located in order to figure out how minutes it took for the child to return to class. In terms of reciting the sequence 2 to 20 and back to 2 or zero, then it´s a matter of HOW MANY SECONDS as measured by the second hand starting at number 12, which marks the beginning of the seconds or the nest minute on the clock.

Teacher___ OK, remember to go backwards as soon as you get to 20. Start again, but wait for the second hand to go around the clock until it gets to the 12. Every interval between numbers in the clock represents five seconds. That is, from 12 to 1, it´s five seconds, which are represented by the tiny dots underneath the numbers 12 to 1. If you recite the numbers 2 to 20 and back to 0 in the time that the second hand gets from 12 to 6, it means that you have done it in 30 seconds because from 12 to 1 is 5 seconds; then five from 1 to 2. Thus, it is five more seconds with each interval. You see, we have 5-10-15-20-25-30 seconds from the 12 down to the number 6. It´s 30 seconds. Can you do it in 30 seconds? Ok, be ready because the time second is now passing by the 11…Ready:

(Also, as the sec. hand passes by 12, the min. hand moves simultaneously)

Jasmin__ 2-4-6-8-10 12-14-16-16-20 20-18 (small pause), 17, Oh no!, 16 14,12,10 (small pause), 8,6,4,2, zero!

Teacher___ That was much better. You did it in 34 seconds. The second hand almost made to 35. Did you see it when you finished saying *zero*?

With these exercises the children gradually incarnate a time sequence as the teacher encourages the children to take time to practice the sequences on a daily basis. Children can work in groups of two, in small groups or individually EVERY DAY 5 TO 15 MINUTES according to their motivation and perseverance. Everything depends on certain social vectors that the teacher must take into account so that children may be able feel a relevant social sense as to why they are practicing this or that sequence. A factor needs to be mentioned, specifically the ***awareness of their time performance***, which they can experience and measure in a very concrete form with the number of seconds it takes them to recite the sequence. As ***time-awareness*** emerges as a self-monitoring factor, it also helps them to challenge themselves as they are no longer competing against other more abled children, but against their own performance on a specific date. This means that the children are able to experience these exercises as a personal challenge and objective, which will allow them to master the number patterns at a deeper level.

A social undertaking would be to make a notation as to where every child is in terms of the number of seconds it takes to recite the numbers. In this case, the teacher would make a daily time chart as to how many seconds each child on such and such a date is able to recite the sequence and if any child is close to reciting it in about 10 to 15 seconds.

T___ Are you ready to recite the sequence of two?

C___ Yes, I´ve done it with the clock and it was faster.

T___ Ok children, stop what you´re doing and let´s see if Jasmin can do the sequence in 40 seconds or less.

C___ 2-4-6-8-10 12-14-16-16-20 20-22- I mean18-16 (small pause), 14-12-10 ... 8-6-4-2, zero!

T___ That was great! You did it in 33 seconds and tomorrow, if you want to practice it at home, we can see if you can do it even faster, but for today, that was excellent! I know it will be difficult, but in a few weeks many of you will be able to do it in less than 10 second, so I hope. Just practice every day, even it seems next to impossible to do it in less than 10 seconds.

Under these conditions it´s possible to motivate the children to practice, say about 5 to 10 minutes a day, so that they can do the sequence of 2 faster and faster every day. The more advanced children will be able to accomplish it in about 8 seconds in a couple of weeks.

It´s important to know that the teacher is the one making most of the decision as to what and how the children learn the time tables. At this stage, it is the opposite of a Montessori pedagogy in which the teacher acts as a mediator, as a supervisor, who has prepared everything needed for the child to guide herself. In the GD Ansatz, it is the teacher who´s guiding every step, but in terms of patterns of thinking and in accordance to the Vygotskyan Zone of Proximal Development. Gradually, the children will become independent of the teacher, but not until they have mastered certain domains. In this, case, they are still learning how to master the sequence of 2 to 20 and backwards, thereby gaining the knowledge and experience of a pattern. This will allow the teacher to introduce the numbers 3 to 30 and backwards to be mastered in about 8 to 10 seconds and likewise, 4 to 40 and backwards, 5 to 50, 6 to 60, ALL THE WAY UP TO 19 to 1900.

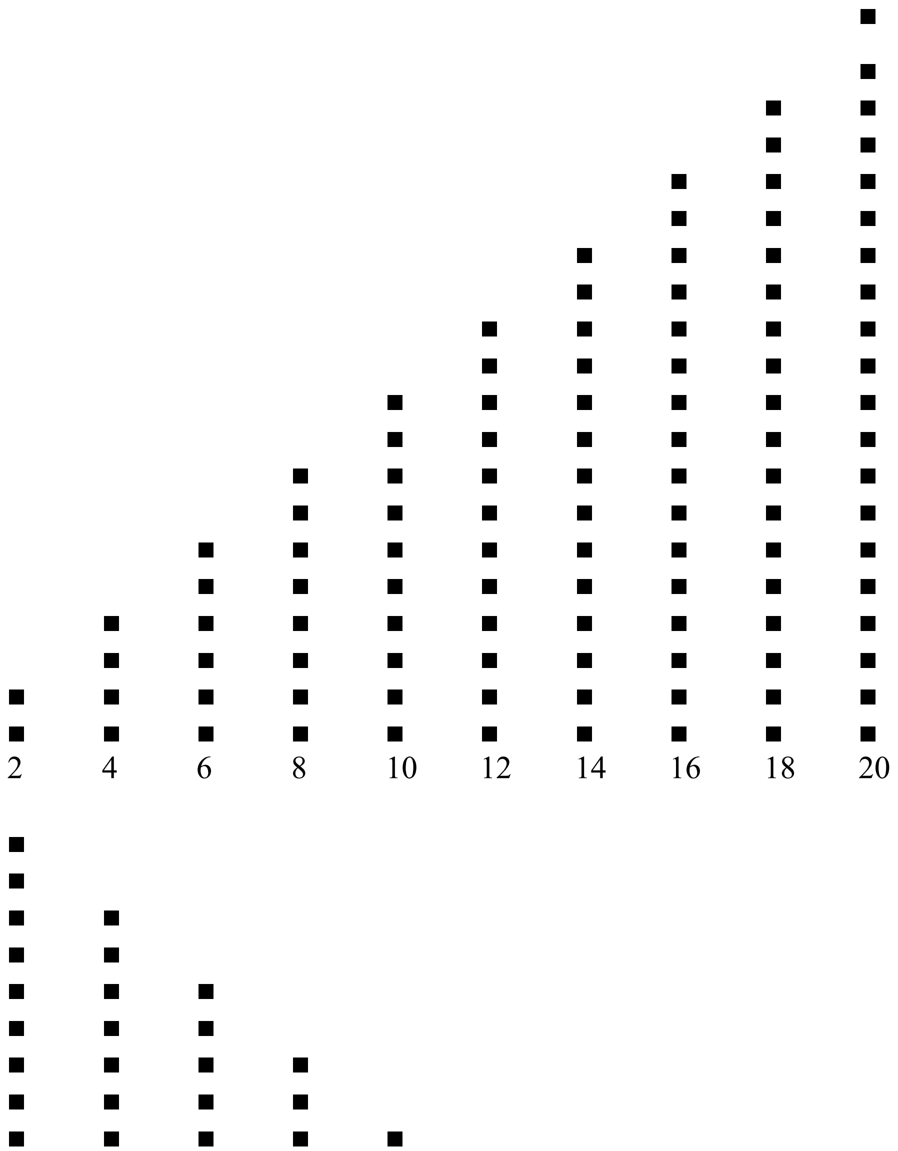
In order to fully master the sequence of 2, there may be other patterns that may be integrated so that the learning doesn´t become too monotonous. It all depends on the creativity of the teacher.

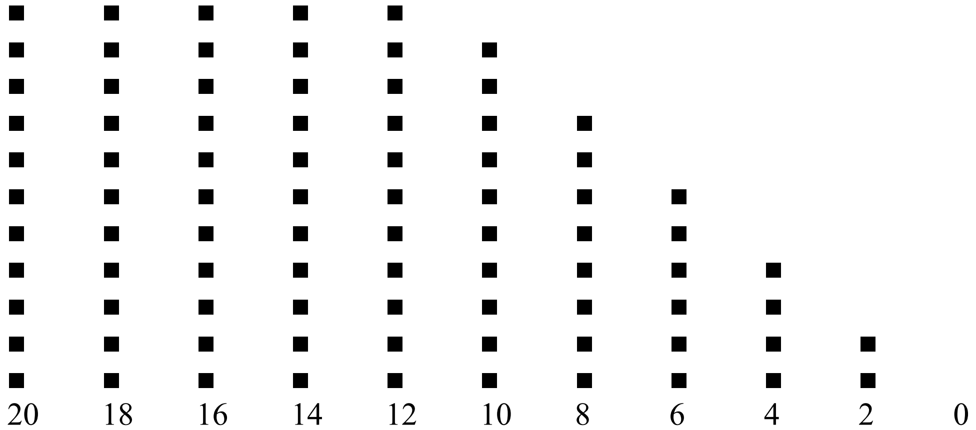
T___ Now, I need your concentration because I believe that the next exercise will help us understand the sequence of 2 much better. We could for example mark the odd numbers with a symbol, such as a big dot as follows:

●2 ●4 ●6 ●8 ●10 ●12 ●14 ●16 ●18 ●20

However, we can focus our attention on the even numbers and just make a graphic representation of them as follows:

Graphic Variation I



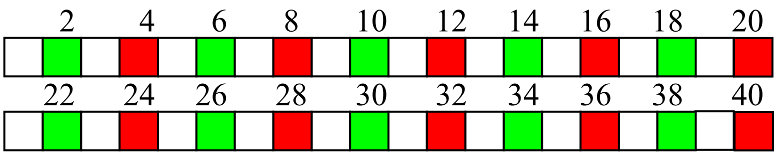


The teacher may ask the children to make the graph on a blank piece of paper as part of their homework. In the GD Ansatz children, in most cased, get a blank piece of paper. There is nothing prepared for them with a few exceptions, at least as to the mastery of the time tables. Many teachers and curricula materials would ask the students to do many facts of addition or multiplication in order to practice them as:

2 + 3 =   2 + 4 =   6 + 5 =   2 + 3 =   8 + 2 =   7 + 8 =

There is nothing wrong in asking the children to practice their facts of addition this way, except that they do not create a number pattern of any kind. The main menu in the GD Ansatz is the creation of thinking patterns for the sake of achieving a challenging objective such as mastering the time tables. With increasing self-confidence and persistence, children will recite the sequence of two in about 6 seconds, but to make the pathway more dynamic, it´s important to introduce as many patterns as possible. It all depends on the creativity of the teacher. Here is for example a graphic variation of the one above:

Graphic Variation II



Naturally the children can expand the sequence to 40, then to 60 or 80. Is there a natural upper limit for this pattern, which instead of reversing seems to keep on going and up to infinity. According to the GD Ansatz the sequence of 1 should be from 1 to 100 and the sequence of 2, the one without reversibility, should be from 2 to 200 and likewise 3 to 300, 4 to 400 up to 19 to 1900.

Ideally, the teacher and parents should have a weekly session in which the parents are informed of the different number patterns that their children are working on. The parents and children should understand that they are free to create their own patterns, even based on the logic of the ones they are working on. Eventually the teacher will introduce many other patterns so that they feel a sense of discovery and infinity. The children, with the help of their parents, are encouraged to create their own patterns according to their intelligence and potential.

As soon as the children work on a specific pattern on a piece of paper, it would be posted on the classroom wall so that all can see the progress of each child. Besides the sequence of 2 forwards and backwards, another very important sequence is that of the 2 up to 200 in a very specific trajectory in which for example the number 22 must be underneath the 2 above. Thus, gradually a new pattern emerges, that of perceiving the number in multiples of 10. The folding of the piece of paper stays the same, i.e., with the same FOLDING PATTERN, but now, instead of the numbers reversing from 20 back to zero, the numbers go up to 200. However, before they do 2 to 100, they must do and understand 1 to 100 as follows:

Sequence of 1 to 100

1	2	3	4	**5**		6	7	8	9	**10**
11	12	13	14	**15**		16	17	18	19	**20**
21...	22	23	24	**25**		26	27	28	29	**30**
31	32	33	34	**35**		36	37	38	39	**40**
etc., up to:										**100**

Sequence of 2 to 200

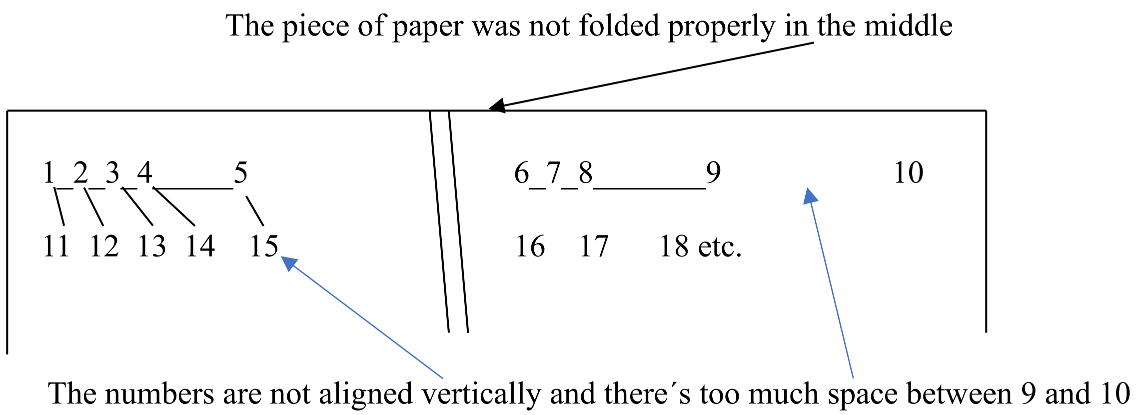
2	4	6	8	**10**		12	14	16	18	**20**
22	24	26	28	**30**		32	34	36	38	**40**
42	44	46	48	**50**		52	54	56	58	**60**
up to										**200**

These are ideal models. At first, however, the children cannot map the numbers with the perfect spacing. That is the reason that the teacher makes it very clear how the numbers should be positioned relative to one another and relative to the spatial arrangement of a piece of paper folded in half lengthwise:

Linear Connections in Multiples of 10 up 100

With the linear connections the children begin to master the spatial intervals between to numbers horizontally and vertically with each unit number being the same. However, at the beginning, most children struggle to makes the vertical and horizontal intervals at an acceptable level.

Erroneous Alignments



Number sense in the pentagonal system must flow logically into the decimal system as soon as we´re dealing with more than 50 units. Up to the number 50, the GD-Structure can deal with it as a pentagonal as well as a decimal system. As the children conquer the numbers up to 100, then it´s imperative to use the pentagonal system as a bridge, as a stepping stone. That is the reason that we still separate the numbers 1 to 10 into two groups, one from 1 to 5 and one from 6 to 10. This pentagonal number sense is still visible within a decimal system dealing with the numbers 1 to 100 in a sequence with multiples of 10 as follows:

The pentagonal system flowing into the decimal system in number patterns

■■■■■	■■■■■	■■■■■	■■■■■	■■■■■
5	5	5	5	5
5	10	15	20	25

■■■■■ ■■■■■ ■■■■■ ■■■■■ ■■■■■

10 20 35

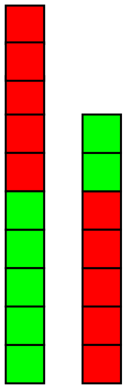
The sequence of 1 to 100 takes the pentagonal system into consideration, but in context of a decimal system:

A pentagonal system within a decimal system in the numbers up to 100

♠♠♠♠♠	♠♠♠♠♠	10
♠♠♠♠♠	♠♠♠♠♠	20
♠♠♠♠♠	♠♠♠♠♠	30
♠♠♠♠♠	♠♠♠♠♠	40
♠♠♠♠♠	♠♠♠♠♠	50
♠♠♠♠♠	♠♠♠♠♠	60
♠♠♠♠♠	♠♠♠♠♠	70
♠♠♠♠♠	♠♠♠♠♠	80
♠♠♠♠♠	♠♠♠♠♠	90
♠♠♠♠♠	♠♠♠♠♠	100

However, in dealing with simple additions, a pentagonal sense is the basis of the addition because all numbers are redistributed into groups of 5, which allows the children to make the addition with more ease:

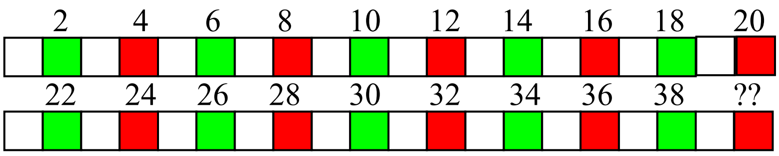
Simple redistribution of 10 + 7 in groups of five



10 + 7 = 5 + 5 +5 + 2 = 17

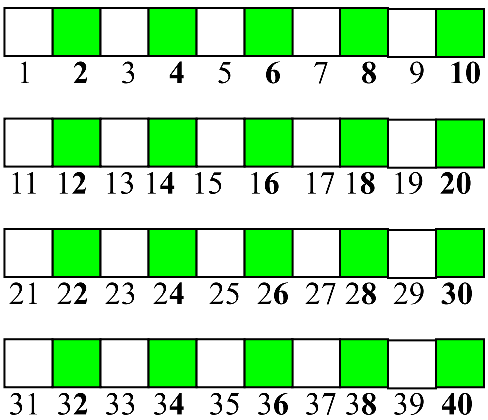
However, as to the question, what comes after 38 in the sequence of 2 to 40, thereby counting in twos, the answer can only be given based on experiences dealing with the sequence 1 to 100, that is, in counting one by one, in context of the decimal system. If the child doesn´t know that in a sequence of two by two, after 38 the number 40 comes, then it is because the child has not yet mastered the fifth GD-S, i.e., the numbers up to 50, based on all the experiences dealing with the pentagonal system and dealing with real life situations with number sense.

Graphic presentation of 2 to 40 counting in twos and with multiples of 10 vertically



Thus, all the GD structures up to number 50 must be mastered in order to introduce the children with more challenging situations such as in counting the numbers in groups of 2 up to 40. The following graphic was designed to make it clear that the odd numbers are not mentioned and that the focus is on the even numbers:

The focus is on the even numbers in counting by two



In this exercise the children recite the odd numbers in a very low-whispering voice and the even number very loud so that eventually they can focus their attention ONLY ON THE EVEN NUMBERS once we´re dealing with the sequence of 2 up to 200.

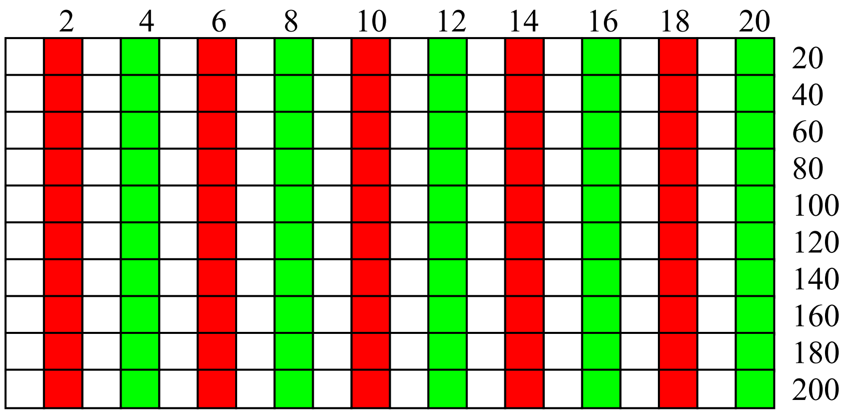
We can help the child determine that 40 comes after 38 and 50 after 48 and likewise 60 after 58, etc., by analyzing the sequence of 1 to 100.

The field of 100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	99	90
91	92	93	94	95	96	97	98	99	100

This sequence implied that the child has mastered the fifth GD-S, that of the numbers up to 50 in terms of the pentagonal system. As the child tries to complete the sequence of 2 to 200 there will be plenty of challenges, but with the help of a mentor, in terms of the Zone of Proximal Development, she will be able to complete it in multiples of 10 as follows:

The sequence of 200



There is an aesthetic quality with the contrasting colors red vs green or yellow vs blue, which makes the assignment more interesting to the child. However, as the child receives a blank piece of paper to fold, then there are no little squares to fill out, just space to be organized with whatever number sequence the child is working on. In this case, it is the sequence of 2 to 200 two by two:

The sequence of 2 to 200 after folding a piece of paper

● 2	● 4	● 6	● 8	● 10	● 12	● 14	● 16	● 18	● 20
● 22	● 24	● 26	● 28	● 30	● 32	● 34	● 36	● 38	● 40
● 42	● 44	● 46	● 48	● 50	● 52	● 54	● 56	● 58	● 60
● 62	● 64	● 66	● 68	● 70	● 72	● 74	● 76	● 78	● 80
● 82	● 84	● 86	● 88	● 90	● 92	● 94	● 96	● 98	● 100
● 102	● 104	● 106	● 108	● 110	● 112	● 114	● 116	● 118	● 120
● 122	● 124	● 126	● 128	● 130	● 132	● 134	● 136	● 138	● 140
● 142	● 144	● 146	● 148	● 150	● 152	● 154	● 156	● 158	● 160
● 162	● 164	● 166	● 168	● 170	● 172	● 174	● 176	● 178	● 180
● 182	● 184	● 186	● 188	● 190	● 192	● 194	● 196	● 198	● 200

The multiples of 10 function as a controlling mechanism. Basically, the children become aware if in their sequence, they´ve made a mistake. Children will always be making mistakes, but as they come to the end of the page, they can see that the multiples are not aligned correctly. So, they know or the mentor tells them that somewhere on that line there is a mistake. In the example below 42 is shown below 20. **Thus, there must be a mistake between 22 and 42**. By looking on the last column, the children can find out if a mistake has been done. This self-control or self-monitoring mechanism is unique in the process to master the time tables. No other method in the world has it. The teacher could have 30 students and all may be working on different sequences, with the more advanced students working on 7 to 700 and yet the teacher is able to monitor the work of all students very fast. **No other method in the world can do this!**

Distortion of the multiples of 10 in the sequence of 2 to 200

2	4	6	8	10	12	14	16	18	20
22	24	26	28	30	**34**	36	38	40	**?**

With practices they will master the sequence 1 to 100 and 2 to 200. The question is, what´s next. It all becomes a logical pattern. Gradually, they understand that the next assignment is to master the sequence of 3 to 300. The children will discover the infinity of the patterns. When the children discover that the pattern horizontally, line by line and vertically, by looking at the multiples of 10 on each column, then there are no limits as to how much they can accomplish. They know for example that under the 2 there must be the number 22. In other words, under each number of each column there is a multiple of 10 and that pattern is valid for any sequence. If they have completed the sequence of 2 to 200, then they can certainly start working on the next sequence, that of 3 to 300 as follows:

Sequence of 3 to 300-A

● ● 3 ● ● 6 ● ● 9 ● ● 12 ● ● 15 ● ● 18 ● ● 21 ● ● 24 ● ● 27 ● ● 30

↓ ↓

● ● 33 ● ● ??

What should a child write after 33? Presumptively she will write 36 because she´s aware that the 35 and 37 do not align in this scheme. Likewise, she will follow the pattern and induce that the next number must be 39 because it has to have a 9 on the units, just like the number above.

The sequence of 3 to 300-B

● ● 3 ● ● 6 ● ● 9 ● ● 12 ● ● 15 ● ● 18 ● ● 21 ● ● 24 ● ● 27 ● ● 30

↓ ↓ ↓

● ● 33 ● ● 36 ● ● 39

Consequently, as the child completes the second line up to 57, and thus be able to write 60 because it coincides with 30 above, with both having a zero on the unit column. Likewise, she will conclude that the next numbers are 63, 66, 69, 72, etc., aligning the unit numbers on each column.

The 3-to-300 sequence

● ● 3 ● ● 6 ● ● 9 ● ● 12 ● ● 15 ● ● 18 ● ● 21 ● ● 24 ● ● 27 ● ● 30

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

● ● 33 ● ● 36 ● ● 39 ● ● 42 ● ● 45 ● ● 48 ● ● 51 ● ● 54 ● ● 57 ● ● 60

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

● ● 33 ● ● 66 ● ● 69 ● ● 72 ● ● 75 ● ● 78 ● ● 81 ● ● 84 ● ● 87 ● ● 90

● ● 93 ● ● 96 ● ● 99 ● ● 102 ● ● 105 ● ● 108 ● ● 111 ● ● 114 ● ● 117 ● ● 120

● ● 123 ● ● 126 ● ● 129 ● ● 132 ● ● 135 ● ● 138 ● ● 141 ● ● 144 ● ● 147 ● ● 150

● ● 153 ● ● 156 ● ● 159 ● ● 162 ● ● 165 ● ● 168 ● ● 171 ● ● 174 ● ● 177 ● ● 180

● ● 183 ● ● 186 ● ● 189 ● ● 192 ● ● 195 ● ● 198 ● ● 201 ● ● 204 ● ● 207 ● ● 210

● ● 213 ● ● 216 ● ● 219 ● ● 222 ● ● 225 ● ● 228 ● ● 231 ● ● 234 ● ● 237 ● ● 240

● ● 243 ● ● 246 ● ● 249 ● ● 252 ● ● 255 ● ● 258 ● ● 261 ● ● 264 ● ● 267 ● ● 270

● ● 273 ● ● 276 ● ● 279 ● ● 282 ● ● 285 ● ● 288 ● ● 291 ● ● 294 ● ● 297 ● ● 300

The children will ask or discover, how does the 4-to-400 work. They´ll see and experience that it works with the same logic as with the 2-to-200 and 3-to-300. However, at the beginning they can use their fingers, touching for example their nose, in order to count by fours:

One, two, three, **FOUR!**, five, six, seven, **EIGHT!**, etc.

After each interval of four, there could be little pause, but all can be don rhythmically, counting by four. They can also count their steps as they walk forwards and jump on each interval of four:

…nine, ten, eleven, **TWELVE!**, thirteen, fourteen, fifteen, **SIXTEEN!**,

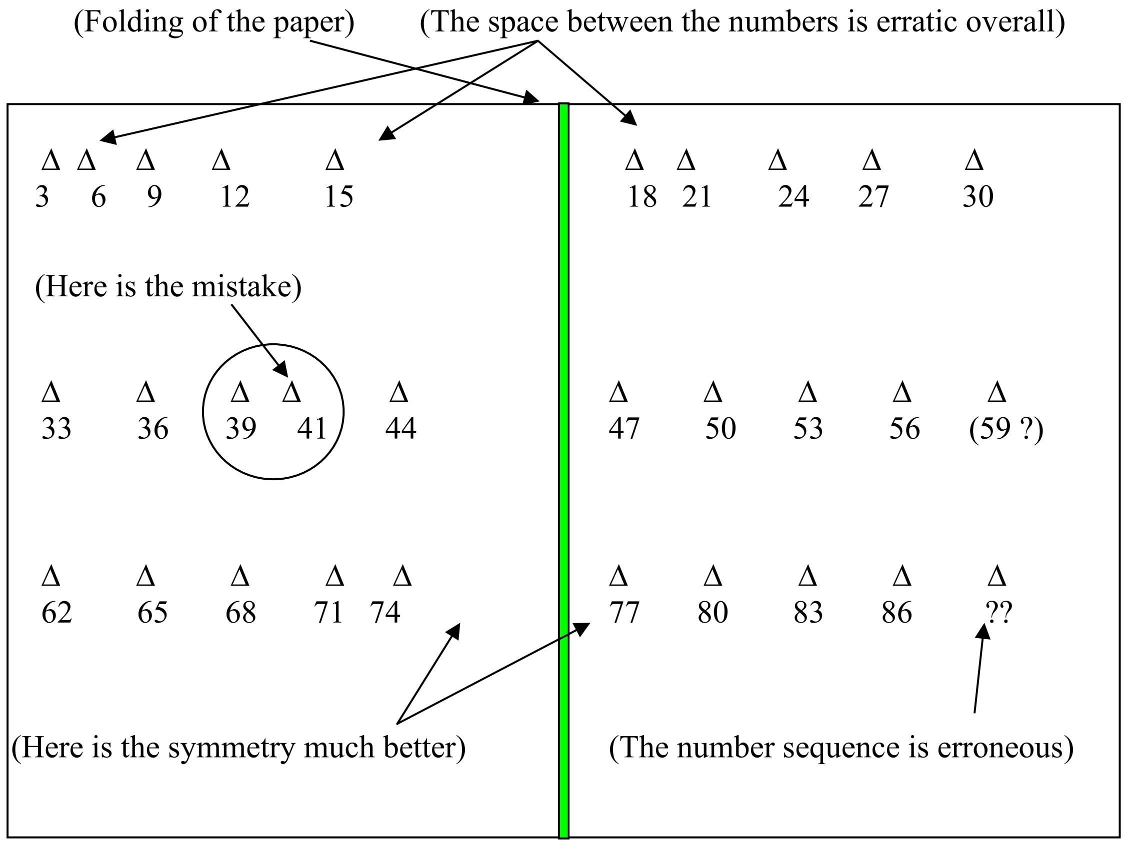
With number bigger than three, it´s important that they touch their nose, forehead or some body part in order to sense their logical progression.

… seventeen, eighteen, nineteen, **TWENTY**, twenty-one, twenty-two, twenty-three, **TWENTY-FOUR, etcetera, all the way to FOUR HUNDRED!**

At the beginning, this exercise is a formidable challenge, but if given in a social context, there will be plenty of motivation and perseverance in order to accomplish the objective. Significant progress will be made in a short time because the instructions are very concrete and the children also ***sense their progress*** and accomplishments, especially if the teacher is able to motivate them with some kind of social interactions within the group or in terms of organizing social structures in which the students can also mentor other students working with those sequences that they have already mastered. As previously mentioned, the teacher or mentor is the main protagonist of the whole GD process.

A variation to the 3-to-300 sequence would be to draw a small triangle for every number, being that a triangle has three lines. What´s important is always to be able to fold the piece of paper in half so that they can follow the number pattern properly:

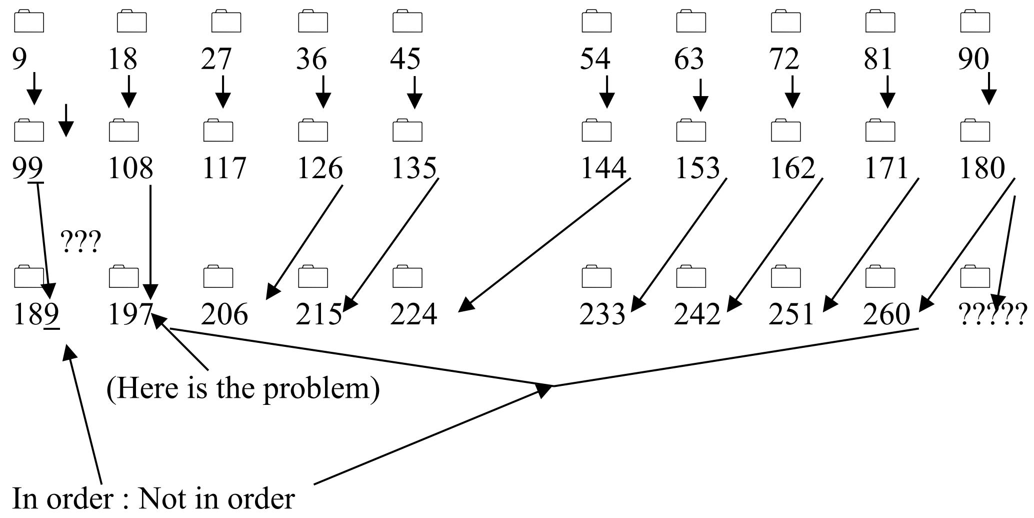
A student´s work on the 3-to-300 sequence with spatial and numerical discrepancies



The child is aware of the problem. She knows that 86 + 3 = 89 because she can feel the mistake as she touches her nose with the corresponding rhythm. However, she´s also aware that the number here has to have a zero on the unit column, that is, 90 and not 89. The child tries to find the mistake, but is unsure, and therefore asks to teacher to locate the mistake. How long would it take for the teacher or mentor to locate the mistake? In reality, only a pair of seconds because the columns are aligned in multiples of 10. The teacher knows right away that the mistake must be found between the numbers

30 and 60, because up to 30, everything is fine. However, the teacher points to the number 39 and what number should then follow afterwards. The teacher shows the child to count touch three times the noses as she says 40, 41, 42 and with that all the rest of the numbers are corrected. Eventually, most of the kids learn how to correct themselves because the patterns allow for a rapid recognition of where the mistake is located.

A more advanced child working on the sequence of 9 to 900



The teacher can see very fast that the mistake is on the third line. However, if the child is very advanced, in most cases she would not ask the teacher, but find the mistake herself. At this level, the student knows where to find the mistake, that it lies between 180 and the next line. In essence, she can control herself and this self-controlling mechanism can give a lot of time to the teacher to concentrate on those students, who need a lot of help. The advanced student is very much conscious of his objective, to write the numbers 9 to 900 without any mistakes and independently from the teacher. It´s a great accomplishment to be able to finish the sequence up to 900 without any mistakes.

This **sense of achievement** must be celebrated in order to motivate the children to master the next challenging sequence, that of 11 to 1,100. All activities in the GD Ansatz should be socially meaningful whenever possible. In this case, the teacher could organize a special day in order to recognize all children as to what sequences and other skills they have accomplished. Children need to be socially recognized with any means possible. It could also be for example, a special trip to the zoo or to go to a theater, or even invite other children to view and exhibit the work from every child. How many people would show up to work every day with the same enthusiasm if their work is never recognized. We as human beings need a social incentive in order to strengthen our self-esteem and so do children. Thus, social recognition is a must in the GD Ansatz.

3.3 The pathway to mastering the multiplication tables

The patterns of sequences will help the children to eventually master the time tables. These sequences require a lot of concentration, persistence and patience, but will eventually pay off with the increasing mastery of the sequences going forwards with those dealing with the numbers 2 to 200, 3 to 300, 4 to 400, etc., up to 19 and 1,900. That is only one strategy as a variation of those dealing with the time sequences as measured in terms of how many second the students can recite 2 to 20, 3 to 30, 4 to 40, etcetera and backwards in less than 10 seconds.

The 15-to-1,500 sequence

⌛	⌛	⌛	⌛	⌛	⌛	⌛	⌛	⌛	⌛
15	30	45	60	75	90	105	120	135	150

⌛ etc. up to 1,500.

In essence, the focus is on the inner logic as a type of arithmetic grammar. In this case, the student is working on the number 15 to 1,500 and knows that the first line must end with 150 because all number in the pattern are multiples of 10 at the end of the first line: 2 to 20, 3 to 30, 4 to 40, etc. up to 15 to 150. It follows that 150 must be the last number of the first line. Let´s assume that the student has mastered the sequences, going forwards, of 2 to 200, 3 to 300, etc., up to 9 and 900. Does that mean that he automatically knows what 7 x 3 is? The student is certainly starting to sense the answer, that 7 x 3 = 21, but he still has to count the numbers one by one such as 7,8,9,10,11,12,13 in order to get to 14, even touching the nose. The teacher will certainly point out that 7 x 3 = 21 because we can count the 7 three times, such as 7 + 7 + 7. Thus, 7 x 3 = 21, but having mastered the sequences going forward, such as 2 to 200, 3 to 300, up to 9 to 900, does not give them the mental capacity to master the time tables. For mastery, the reversibility of the numbers as measured in less than 10 seconds, going forwards and backwards (such as 2 to 20, 3 to 30, 4 to 40, etc.) will certainly help them to reach mastery of the time tables up to 10 if they can do them in less than 10 seconds. They will certainly know that 7 x 3 = 21 or 8 x 6 = 48 right away because mastery has been achieved in the reversibility sequences in less than 10 seconds.

Beginning in second grade, the main objective would be to master the first reversibility sequences along with those dealing with the forward trajectories of 2 to 200, 3 to 300, etc. Here are some examples:

The 3-to-300 sequence

Δ	Δ	Δ	Δ	Δ		Δ	Δ	Δ	Δ	Δ
3	6	9	12	15	(Pause)	18	21	24	27	30
Δ	Δ	Δ	Δ	Δ		Δ	Δ	Δ	Δ	Δ
33	36	39	42	45	(Pause)	48	51	54	57	60

etc. up 300

⌑	⌑	⌑	⌑	⌑	⌑	⌑	⌑	⌑	⌑
4	8	12	16	20	24	28	32	36	40
⌑	⌑	⌑	⌑	⌑	⌑	⌑	⌑	⌑	⌑
44	48	52	56	60	64	68	72	76	80

etc. up 400

Students can experiment with their own visual representations of the numbers, such as the following:

A particular student may choose this display of the 5-sequence

●●●●● ●●●●● ●●●●● ●●●●● etc., up to 500
5 10 15 20

or:

● ● ● ● ● ● ● ●
● ● ● ●
● ● ● ● ● ● ● ●
5 10 20 25

etc. up to 500

The 10-sequence along with the 5- and 2-sequence are very easy to master. Thus, very early on the students may choose to master them. Here is an example of the 10-sequence:

The 10-sequence up to 1000

●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
●●●●● ●●●●● ●●●●● ●●●●● ●●●●●
10 20 30 40 50 etc. up to 1000

With the 6-sequence there are interesting and funny possibilities:

●●● ●●● ●●● ●●● ●●●
●●● ●●● ●●● ●●● ●●●

∆∆ ∆∆ ∆∆ ∆∆ ∆∆
(two triangles = 6)

or:
∴ ∴ ∴ ∴ ∴ ∴ ∴ ∴
6 12 18 24

etc. up to 600

The students may discover the infinite possibilities, and most of the time they´re discovered with the power of the pentagonal system:

The 7-sequence

7 14 21 28

7 14 21 28

etc. up to 700

As soon as possible, the students will be able to do the sequences without the corresponding graphic representations:

1-to-100 sequence with only the numbers

The paper is colored in the middle in order to separate the numbers in groups of 5

1	2	3	4	5		6	7	8	9	10
11	12	13	14	15		16	17	18	29	20
21	22	23	24	25		26	27	28	29	30
31	32	33	34	35		36	37	38	39	40

Etc. up to 100

2-to-200 sequence

2	4	6	8	10		12	14	16	18	20
22	24	26	28	30		32	34	36	38	40
42	44	46	48	50		52	54	56	58	60

Etc. up 200

3-to-300 sequence

3	6	9	12	15		18	21	24	27	30
33	36	39	42	45		48	51	54	57	60
63	66	69	72	75		78	81	84	87	90

bis 900

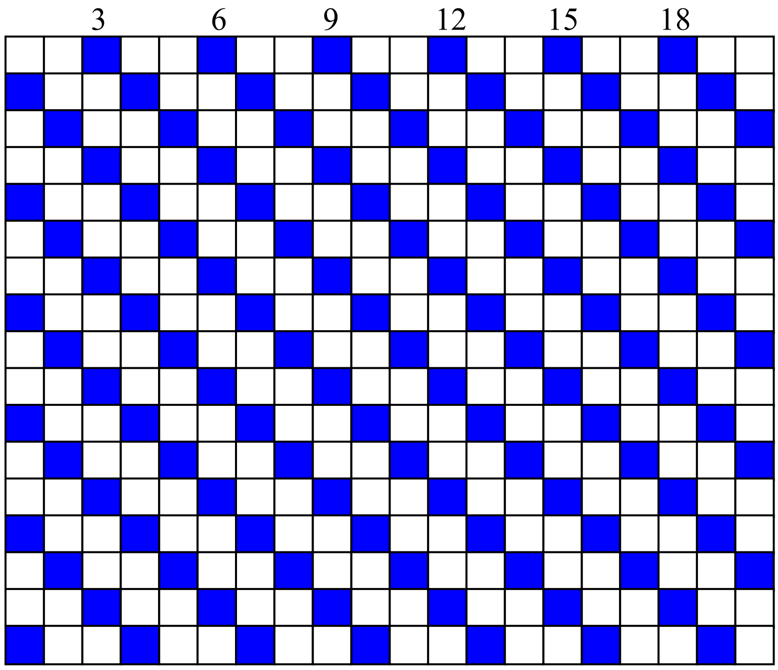
Likewise, the students will follow the pattern with the 4-to-400, 5-to-500, 6-to-600 sequence, etc. The students begin to experience the infinity of the numbers in a clear and distinct way. While some students may be working on the 1-to-100 sequence, other more advanced students may be working with 9 to 900 or even more challenging sequences.

99-sequence

99	198	297	396	495	594	693	792	891	990
1,089	1,188	1,287	1,386	1,485	1,584	1,683	1,782	1,881	1,980
etc. up to 9,900									

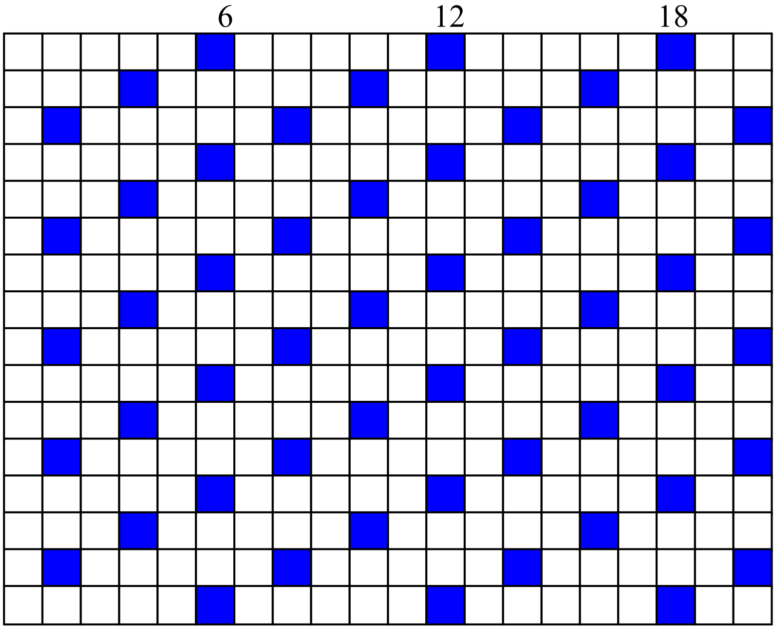
To make the sequences more palatable, students more work on other variations to the patterns, especially with grid paper as follows:

The 3-squence in graphic form



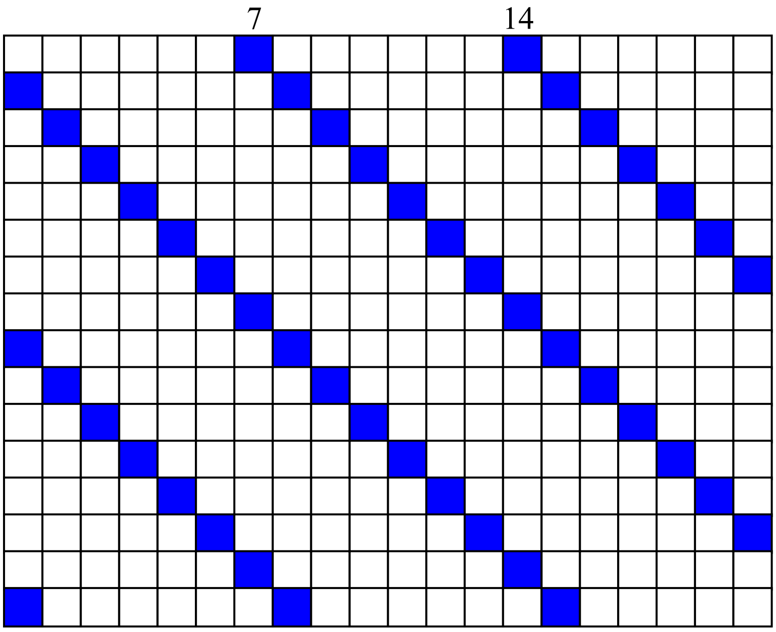
And for the 6-sequence it would be similar:

The 6-sequence on a grid in graphic form



For each sequence there is a definite graphic signature, that the children can discover. They experience it with their corresponding number configurations:

The 7-sequence on a grid in graphic form



It´s very interesting for the children to work on these graphic configurations, which can be colored arbitrarily. As they work on these patterns, they would take time to practice the reversibility sequences in order to master the time tables as follows:

Reversibility Assignments

Objective: 1 to 10 forwards and backwards to zero in 5 to 7 seconds:
[1-2-3-4-5 (Pause) 6-7-8-9-10 10-9-8-7-6-5 (Pause) 4-3-2-1-0]

Objective: 2 to 20 forwards and backwards to zero in 5 to 7 seconds:
[2-4-6-8-10 (Pause) 12-14-16-18-20 20-18-16-14-12-10 (Pause) 8-6-4-2-0]

Objective: 3 to 30 forwards and backwards to zero in 7 to 9 seconds:
[3-6-9-12-15 (Pause) 18-21-24-27-30 30-27-24-21-18-15 (Pause) 12-9-6-3-0]

Objective: 4 to 40 forwards and backwards to zero in 8 to 10 seconds:
[4-8-12-16-20 (Pause) 24-28-32-36-40

40-36-32-28-24-20 (Pause) 16-12-8-4-0]

Objective: 5 to 50 forwards and backwards to zero in 5 to 7 seconds:
[5-10-15-20-25 (Pause) 30-35-40-45-50

50-45-40-35-30-25 (Pause) 20-15-10-5-0]

Objective: 6 to 60 forwards and backwards to zero in 8 to 10 seconds:
[6-12-18-24-30 (Pause) 36-42-48-54-60

60 54-48-42-36-30 (Pause) 24-18-12-6-0]

Objective: 7 to 700 forwards and backwards to zero in 9 to 12 seconds:
[7-14-21-28-35 (Pause) 42-49-56-63-70

70-63-56-49-42-35 (Pause) 28-21-14-7-0]

Objective: 8 to 80 forwards and backwards to zero in 8 to 11 seconds:
[8-16-24-32-40 (Pause) 48-56-64-72-80

80-72-64-56-48-40 (Pause) 32-24-16-8-0]

Objective: 9 to 90 forwards and backwards to zero in 7 to 9 seconds:
[9-18-27-36-45-(Pause) 54-63-72-81-90

90-81-72-63-54-45 (Pause) 36-27-18-9-0]

Objective: 10 to 100 forwards and backwards to zero in 5 to 7 seconds:
[10-20-30-40-50 (Pause) 60-70-80-90-100

100-90-80-70-60-50 (Pause) 40-30-20-10-0]

Objective: 11 to 110 forwards and backwards to zero in 6 to 8 seconds:
[11-22-33-44-55 (Pause) 66-77-88-99-110

110-99-88-77-66-55 (Pause) 44-33-22-11-0]

Objective: 12 to 120 forwards and backwards to zero in 8 to 10 seconds:
[12-24-36-48-60 (Pause) 72-84-96-108-120

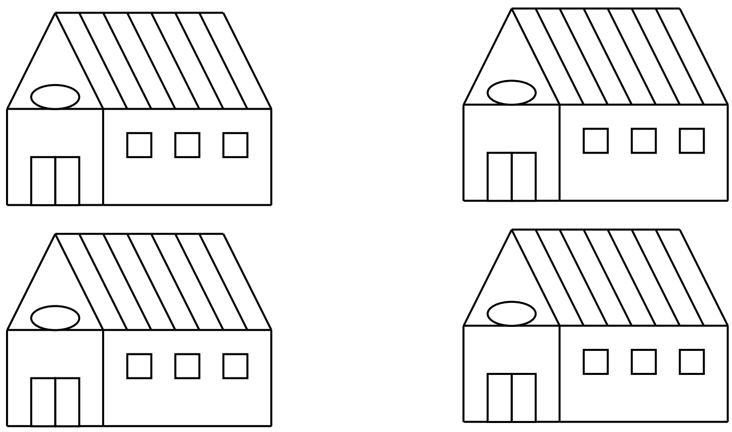
120-108-96-84-72-60 (Pause) 48-36-24-12-0]

Some observers may object to the drilling of the sequences, but in reality, we are drilling with number patterns with an inner logic. Students will not do them unless they are prepared and unless they have mastered the numbers up to the fifth GD-Structure. Kids in first class may start with some of the reversibility exercises and it´s all done within a social undertaking by officiating celebrations once the children have mastered x sequence. Most of the children in third grade should be able to master the reversibility sequences and thus be able to master the time tables from 1 to 12. The advantage of the GD-Ansatz is that children no longer need to memorize the time tables in isolation. If for example, they can recite the sequence of 7 to 70 and back to zero in 10 seconds, then obviously, they will be able to know right away that 3 x 7 = 21 because of the sequence 7-14-21.

Nevertheless, children should limit their time practicing the reversibility exercise to about 5 to 15 seconds per day. They can work in pair or in small groups. They keep records of the speed in which they did the sequences with the objective of reciting them faster and faster every day. Regardless as to how fast or slow they are, there must be social recognition to motivate their progress. Children must be constantly praised for their efforts and accomplishments not matter how small they may be. In due time, they will be able to master the time tables and understand for example what 3 x 4 means. If we are dealing with 7 + 7 + 7 + 7, then the children also need to know what it means. Is it 4 x 7 or 7 x 4? The children do have a sense of relationship, that a multiplication is a function of the addition.

The GD-Ansatz is not a mere recitation of the number sequences as if it were a drill. The children must understand the relation with the addition and it´s up to the creativity of the teacher to relate the numbers to real world circumstances. Where in real life to the number sequences have any significance? There are infinite ways to show how numbers help people to organize their daily experiences, but in a classroom setting, the least that teacher would do would be to relate the numbers with the illustration of a house or some other geometric form. Children could, for example, be asked to draw four houses in order to see the social and technical value of number patterns that deal with multiplication facts:

Applied number pattern in 4 houses



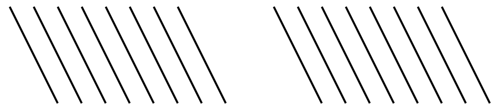
A) How many square windows do we see in the four houses?
Answer: ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐

3 + 3 + 3 + 3

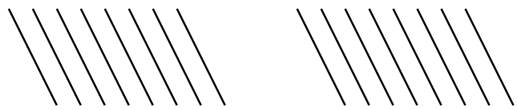
Thus: “**4-times** the-3” = 12[18] or “the-3__ **4-times**”[19]

B) How many diagonal lines do you see?

Answer: Either as “**4-times** the-8” or “the-8__**4-times.**”



8 + 8



8 + 8

Another objective would be for the children to experiment with their own arithmetic variations. That way, they´ll start becoming more and more independent from the teacher. They may, for example, draw clouds, buildings, homes, etc., and as they do, they´ll start sensing the infinity of the numbers in their multiplicative relations. The reversibility exercises should not be done in isolation from their daily social activities. The teacher, for example, may ask for a volunteer to recite the 7 to 70 and backwards to find out if a student is faster than the day before or to ask if any other student from the group could do it faster. For the sequence 2 to 200 the teacher may give them the option to make the corresponding illustrations with the objects of their choice:

Creative illustrations for two sequences

♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥
2	4	6	8	10	12	14	16	18	20
♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥	♥♥
22	24	26	28	30	32	34	36	38	40
etc. up to 200									
*	*	*	*	*	*	*	*	*	*
6	12	18	24	30	36	42	48	54	60
*	*	*	*	*	*	*	*	*	*
66	72	78	84	90	96	102	108	114	120
etc. up to 200									

To make things more interesting, the students could analyze the relationships between the additions and multiplications:

---

[18] According to the ***a**b*-thesis where ***a*** is a multiplier and *b* is a multiplicand.
[19] According to the *a**b***-thesis where *a* is a multiplicand and ***b*** is a multiplier.

Additive vs. Multiplicative Relations

□□□□	□□□□□	□□□	□□□□□	□□□□□□□	□□
4 +	5 +	3 +	5 +	7 +	2 +

□□□	□□□□□	□□□	□□□□	□□□□□	□□□
3	5 +	3 +	4 +	5 +	3 +

□□□	□□	□□□□□□□
3 +	2 + +	7

4 + 5 + 3 + 5 + 7 + 2 + 3 + 5 + 3 + 4 + 5 + 3 + 3 + 2 + 7 =

(4 + 4) + (5 + 5 + 5 + 5) + (3 + 3 + 3 + 3 + 3) + (7 + 7) + (2 + 2) =

(2 x 4 = 8) +(4 x 5 = 20) + (5 x 3 = 15) + (2 x 7 = 14) + (2 x 2 = 4) =

8 + 20 + 15 + 14 + 4 =

```
     8              8
    20         20 + 0
+   15  ---->  10 + 5  ---->  40 + 8 + 5 + 4 + 4 =
    14         10 + 4           48 + 5 + 8 =
     4              4           56 + 5 = 61
```

We should analyze the additions analytically, that is, starting with the bigger wholes, as oppose to the synthetic approach of starting with the smaller entities:

Synthetic Addition

```
   2 4 6
+  3 2 4
   5 6 3
```

A) Starting with the ones:

```
     1 <--
   2 4 6
+  3 2 4
   5 6 3
       3
```

B) Then the tens:

1 1
2 4 6
\+ 3 2 4
5 6 3

3 3

C) And finally the hundreds

1 1
2 4 6
\+ 3 2 4
5 6 3

1,1 3 3

This synthetic approach may be practical, but it´s wiser to be more analytic, more holistic by identifying the biggest entity in the addition. This will allow the students to keep in mind a comprehensive overview of the whole process. With the synthetic approach, the students may not have a sense as to why they got a specific answer because it´s difficult for the mind to have an overview as the numbers are being added from the ones to the tens to the hundreds, etc. As they get an x answer, they do not have a certainty that the answer is correct. They only know that they got an answer, but they do not know why. Contrary to that, an analytic approach is much more dynamic and enlightening. In the GD Ansatz, one begins with the biggest number of the addition from left to right:

Dynamic Addition I

	2	4	6
+	3	2	4
	5	6	3

The pupils may start with either 200, 300 or 500, but the teacher should motivate them to choose the 500 as the starting number to be added because it´s the biggest whole of the three options. They are always reminded to look and start with the biggest number because it´s more efficient. The teacher should point out that the 5 does not represent five units, but 500 and that we don´t see the two zeros because of the 6 and 3 in place: basically the 6 and 3 are hiding the two zeros of 500. Thus, mentally the pupil makes the addition of 500 + 300 + 200 = 1000:

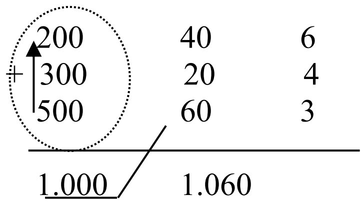
Dynamic Addition II

	200	40	6
+	300	20	4
	5 00	60	3

1.000

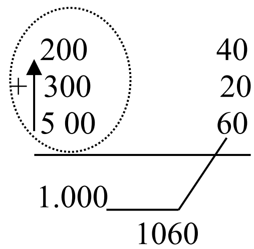
The pupil then must decide between 60 and 40. Hopefully the pupils may add 60 to 1000 because 60 is bigger than 40:

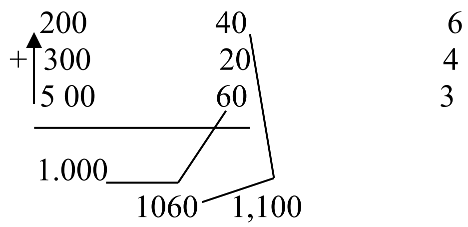
Dynamic Addition III

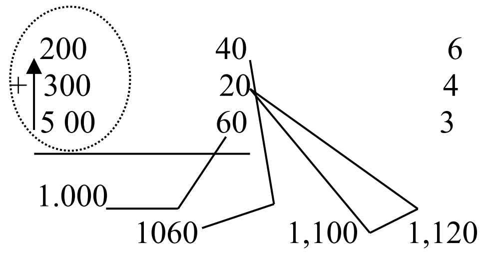


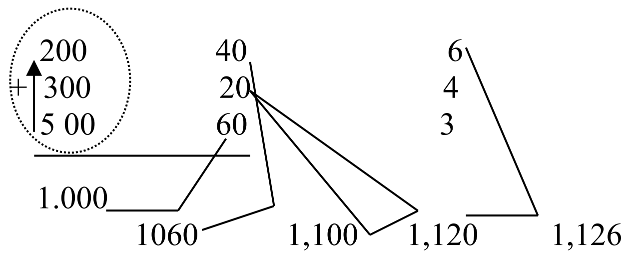
The whole mental addition should look as follows:

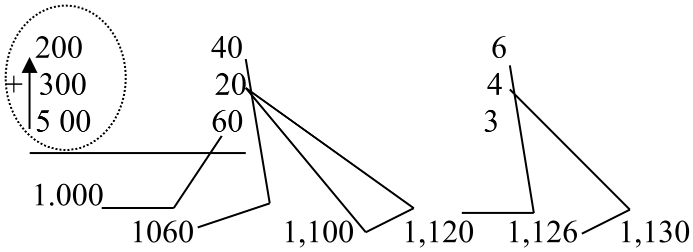
Dynamic Addition IV

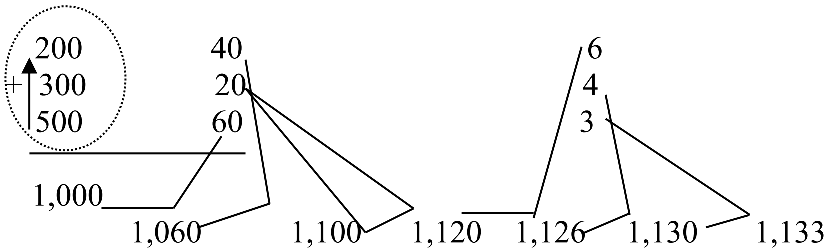






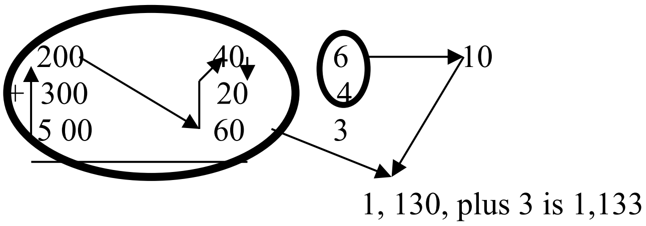






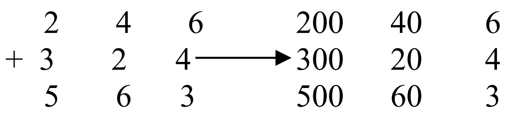
This analytic process is not a one-way street like in many algorithmic formulas. For example, upon mentally getting to 1,120 the pupil has the choice to add 10 to it by combining 6 and 4. Thus, the child could say to herself: I have 1,120 plus 10 is 1,130, plus 3 is 1,133.

The mental pathway



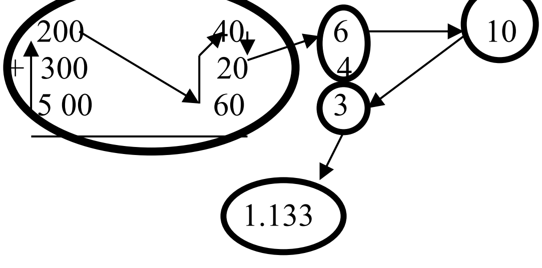
At the beginning, the teacher may present the true value of every digit as follows:

Mental Representation



(1.000 +120 +13) = (1.120 + 10 +3) = 1.130 + 3 = 1.133

The allows the pupil to add from whole to part in a holistic order:



3.4 The restructuring process of the commutative property

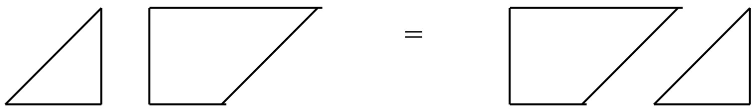
According to the commutative property the order of operations does not change the result. Thus;

$$ab = ba, \text{ or } a + b = b + a$$

The result is the same, but not the process according to Wertheimer, one of the founders of gestalt psychology.

Commutative Principle I

a + b = b + a

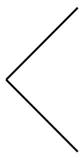


The figures on the left and right do not look alike. For example, lower left angle disappears in the right figures into a right angle. Therefore, the equation a + b = b + a is not qualitatively the same (compare Wertheimer 1964, p. 84-85) as seen in the following example:

Commutative Principle II

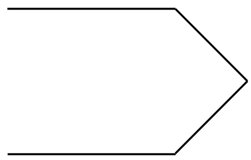
If

a =



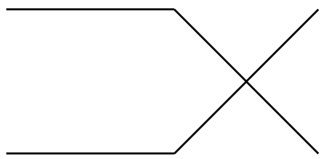
and

b =



then it means that a + b =

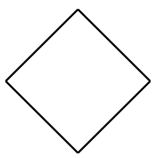
However, b + a means this:



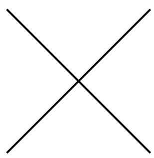
We can observe the same exchange problem (Wertheimer 1964, S. 86) in the next display:

Commutative Principle III

a b



b a



In everything we do, we must take into consideration the inner structures in their functions. We are dealing with rearrangements of perception and we must consistently keep in mind the corresponding principles. For psychological reasons the multiplication process should be more important or as important than the result. In pure arithmetic terms we can ascertain that 3 x 4 = 12 and 4 x 3 = 12 as long as the result is regarded as the only factor. However, based on the research by Max Wertheimer we should regard the commutative property with caution and be able to analyze it so that the pupils become aware of its consequences, especially in regards to the inner process.

3.4.1 The Multipliers

The latest books on arithmetic for the elementary school are designed in accordance to the commutative principle of ab = ba, as seen in *Duden: Mein erstes Rechenbuch*. Most researchers identify the first factor as the multiplier (Kolbinger 1991, Padberg 1994, Wittmann / Müller 1990). Thus, 3 x 4 = 12 as follows:

4 4 4

● ● ● ● ● ● ● ● ● ● ● ●

as opposed to 3+3+3+3:

● ● ● ● ● ● ● ● ● ● ● ●

That implies a relevant and psycholinguistic convention. Syntactically we say *three times four*. This would be the equivalent of saying: *let us multiply the four three times or let us repeat it three times*. There may be historical reasons why in Germany or in the USA it´s done that way, but it´s nevertheless a convention and not a law. There´s nothing to prohibit a pupil for example to make of the first factor the multiplicand and regard the second one as the multiplier. Such a debate in the classroom will allow the pupils to become aware of the consequences of regarding the first factor either as the multiplier or multiplicand. Either way, they should be aware because it´s a psycholinguistic problem. Thus, in the GD *Ansatz* the process is more important as the result and as such, the teacher must consider the equation $a + b = b + a$ not as a fact, but as a convention. In the case of 3 x 4, we may postulate that it´s different in its inner structure than 4 x 3. In the first case it means:

● ● ● ● ● ● ● ● ● ● ● ●

And in the second case it means:

● ● ● ● ● ● ● ● ● ● ● ●

In English or German, the first factor is considered to be the multiplier. If so, then the teacher should define the multiplication table for the five as follows:

1 x 5 = 5	5
2 x 5 = 10	5 + 5
3 x 5 = 15	5 + 5 + 5
4 x 5 = 20	5 + 5 + 5 + 5
Etc.	

In such cases, the 1, 2, 3 and 4 should convey a repeating function, that is, we should repeat the five one time, two times, three times, etc. This linguistic convention is not unique. In Mexico the sequence is presented as follows:

5 x 1 = 5
5 x 2 = 10
5 x 3 = 15
5 x 4 = 20
usw.

Syntactically we say *cinco por una cinco, cinco por dos diez, cinco por tree quince*, etc. This implies that the first factor is a multiplicand. In this convention we would interpret 3 x 4 as follows:

● ● ● ● ● ● ● ● ● ● ● ●

In the USA or German 3 x 4 is conventionally interpreted as follows:

● ● ● ● ● ● ● ● ● ● ● ●

If the teacher were to maintain that $ab = ba$, then it would be confusing, which factor is the multiplier and which one the multiplicand. Thus, it´s better to analyze the inner structures of a multiplication and define in clear and concrete terms which one is the multiplier in order to avoid any confusion in the future.

In the case of the division, we likewise must be as concrete as possible. We say for example *8 divided by 2*. What do we mean? It should mean that we have 8 objects such as 8 hearts

♠♠♠♠♠♠♠♠

and it should be clear that they are divided into two people: (🚹 🚹). In this case the two is concrete, but it´s also the divisor. As we say for example 12 ÷ 3 then the three may represent the divisor and the twelve the dividend:

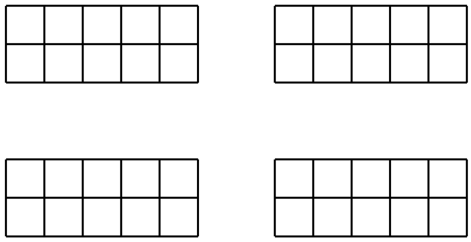
https://www.splashlearn.com/math-vocabulary/division/divisor

This means that in the division the first factor is considered the dividend while in the multiplication it´s the multiplier. It seems to be contradictory in its logic. In the division the logic is nevertheless very concrete, taking is first factor as the number to be divided by x factor, while in the multiplication the logic seems to be more abstract as it portrays the first factor as the multiplier by convention, at least in the USA and Germany.

The inner structures of the division seem to be more concrete because the first factor is considered to be the dividend. With its logic, we can proceed to show its reversibility with the multiplication with the logical consequence that in such reversible multiplication the first factor has to be a multiplicand, which goes against the convention of treating it as a multiplier. In effect we can say that 8 ÷ 2 = 4, because 4 x 2 = 8, thereby ♠♠♠♠ ♠♠♠♠ where 4 is sensed as the multiplicand.

What is a reversible operation? It means towards and backwards as in $a + b = c$, because $c – b = a$ Thus 8 + 4 = 12 because 12 – 4 = 8. Likewise with a division: $a ÷ b = z$ because $zb = a$. In this sense, there is no harmony in a reversible operation in Schröter (1987) because the first factor in the reversible operation is portrayed as a multiplier:

Division task	Reversible task
40 : 4 = ?	4 x 10 = 40
	(and not 10 x 4 = 40)



Schröter 1987, p. 208

In other words, the reversibility in the division should follow the clear logic of the addition. This implies that the first factor in the multiplication should be a multiplicand as opposed to a multiplier as Schröter tried to *enforce*. This is the clear logic that Schröter rejected:

$a \div b = z$ because $zb = a$

Instead of that beautiful logic, we have a procedure that is difficult to illustrate because its reversibility is rather awkward:

$$a \div b = z \text{ because } bz = a$$

The GD Ansatz subscribes to the thesis that the pupils should be able to arbitrarily manipulate a multiplication in which the first factor is either a multiplier or multiplicand. That brings a higher awareness and a deeper understanding of the multiplication process. Would it then be possible to consider the first factor as the multiplicand? In such a case, I would add the definite article *the* to the first factor and say *the three, four-times!*: 3,3,3,3.

Likewise, I would reform the American and German convention by adding a definite article to the multiplicand and say for example *3x the-4* as opposed to the way these two factors are portrayed in most books: *3 x 4*.

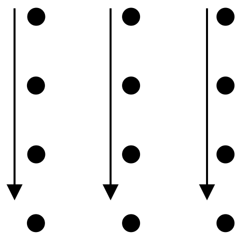
Thus, I propose that we improve the American and German convention in context of an additive conception of the process. In *3x the-4* the GD Ansatz implies an additive function of the multiplication facts or a repeating function. In the Spanish semantics we use the word *by* (por) instead of *times*, which makes it easier to denote the first number as the multiplicand.

There would be theoretical advantages in perceiving the commutative property from a more psycholinguistic reality. Instead of saying that $ab = ba$, we would then add that the process is different, although the end result is the same. The GD Ansatz then promotes a deeper level with the ability of the pupil to arbitrarily manipulate the first factor as either a multiplier or multiplicand.

Also, in the GD Ansatz the pupils are able to view the second factor as the multiplicand in accordance to the German-American convention:

A) As *3x the-4* in which *the-4* is regarded as the multiplicand:

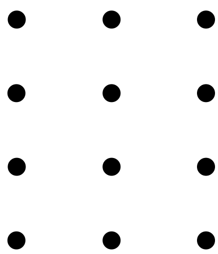
(vertical perception)



3x the-4 (three times the four)

B) As *the-3, 4x* (spoken: *the three, four times*). This is a GD convention in which the first factor may be regarded as the multiplicand within a horizontal perception:

(horizontal perception)



The three, one time or *the-3, 1x*

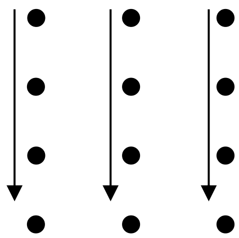
The three, two times or *the-3, 2x*

The three, three times or *the-3, 3x*

The three, four times or *the-3, 4x*

C) As *the-4, 3x* (spoken: *the four, three times*) according to the GD convention in which the first factor is conceptualized as the multiplicand within a vertical perception:

(vertical perception)

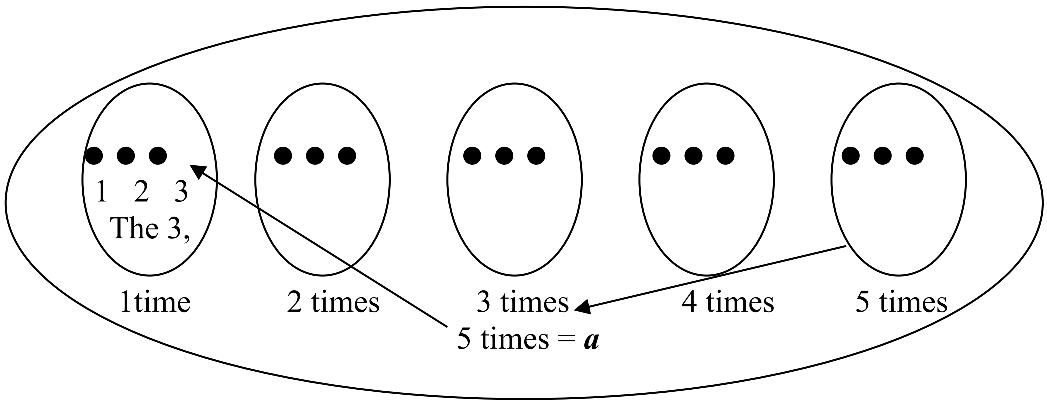


The four, one time, two times and three times

Thus, the GD Ansatz postulates a ***first-multiplicand thesis***, that is, a thesis in which the first factor arbitrarily becomes the multiplicand in order to contrast it with the German-American convention in which the first factor is seen as the multiplier. We could also call it the first-multiplier thesis in which the first factor is seen as a multiplier. By introducing the two opposing perspectives, we expect that the pupils will generate a deeper understanding of the multiplication process as a function of the addition.

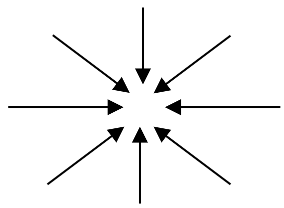
Both conventions are equally valid as suggested by the commutative property. The ability to view the first factor as either the multiplier or multiplicand may lead the pupil to be more versatile and flexible, more three-dimensional as opposed to viewing things in one or two dimensions. In the German-American convention we may have the following vectors:

The psychological field of perception with the first factor as the multiplier



One may theorize that in this perceptual process the centripetal forces go inward. The perception is directed inwardly.

Encapsulated perception going inward into the first factor as the multiplier

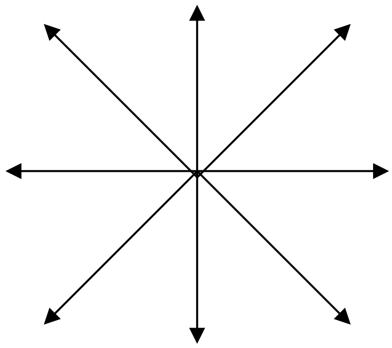


The child is trained to perceive the first factor as the multiplier by asking, for example: how many groups do you see? Then the focus of the attention would be the objects within each group. In this case, the teacher can explain:

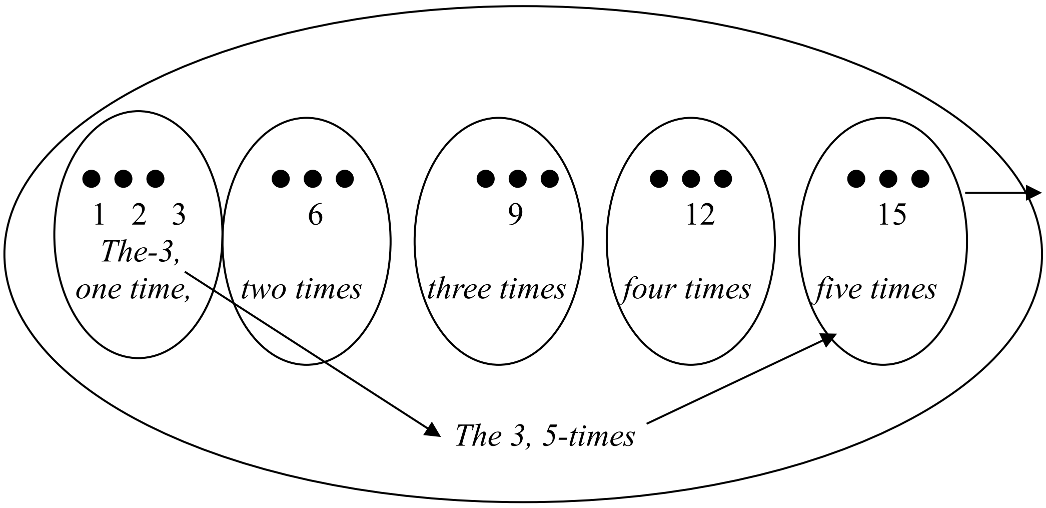
> T__ Yes, we have five times the three, thus, *5x, the-3*. Then we have six times and seven times the three:
>
> ●●● ●●● ●●● ●●● ●●●
> *5x,* ***the-3***
>
> ●●● ●●● ●●● ●●● ●●● ●●●
> *6x,* ***the-3***
>
> ●●● ●●● ●●● ●●● ●●● ●●● ●●●
> *7x,* ***the-3***
>
> Etc. in the direction of infinity.
>
> However, we can also focus our attention on the three objects and say for example, *the-3, 5x,* that is, *the three, five times, the three, six times and the three, seven times*.

With the GD convention, the pupils experience a mathematical variant. The objective is for the pupil to arbitrarily manipulate the first factor as either the multiplier as in the German-American convention or as a multiplicand as postulated by the GD thesis. In this sense, we are dealing with a more flexible perception, one able to interchange the figure for the ground and the ground for the figure. In the German-American convention the first figure is a multiplier, whereas in the GD convention, it´s a multiplicand and the pupils can manipulate both conventions at will. Here is the theoretical directionality of the vectors going outwards, instead of inwards, in this case, going outwards towards infinity, starting from a center:

The GD convention in its directionality with the center as a multiplicand



The directionality is going outwards from its center



The objective is for the pupils to master the two perspectives, one as seen from the German-American and one from the GD convention; one in which the first factor is a multiplier and one in which it´s a multiplicand. However, the GD convention may be a more natural phenomenon because if we see two objects, wouldn´t be logical to then add another group of two and a third group of two, thereby perceiving it as two, the number two times three groups? This would make the first factor, the two, as the multiplicand in accordance to the topology of the field according to Lewin (1982). The child sees two items ● ● and adds two more (● ● ● ●), etc.

The pupils could keep on adding groups of two[20] and conclude "2 and 2 equals 4, thereby adding more groups of two:

● ● ● ● ● ● Here I have six,

[20] The phrase, one group one time, two times, three times, etc. may be sensed in a way that the first factor may be viewed as the multiplicand in opposition to the German-American convention. This expression may solicit or invite the pupil to regard the first factor as the multiplicand

●● ●● ●● ●● and with two more I have eight.

●● ●● ●● ●● ●● Etc.

●● ●● ●● ●● ●● ●●

●● ●● ●● ●● ●● ●● ●● ●●

●● ●● ●● ●● ●● ●● ●● ●● ●●

●● ●● ●● ●● ●● ●● ●● ●● ●● ●●

●● ●● ●● ●● ●● ●● ●● ●● ●● ●● ●●

into infinity and likewise with three items:

●●●

●●● ●●●

●●● ●●● ●●●

●●● ●●● ●●● ●●●

●●● ●●● ●●● ●●● ●●●

If the child starts with two or three items, wouldn´t it be more natural to focus the attention on the number of items within the group? It may be so! The child could say, *Well, I have now two and two and two more, I have six. Thus, two times three is six.*

Pupils may be trained to manipulate the German-American as well as the GD convention at will. Even, within a convention, they can also illustrate the corresponding fields differently as in the German-American convention in which the first factor is a multiplier:

*4x,the-**6*** vs. The *6, **4x***

●●●●●●
●●●●●●
●●●●●●
●●●●●●

*4x,the-**6*** (horizontal view)

●●●●
●●●●
●●●●
●●●●
●●●●
●●●●

*6x,the-**4*** (horizontal view)

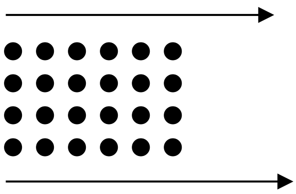
Note:

If the pupils in the first case were to see 6x 4 items (vertically) and does it consistently, it should not be viewed as an error.

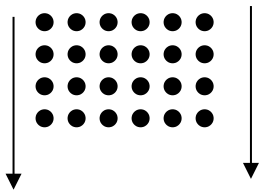
Wittmann / Müller 1990, p. 109

The comments by Wittmann und Müller (1990) indicate of the possible restructuring in which a pupil may sense the first factor to be the multiplicand. There´s a need to research the issue, whether the first factor has a tendency to be perceived naturally as a multiplicand in accordance to the GD convention. If that were to be the case, instead of a convention, we would be able to ascertain a thesis, i.e., that the first factor should be taught to be as a multiplicand because a child may naturally perceive or sense it to be so. We could start with a thesis, that the mind prefers to see the dots more from a vertical perspective as opposed from a horizontal one. I would call it the verticality thesis:

From a horizontal arrangement:



To a vertical arrangement:

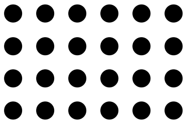


In either case, the pupils should be trained to perceive the arrangements from one particular perspective, either from the German-American or the GD convention and within a vertical perspective. The pupils should be able to master one of the two. And as soon as they become consistent in one of them, then the opposing convention should be introduced in order to achieve a deeper understanding of the multiplication process. If we are dealing with pupils in Germany or the USA, we would then follow the classical convention of perceiving the first factor as a multiplier, but if we were in Mexico, it should be the GD way of perceiving the first factor as the multiplicand because the word *por* does not mean times, but rather *by*:

3 x 1 = 3 (spoken: *tres por uno, tres*, literally *three by one three* and not *three times one three*)
3 x 2 = 6 (*tres por dos, seis*, literally *three by two six*)
3 x 3 = 9 (*tres por tres, nueve*, literally, three by three nine)
3 x 4 = 12 (*tres por cuatro, doce*, literally, *three by four twelve*)
Etc.

This *tres por uno, tres, tres por dos, seis, tres por tres, nueve, tres por cuatro, doce*, etc., allows the teacher to denote the first factor as the multiplicand because *por* (by) does not denote a multiplication as in the word *times* in the German-American convention. Thus, in Mexico it´s would be natural to teach the first factor as the multiplicand on either a vertical or horizontal arrangement:

Vertical and horizontal



The-6, 4 times (horizontal view) or *6 por 4* in Spanish

The-4, 6 times = 24, or *4 por 6 = 24*
(vertical view)

Thus, in Mexico, it would be worth it to start with the GD convention in which the pupils are trained to view the first factor as the multiplicand because the word por (by) is neutral, and allows a natural linguistic phrase with it as opposed to the word times, which denotes the first factor as a multiplier.

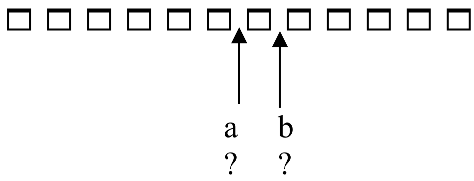
Either way, after one convention has been mastered, the opposing convention will be introduced so that the pupils have a deeper understanding of what a multiplier and multiplicand represent.

3.4.2 The vertical tendency in a point-grid like configuration in pro of the GD convention

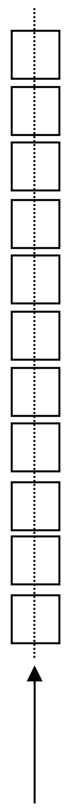
The GD convention may be found to be a more natural one if the human mind were to view things more from a vertical vs a horizontal perspective. Such a natural verticality tendency may be valid because in our human experience, more things may be view from a vertical as opposed to a horizontal perspective. When a child sees a tree, he or she is looking at it from a vertical perspective. People, trees, plants, and most of the objects in our environment may be seen from a vertical perspective, but we might also see a row of houses, or trees. A train, on the other hand, is perceive as a long row of moving wagons, that is, from a horizontal perspective. The question, is, which tendency is more prevailing in our environment, a horizontal or a vertical one?

On the other hand, when we have a row of items, like up, one by one, it is not easy to detect the middle right away. Given the same figures, if seen as a row in a horizontal position vs the same on a vertical position, it´s easier to sense the middle in a vertical position:

Where is the middle?

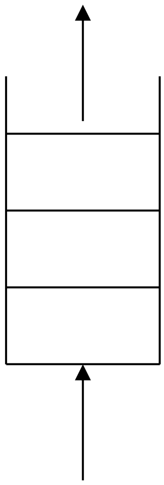


The middle is easier to detect if given in vertical position



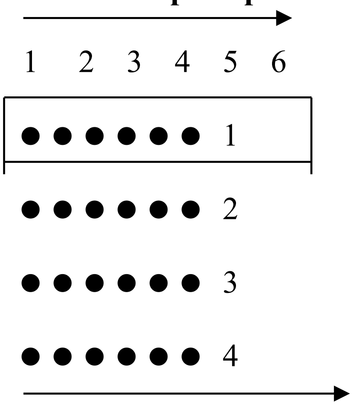
That is more or less the midpoint of the column

Given a horizontal row of things, one is not very sure where the midpoint should be and if we´re dealing with a big number of things, it would be even more difficult to calculate the midpoint by looking at them. However, if we perceive items vertically, it seems to be easier to sense where the midpoint should be:



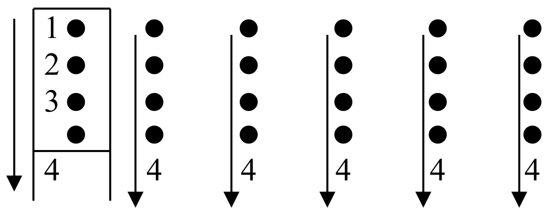
The GD thesis assumes a tendency to detect things from a vertical position as opposed to a vertical one. This tendency may be significant when dealing with a grid of dots, either presented horizontally or vertically. The GD thesis assumes that the human mind can detect things better if they are presented in a vertical position as opposed to a horizontal one. In other words, vertical vectors may have more influence than the corresponding horizontal vectors. A teacher may use the law of proximity to show that groups of dots should be viewed horizontally as follows:

Horizontal perception



However, using the same law of proximity, the teacher may show that it´s easier to perceive the same dots on vertical positions and interpret the corresponding multiplication in which the first factor is a multiplicand:

Vertical perception of groups as seen from the GD convention of the multiplication:

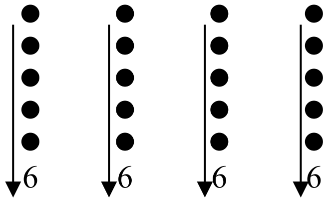


The-4, 6 times = 24

According to the GD convention, the pupils should be trained to interpret the grid of dots as follows:

1___ Here on the left we see 4 small plates going downwards
2___ The 4 repeats itself.
3___ Thus we have, 4,4,4,4,4,4 or *the-4, 6 times*.

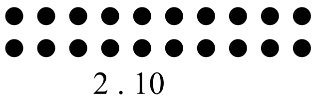
In essence, we can train the pupils to regard the first factor in a multiplication as the multiplicand and in order to avoid any confusion, the multiplicand should always be accompanied with the definite article *the* as in *the-4, the-6, the-7*, etc. Thus, if the pupils want to illustrate *the-6, 4-times*, which we abbreviate as *the-6, 4t*, the pupils should be able to illustrate it as follows as seen from a vertical position:

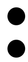


The-6, 4t = 24

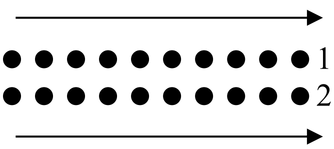
Mastery of both the German-American and GD conventions, along with the verticality thesis, should allow the pupils to gain a deeper understanding of the multiplication process. That way, optical mistakes could be avoided. For example, in *Mein erstes Rechenbuch* (My first arithmetic book, Duden 1994) we have the following point constellation with its corresponding multiplication in which the first number is presented as a multiplier:

(● = shells)
Wieviel sind 2 mal 10 Muscheln?
(How much is 2 times 10 shells?)

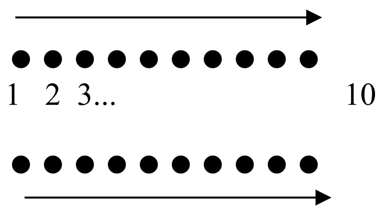


In this illustration, the author did not mean to illustrate the 2 in a vertical way as: 

However, in this illustration it´s easier to detect the 2 vertically. The author´s intention was to represent the 2 as a multiplier horizontally as in two rows, each have 10 shells.



It would have been much more effective if the author had considered the Law of Proximity in gestalt psychology, presenting the 2 as a multiplier with two horizontal rows as follows:



As to the GD verticality thesis, I would have presented two columns vertically and clearly state that the 2 is a multiplier, by accompanying the multiplicand with a definite article:

● ●
● ●
● ●
● ●
● ●
● ●
● ●
● ●
● ●
● ●

Here are two columns, each with 10 shells
Thus, *2-times, the-10* or we can also say *the-10, 2-times*

Another example comes from the representation as given in lecture on February 5, 1996 by Prof. Bauer at Universität Passau in Germany. He described the multiplication as a temporal continuation of the addition as follows:

> A truck drives three times to a building site. Each time it loaded four boxes. How many boxes did it take altogether?[21]
>
> A) Graphic description (zeichnerische Darstellung):
>
> ☐ ☐ ☐ ☐
> ☐ ☐ ☐ ☐
> ☐ ☐ ☐ ☐
>
> B) Arithmetic description (rechnerische Darstellung): 4 + 4 + 4 = 12
>
> 3 . 4 = 12

According to the GD convention and to the GD verticality tendency, a child would perceive the graphic description as follows:

☐ ☐ ☐ ☐
☐ ☐ ☐ ☐
☐ ☐ ☐ ☐
3 3 3 3

The-3, 4-times = 3 + 3 + 3 + 3

One can debate, from the linguistic point of view, the description as *the-3, 4-times*, according to the GD convention or *4-times, the-3*. It all depends on the linguistic convention of the country. However, the Law of Proximity in gestalt psychology should be followed in order to avoid any optical confusion even if the teacher wants to pupils to perceive the first factor as the multiplier. According to the GD thesis, it´s much better to teach both conventions so that the pupils may reach a deeper and more flexible understanding of the multiplication process.

At any rate, we should follow all the laws of gestalt psychology as presented by Katz 1944) and we should also take into consideration any field vectors in order to analyze any tendencies, especially in

---

[21] Original text in German: *Ein Lastwagen fährt dreimal zur Baustelle. Jedes Mal hat er vier Kisten geladen. Wie viele Kisten fährt der Wagen insgesamt?*

regards to what I consider to be a tendency to view things from the verticality point of view. In the case of Prof. Bauer at Universität Passau, the Law of Proximity could had been easily presented as follows:

A truck loading four boxes three times as per the Law of Proximity
and from a horizontal point of view

□□□□4

□□□□4

□□□□4

The same description with the Law of Proximity, but perceived vertically, either as *3-times, the-4* (German-American convention) or *the-4, 3-times* (GD convention):

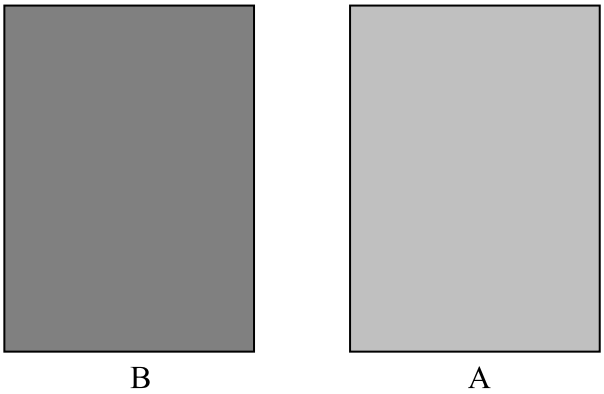
□ □ □
□ □ □
□ □ □
□ □ □
4 4 4

If the arithmetic books want to promote horizontal vectors in their arithmetic descriptions, they should at the very least take the Law of Proximity into account. Otherwise, optical confusion may be the outcome. Given a German-American convention, it assumes that the pupils would first be able to perceive the entire description in order to know how many groups. In this case, we are dealing with three groups of four boxes each. That implies a conscious perception, thereby identifying how many groups we´re dealing with and then knowing how many items each group has. That is the German-American convention. However, the pupil may decide to start with the number of items in the first group and be fixated on it as the focus of attention. In this case, the first factor could become the multiplicand and the second one the multiplier as in the GD convention. What´s essential is that the pupils perceive the configuration in a consistent way.

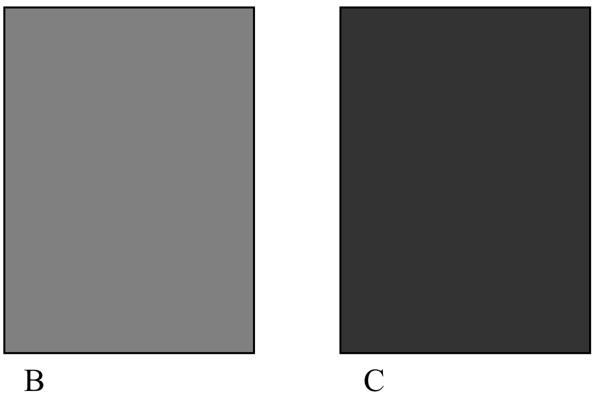
3.4.3 Holistic relativity in perception

A whole can also be a part of something bigger. If we were to see three houses, one after another, do we see three houses or do we see one house, and then a second and a third one? It all depends on how we look at things, either holistically or as part of a bigger whole. In other words, do we see the first factor of a multiplication as a multiplier or multiplicand. It all depends on how we look at things. A word for example is a part of a sentence, but it´s also a whole relative to a syllable or a letter. Thus, it´s all relative. Even in classical conditioning, it´s shown that it can be relative. For example, a dog may be conditioned to choose a door, next to a lighter one. Thus, the dog may be trained to choose door B, but not in absolute terms, but because door B is the darker of the two doors. That can be seen in the next experiment when the dog is confronted with a new pair of doors, in which door C is darker than door B. In this case, the dog would choose door C although it was not trained to choose this door, but it chooses this new door because it´s darker than B. Although the dog was conditioned to choose door B, now suddenly in the second experiment, chooses door C, one which the dog had not seen at all.

Relativity in the Pavlovian experiment

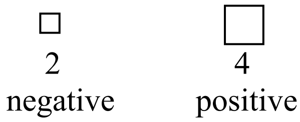


In the classical conditioning phase, the dog chooses door B because it gets a stimulus.



In a second experiment door B is rejected in favor of a darker door. This shows that the dog had been in reality choosing door B relative to door A in the first experiment. The dog had been conditioned to choose in relative terms. It´s been shown that relativity also has its limits. The second experiment would work if it did not deviate too much from the classical conditioning in the first experiment (Bryant 1974). It was shown that within a small transposition in the second experiment, the relativity was still valid. Four-year old children would react positively to a size 4 vs a size 2:

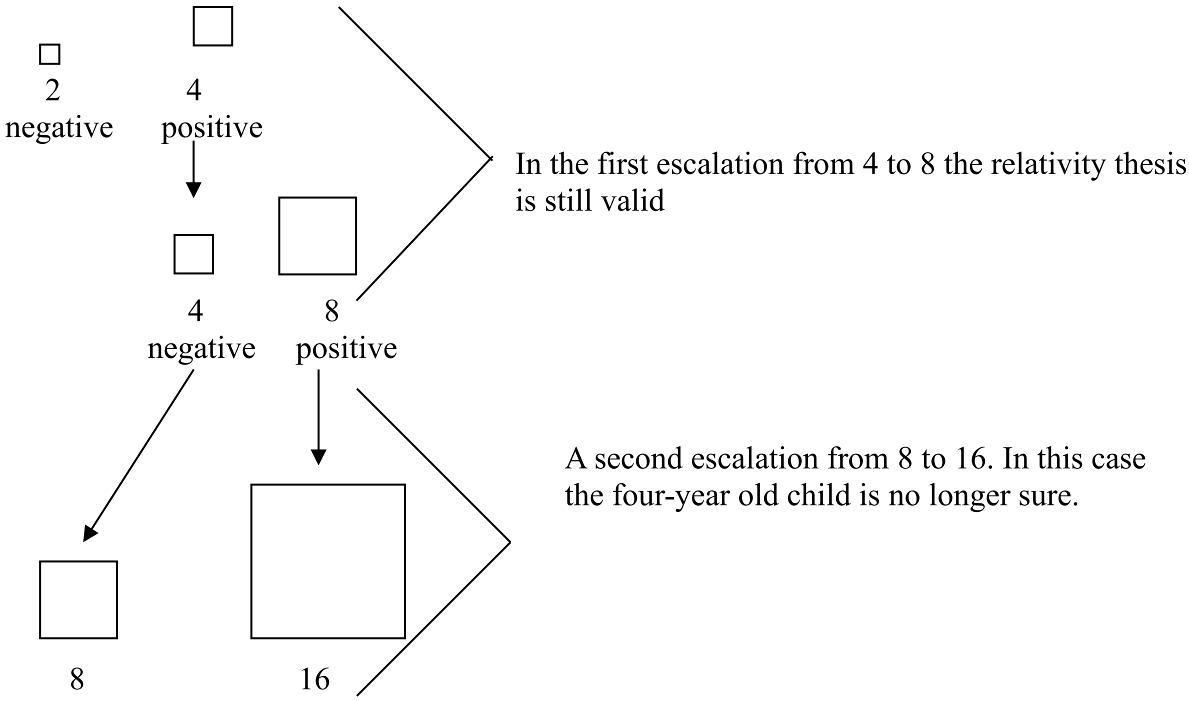
Implementation with children I



A four-year old child chooses 4 relative to 2.

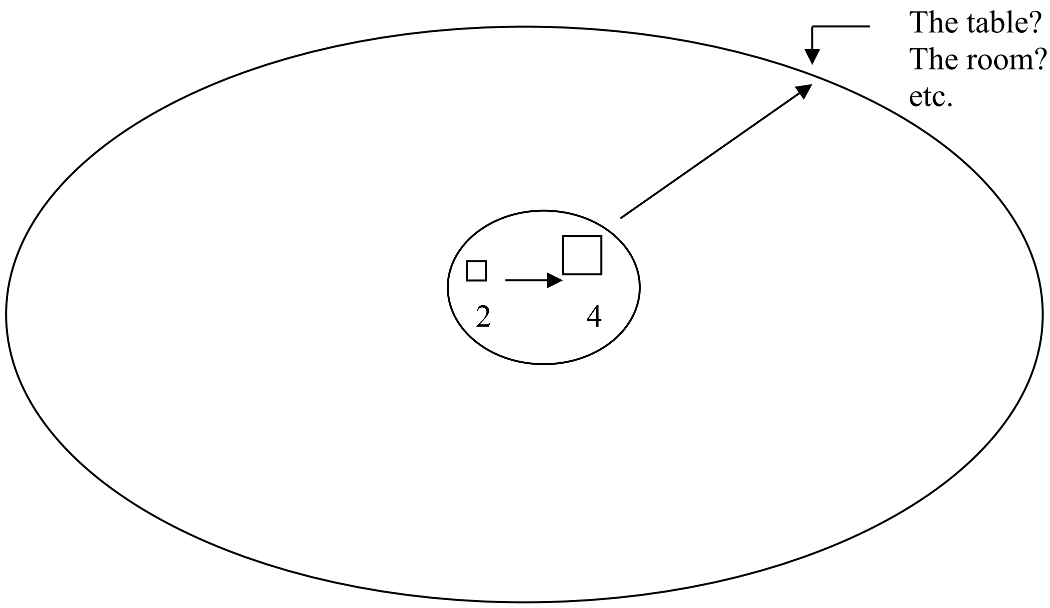
It was found that in the first doubling the children reacted relative to the size of the other number. Thus, if confronted with a pair of 4 vs.8 in size, the four-year old children would choose 8, although they were conditioned to choose size 4 because 4 is twice as big as 2, just like 8 is twice as big a 4. However, when the escalation is too far from the first conditioning, the relativity thesis was no longer valid.

Implementation with children II

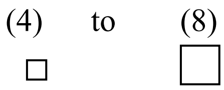


Within the first escalation from 4 to 8 the whole perspective of a child is still relatively intact. The relations are still within reach. The perceived field of things are still comprehensible, but with more escalations the whole configuration becomes too abstract. What we do know is that within the contrast between 2 and 4, the child chose 4 relative to 2 and likewise 8 relative to 4. According to Bryant (1974) we are dealing with two relations:

Bryant´s Relations of two main factors

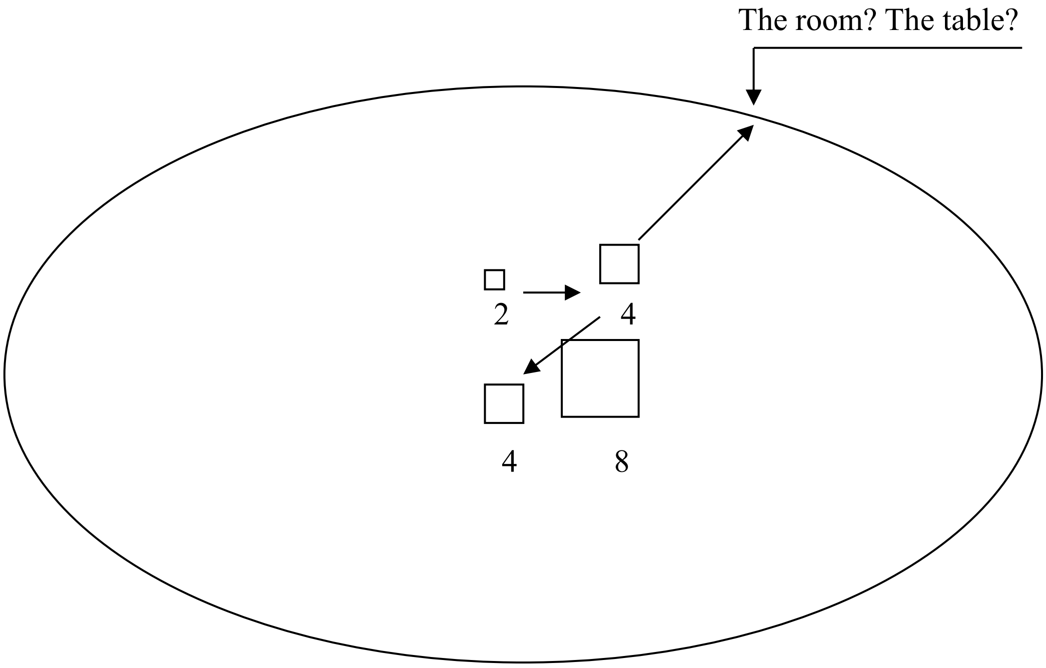


This means that the human mind must take into consideration at least two relations: a) the inner relation of the two factors in question, size 2 vs size 4, and b) these two factors against the background. That is a possible thesis. According to the GD thesis, in the first escalation there is no bewilderment and no optical confusion between the two factors with their surroundings:



In this case, the surroundings to not play an important role. There is no deviation within the inner relations:

Inner relations

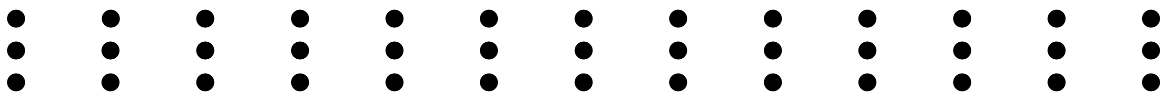


As the figures become greatly magnified in the first transformation the background is still rather static. However, by thy third or fourth transformation, the relations are too big to perceive in absolute or relative terms. Sure, from 4-to-8, the field of perception is clear, but not anymore if we double it to 8-to-16, 16-to-32, and definitely not form 32-to-64. Beginning in the third transformation the animals would not be able to react in any way. Going from 2 to 4 into 4 to 8 is still within a perceptible field, going into the third or fourth transposition will make the relations irrelevant and uncomprehensible. In the third, fourth or fifth transpositions, the relativity will be lost because it´s too distant from the conditioning effect of the first two factors.

Experiments with 8-year-old children show that within a third transformation it may be possible to perceive the relativity of things, but that in itself implies that the children are capable of higher abstract thoughts (Kuenne 1946). Surely, it should be possible for children, even at an early age of 4, to find some kind of a pattern, such as finding out the bigger of the two figures.

As long as the relations are perceived in a clear way, and as long as the distance to the first conditioning is not too distant, it should be possible to ascertain the relativity. For example, given two pairs such a 3-to-4 and 4-to-5, children would be conditioned to choose the bigger number. In such a case, the relativity was kept even at great distances, for example within the third, fourth or fifth transpositions because the children understood that the essence of the exercise was to find the bigger number Bryant (1974). Thus, if confronted for example with5-to-6, 6-to-7, 7-to-8, etc., they were able to choose the bigger number, because that was in the conditioning effect. In essence, children would understand that the issue here was to find out which number was bigger. Thus in 4-to-5, 5-to-6 or even 6-to-7, children would choose the bigger number, even within the third or fourth transformation. That may be possible, because it became clear in the conditioning stage

In the GD thesis, how a child perceives 3 + 3 + 3 + 3 depends on the conditioning effect, i.e., on the linguistic and perceptual convention. If the child is trained to say *times* as in *4 times three equals 12*, then we´re dealing with the German-American convention. However, according to the GD thesis, this convention could be invalidated with some perceptions if the items are given within a verticality effect such as the following:



In this case, the child may be influenced verticality of the groups and instead of counting the number of groups and say 13 times 3, in accordance to the German-American convention, he or she may perceive it as 3 times 13, where 3 becomes a multiplicand as per the GD convention. Thus, although the children may be conditioned to view the first factor as a multiplier, in this case, the children would rather view the items within the first group as the factor to be multiplied, in this case, they would see 3 dots and then count the number of groups, making it into 3 times 12.

Whether they view the first factor as a multiplier or multiplicand, according to the GD thesis, it´s practical to add the definite article *the* to the multiplicand in order to make it very concrete that it is in fact a multiplicand and the other number the multiplier. Thus, they could interpret these groups as *3-times the-4*, which could be abbreviated as *3-t, the-4*, in which the first factor is clearly the multiplier as per the German-American linguistic convention, or they could view it as the *the-4, 3-times* (abbreviated as the-4, 3-t) and spoken as *the four, three times* as per the GD convention.

4.0 Algorithms in fractions are formulas that have nothing to do with reality:

A critical view according to Gestalt-Dialektik:

a holistic philosophy

of education[22]

∞

Algorithms in fractions are formulas that have nothing to do with reality

In the division of fractions, at least in the way I was taught in Acámbaro, Gto. México, I learned to cross-multiply diagonally as follows:

$$\frac{1}{2} \div \frac{1}{4} = \frac{4}{2} = 2$$

In the USA the numerator and denominator of the second fraction are flipped and then the two are multiplied horizontally:

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$$

In the final analysis, we are dealing with a formula in both countries, albeit a formula that says nothing about reality. **The students just get a result, but they do not know its meaning.** However, I could give some credit to the American algorithm whereby in the second fraction (as a divisor) the numerator and denominator are flipped. This would make sense, if and only if, the teacher were to explain to the students the logic behind it, such as multiplying one-fourth by its reciprocal:

$$\frac{1}{4} \times \frac{4}{1}$$

And if one-fourth is multiplied by four-ones, so should one-half as well:

$$\frac{1}{2}\left(\frac{4}{1}\right)$$

In such a case, we would end up having the following scenario:

$$\frac{1}{2}\left(\frac{4}{1}\right) \div \mathbf{\frac{1}{4}}\left(\mathbf{\frac{4}{1}}\right) =$$

In mathematical terms, we may then be able to transform {***one-fourth times four-ones***} into ***one***:

[22] http://www.gestaltdialektik.com/index.php?area=philosophicum&areaSub=introduction_gestaltdialektik

$$\mathbf{\frac{1}{4}\left(\frac{4}{1}\right)} = 1$$

Thus, we have the following scenario:

$$\frac{1}{2}\left(\frac{4}{1}\right) \div 1$$

Furthermore, we should also take into consideration that everything that is divided by itself is equal to itself:

X ÷ 1= X

3 ÷ 1 = 3 7 ÷ 1 = 7 49 ÷ 1 = 49 etc.

Thus,

$$\frac{1}{2}\left(\frac{4}{1}\right) \div 1 = \frac{1}{2}\left(\frac{4}{1}\right)$$

In essence, we go from:

$$\frac{1}{2} \div \frac{1}{4}$$

into:

$$\frac{1}{2}\left(\frac{4}{1}\right) \div \frac{1}{4}\left(\frac{4}{1}\right)$$

and then into:

$$\frac{1}{2}\left(\frac{4}{1}\right) \div 1$$

Thereby:

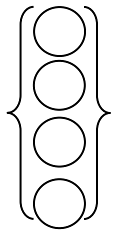
$$\frac{1}{2}\left(\frac{4}{1}\right)$$

and finally into:

$$\frac{1}{2}\left(\frac{4}{1}\right) = \frac{4}{2} = 2$$

Once I asked a university student out of Nicaragua if he could explain to me the result of one {*half-divided by one-fourth*}. He responded by saying that two represented whole numbers, which is completely illogical. If we say that 2 represents whole numbers, then it follows that 2 must represent two real objects as in *two pizzas*. How can we get two pizzas out of {*one-half of a pizza divided by* *one-fourth of a pizza*}? Likewise, how can we justify, for example, {*one-half by one-eight equals four*} in terms of four objects? Whole numbers are called whole numbers because they must reflect some kind of quantity in real terms as in the following example:

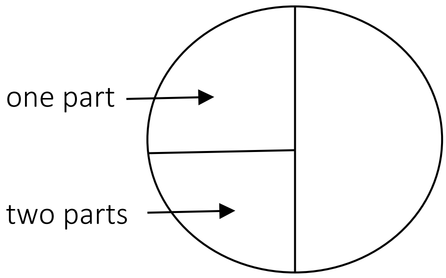
$$\frac{1}{2} \div \frac{1}{8} = \frac{1}{2} \times \frac{8}{1} = \frac{8}{2} = 4$$



Later on, another student explained to me that the answer becomes “undefined” because it has no concrete definition. If the result remains mathematically undefined, we would have to conclude that the 4 remains in the air without having any relation with life itself. This, in my opinion, is ridiculous.

Antithesis: a practical and linguistic reasoning of the fractions

There are people who are very lucid in their mind because they can relate to everything in a very clear and tangible way, even after years of not having studied mathematics. Once I asked a real estate manager if she could give me a logical explanation as to why {*one-half divided by one-fourth equals two*}. Right away she answered that if we were to take one-half of a whole (for example of a pizza) one could further divide that half into two parts:



This holistic interpretation of the fractions is the only one that makes practical sense. In other words, when I say:

$$\frac{1}{2} \div \frac{1}{4}$$

we are dealing with a syntactical conception to the problem, which requires a linguistic interpretation and not a mathematical one. The root of the problem in the division of fractions can best be reflected in terms of language and not in mathematical terms. According to Gestalt-Dialektik (GD), my own holistic philosophy of education (www.azul-celeste.com), arithmetic is a function of language and not the other way around. In this case, language takes precedence over the arithmetic functions because number sense is based on syntax or syntactical equations and syntax is a part of grammar; syntax plays the main role, the main figure, and not a numerical configuration as portrayed by the classical algorithm in the division of fractions.

According to my antithesis, we must reconfigure the problem linguistically and instead of saying:

{*One-half divided by one-fourth*}

we should ask:

{*How often does one-fourth fit into one-half?*}

Thus, it is no longer valid to say {*one-half divided by one-fourth*} because it leads to a fallacy: the 2 would have to be interpreted in terms of whole numbers as in two objects. The only way to interpret {*one-half times four-ones*} (1/2 x 4/1) is by referring to 2 as in *two pizzas* or any *two objects*. There is no other interpretation to {*one-half times four-ones*} and thus 2 as in *two pizzas* cannot be the answer to the original question (*What is one-half divided by one-fourth?*). Under this GD antithesis, we want to find out how many times does **one-fourth** fit into **one-half**. How many times? Two times! Therefore:

$\frac{1}{2} \div \frac{1}{4} =$ 2-t ¼ in ½ (i.e., "twice fits one-fourth in one-half")

Thus, saying for example {*one-half divided by one-fourth*}:

$$\frac{1}{2} \div \frac{1}{4}$$

is linguistically and logically awkward. Instead of looking at the issue in terms of *dividing one-half by one-fourth,* we should be asking: *How often does one-fourth fit into one-half?*
Therefore:

$\frac{1}{2} \div \frac{1}{4} =$ 2-t ¼ in ½ (i.e., "twice fits one-fourth in one-half")

Another way of interpreting the problem is as follows: *How often does one-half can be divided into fourths?* The answer is *twice* because:

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Likewise, we could interpret the following problem:

$2 \div \frac{1}{2} =$ 4-t ½ in 2 ("four times fits ½ in 2")

in accordance to the following illustration:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

one two three four "times"

Therefore, 2 ÷ ½ = 4-t ½ (four times ½ in 2) because 4 x ½ = 2 where the 4 becomes the multiplier and not the multiplicand.

4.1 Contribution by Stefanie Grotz[23]

A dialogue as dialectical foundation in order to refine the Ansatz (the procedure) of GD

Yes, so it is very good, there you explain what the position of GD is and why. However, one should also clarify that our main concern is to clarify fundamental problems of mathematics. When somebody wants to go deeper into mathematics, then at some point it is clear that {*one-half divided by one-fourth*} is two and one must accept the rule just as it is given in order to be able to process everything in a simple manner. **One cannot reconstruct every operation based on the logic of the addition; that would be very uneconomic.** Nevertheless, it is fundamentally important to understand at the beginning why we do things in certain ways and why the rule functions. In that respect there isn´t enough emphasis in education.

However, I would no longer be able to say that {*one-half divided by one-fourth is two*}. That would be unprofessional because at some point one must be able to reflect a professional language.

4.2 Answer from Gustavo Vieyra

A generalization-synthesis according to Gestalt-Dialektik

You write: "One cannot reconstruct every operation based on the logic of the addition; that would be very uneconomic."

That is exactly the point! **One should, in any case, be able to reconstruct every operation based on the logic of the addition**, especially in context of a mental *blitz* in a manner that is highly economic and practical.

The student should be able to experiment with the **"right predicative language"** and with all the corresponding mental demands in order to internalize a new way of thinking. At a certain abstract level, it is fully OK to determine that {*one-half divided by one-fourth equals two*}, but only as generalization of the GD-Postulate:

$$\frac{1}{2} \div \frac{1}{4} = \text{2 times the ¼ into ½} \text{ (thus: "twice fits a fourth into a half")}$$

At the beginning the children must internalize unconditionally the entire **predicative process** based on the logic of the addition. Only then it is fully OK to abstract the 2 through a **mental-predicative *Ansatz***. What does it mean to abstract the 2 in accordance to a mental-predicative *Ansatz*? The students must understand that the 2 does not represent a normal 2. It is, for example, wrong to determine that the 2 in 24 is 2. One makes the addition 9 + 9 + 6 and says, *it is equal to 24*. Up to that point everything is OK. However, when we´re confronted with a more complex addition such as 19 + 29 + 16 then we have syntactic problems. In the USA we may be able to present an addition vertically:

$$\begin{array}{r} 1\,9 \\ +\,2\,9 \\ \underline{1\,6} \end{array}$$

[23] As a former colleague of GD, Stefanie Grotz is a graduate from the University of Passau, Germany in Culture and Economics (KulturWirtschaft) with a Master´s Degree.

In the basic algorithm for additions, we say then {*9 + 9 + 6 equals 24; we put the 4 and we carry the 2*} as follows:

$$\begin{array}{r} ^{(2)} \\ 19 \\ 29 \\ \underline{16} \\ 64 \end{array}$$

That is precisely the linguistic problem because the 2 in 24 is not a 2, but rather a 20. Linguistically, mathematically and logically speaking it is a fallacy to say {*I put the four and carry the 2*}. It is false and illogical to make such a statement. Thus, I propose that we think differently, thereby internalizing a new "**mental-predicative *Ansatz***" (in terms of a paradigm shift so to speak) in order to eventually be able to internalize something so illogical. In other words, let the student be aware of the illogical principles at hand. In order to get to a higher level of abstraction one must start with the **logic of the addition**, such as in the following example:

> {*Twenty-nine plus nineteen equals twenty-nine plus ten, plus nine; that is equal to forty-eight*}. Numerically: *29 + 19 = [29 + 10 = 39] + 9 = 48*
>
> {*Forty-eight plus sixteen equals forty-eight plus ten, plus six . . .*}
> Numerically: *48 + 16 = [48 + 10 = 58] + 6 = 64*

This long addition is not economic, but it is an effective approach in order to introduce later on the illogical, but faster algorithmic format. If all the factors have been holistically internalized, that is to say, if the students are conscious of the holistic principles at hand, then it would be acceptable to generalize an illogical algorithm, but only as an end product in context of the **mental-predicative *Ansatz* in accordance to the philosophy of Gestalt-Dialektik.**

In other words, what is it that we need to accomplish in order to make sure that the 2, as an answer to the aforementioned fraction, **does not represent a whole number**, that is, **a 2 as in two pizzas**? In order to bring the student to a full and complete understanding of the predicative approach of GD we must introduce fractions in accordance to the logic of the addition as follows:

$\frac{1}{2} \div \frac{1}{4} =$ 2-t the ¼ into ½ (**a predicate of**: "two times fits one fourth in one half")

Making the claim that {*one-half divided by one-fourth equals two*} does not give any clarity in the process. The student gets in theory the right answer; however, the answer is laden with the wrong interpretation because the *two* must be represented as *two objects*, which is completely illogical. One cannot get, for example, two pizzas as the answer to the question, *what is one-half of a pizza divided by one-fourth?* That would be like making something out of nothing. It´s impossible to get two pizzas or for that matter any whole units out of the division of fractions. What we get are numbers that function as a **multiplier**, as opposed to a **multiplicand** of a multiplication. In other words, the answer to a division of fraction is an **implied multiplication** in which the first term is a **multiplier**. Thus, when we say 2 as in

$$\frac{1}{2} \div \frac{1}{4} = 2$$

then that *2* is referring to *two-times fits ¼ into ½* in the sense of (2) (¼) = ½. Regardless of how we want to analyse it, the *2* is always functioning as a multiplier. Thus, I propose a mental connotation by introducing the symbol *t* in order to give out a short, quick and precise answer such as for example {*one-half divided by one-fourth equals two-t*} as follows:

$$\frac{1}{2} \div \frac{1}{4} = 2\text{-t}$$

This approach is a logical *connotative-predicative-linguistic-mental solution* to the division of fractions.

When the children understand what the 2 means, in other words, **the 2 as a multiplier** (two times) and not as a multiplicand, then the predicative-connotative short version is a good solution to a fast and quick answer: {*one-half divided by one-fourth is two-t*}.

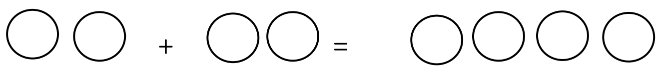
4.3 Thesis of Gestalt-Dialektik

Most students at the end of high school and at the university level in Germany, the USA and Mexico may or may not be able to comprehend the meaning of 2 in: {*One-half divided by one-fourth is two*}

If they do, then it is because they learned their arithmetic facts in their elementary school in terms of a broad holistic philosophy of **number sense**. However, if the students learned the fractions and other arithmetic facts via the classic mechanical approach, e.g., by **flipping one-fourth into four-ones and multiplying horizontally**, then most of them would not be able to illustrate with any logic the meaning of the fraction. In others words, if asked to draw some graphic drawings in order to show the procedure as to how they ended up with two as the answer, most students will not be able to make any logical sense of the process and thus, they will get confused with some illogical conclusion. And why is that? That is because they would not have comprehended, holistically speaking, that *2* as the answer to the fraction cannot mean something like *two pizzas*, but rather as a **simple multiplier** in the sense of *twice* as in {*one-fourth fits into one-half twice*}.

According to this thesis, we may be able to go to any university, high school and especially to students in the upper elementary school grades and show them how to illustrate for example:

2 + 2 = 4



Having done this, we would then ask the students to make a similar illustration as to the fraction:

$$\frac{1}{2} \div \frac{1}{4}$$

If the students had experienced, in the elementary school years, the facts of arithmetic via a holistic philosophy of number sense, then they may come up with a logical illustration. Otherwise, most will get confused and won´t be able to show any logic behind the numbers, especially those upper elementary school grade students who learned the division of the fractions by flipping the second fraction and multiplying horizontally. The most basic and simple solution is to be practical and give for example the following interpretation:

> Look here is a pizza and if I were to divide it in two then we have two halves. Now, how many fourths fit into one-half? . . . (pause) . . . Two! Thus, *one-half divided by one-fourth is two*, as in *two times the one-fourth fits into the one-half!*

I predict that most university students will even have a lot of trouble explain more complex fractions such for example:

$$\frac{1}{2} \div \frac{1}{3}$$

By flipping the 1/3 into 3/1 and multiplying across they may come to the right answer, but nevertheless most university students, especially those in the humanities, will not be able to explain the logic of the answer. They will be fooled by the classic algorithm:

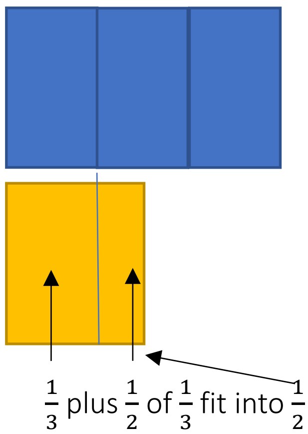
$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2} = 1\frac{1}{2}$$

Here is the logic according to my GD holistic interpretation:

a) How many times does $\frac{1}{3}$ fit into $\frac{1}{2}$?

b) $\frac{1}{3}$ fits into $\frac{1}{2}$ one and a half times!

$$\frac{1}{2} \div \frac{1}{3} = 1\frac{1}{2} \text{ times fits } \frac{1}{3} \text{ into } \frac{1}{2}$$



In accordance to the “GD predicate Ansatz”, we must then write:

$$\frac{1}{2} \times \frac{3}{1} = \frac{3}{2} = 1\frac{1}{2}t$$

Orally we would say for t “times” or $1\frac{1}{2}$ *times* fits $\frac{1}{3}$ into $\frac{1}{2}$.

4.4 Critical review of Mathe-Stars 6: mechanical and senseless

The division of fractions on pages 36-39 from **Mathe-Stars 6**[24] is mechanical and senseless from a practical point of view. **The algorithmic representation has nothing to do with logic or reality.**

****

Division of fractions (1)

1 Reciprocal value

a) For each fraction one can get a reciprocal value.

Fraction: $\frac{3}{4} \rightleftarrows \frac{4}{3}$ reciprocal

Form the reciprocal value:

$\frac{2}{3} \quad \frac{3}{\square} \quad \frac{6}{8} \quad \frac{\square}{\square} \quad \frac{9}{5} \quad \frac{\square}{\square} \quad \frac{7}{10} \quad \frac{\square}{\square} \quad \frac{2}{4} \quad \frac{\square}{\square}$

b) Natural numbers may be transformed into their respective reciprocal values.

$5 = \frac{5}{1}$ $\qquad$ reciprocal *value* $\frac{1}{5}$

Form the reciprocal value:

$8 \ \frac{\square}{\square} \qquad 6 \ \frac{\square}{\square} \qquad 3 \ \frac{\square}{\square} \qquad 2 \ \frac{\square}{\square} \qquad 9 \ \frac{\square}{\square}$

2 Division of fractions

One divides by a fraction by multiplying by multiplying by the reciprocal.

$\frac{1}{3} : \frac{2}{5} = \frac{1}{3} \cdot \frac{5}{2} = \frac{1 \cdot 5}{3 \cdot 2} = \frac{5}{6}$ $\qquad$ $\frac{1}{4} : \frac{7}{9} =$

$\frac{1}{7} : \frac{3}{5} = \frac{1}{7} \cdot \frac{5}{3} =$ $\qquad$ $\frac{1}{6} : \frac{1}{5} =$

$\frac{1}{5} : \frac{1}{3} =$ $\qquad$ $\frac{2}{5} : \frac{3}{7} =$

$\frac{1}{2} : \frac{5}{7} =$ $\qquad$ $\frac{1}{5} : \frac{2}{3} =$

$\frac{1}{3} : \frac{4}{7} =$ $\qquad$ $\frac{2}{9} : \frac{5}{6} =$

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[24] Mathe-Stars 6 (2015). Werner Hatt, Stefan Kobr, Ursula Kobr, Elisabeth Plankl & Beatrix Pütz (eds.). Berlin: Cornelsen Schulverlag GmbH, www.oldenbourg.de

The representation of the division of fractions on pages 36 to 39 is mechanical and thus has nothing to do with the practical reality of a student. It deals with an amount of numbers that have no practical meaning for the students at all.

4.5 Critique on an X Quadrat MATHEMATIK 6 BADEN-BÜRTTEMBERG: At the beginning is the approach holistic, but at the end relatively mechanical

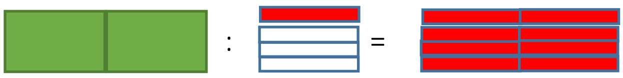
What about a more holistic perspective in order to make a better representation of the natural numbers? The approach on page 31 in the book *X Quadrat MATHEMATIK 6 BADEN-BÜRTTEMBERG*[25] is excellent at the beginning because of its holistic representation:

---

2 We divide surface areas by portions. Look carefully what happens:

a) How often does the red fit into the green area?

2 wholes are divided into fourths: $2 : \frac{1}{4}$



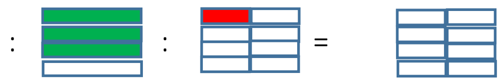
$$2 : \frac{1}{4} = 8$$

b) How many red fifths fit in 3 green wholes?

3 wholes are divided into fifths.

$$3 : \frac{1}{5} = ? \quad ? \quad ?$$

C) How many red eights fit in 3 fourths of a whole?
3 fourths are divided into eights.



$$\frac{3}{4} : \frac{1}{8} = ?$$

X Quadrat MATHEMATIK 6 BADEN-BÜRTTEMBERG 2017, p. 31

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During this process the students are able to experience the division of fractions in a real holistic process, beginning with a practical context and with the right linguistic application of the sentence: **how often does a fraction fit into a whole?** Thus, wholes are divided into fourths, fifths, eights in such a way that the logic of the sentence structure becomes crystallized as in: **how often does a fraction fit into a whole?** However, after offering a wonderful holistic and logical perspective the book deteriorates into a senseless mechanization of the division of fractions with the description of the

[25] X Quadrat MATHEMATIK 6 BADEN-BÜRTTEMBERG. Publisher: Dieter Baum & Hannes Klein. Berlin: © 2015 Cornelsen Schulverlag GmbH www.cornelsen.de

classical algorithm in the form of a math formula on page 31:

4.5.1 Division of fractions

> The division of a fraction can be replaced with the multiplication of the reciprocal value. One is able to get the **reciprocal value** of a fraction by inverting the numerator with the denominator.

There is no holistic explanation as to why the division is solved via the reciprocal value. The formula is just introduced without any practical application and above all, without any connection to reality in terms of **number sense** or a real-world representation. On page 32 an algorithmic exercise is presented by asking the students to **determine** the reciprocal value and by **solving** certain fractions:

Exercises: 1 Determine the corresponding reciprocal value.

a $\frac{1}{4}$; $\frac{2}{3}$; $\frac{2}{5}$; $\frac{3}{7}$ c $\frac{3}{4}$; $\frac{5}{8}$; $\frac{8}{15}$; $\frac{11}{12}$

b $\frac{27}{41}$; $\frac{35}{12}$; $\frac{171}{100}$; $\frac{45}{92}$ d $\frac{1}{4}$; $\frac{2}{3}$; $\frac{2}{5}$; $\frac{3}{7}$

2 Solve.

a $\frac{3}{7} : \frac{4}{5}$ c $\frac{2}{6} : \frac{5}{9}$ e $\frac{2}{6} : \frac{5}{9}$

b $\frac{1}{4} : \frac{1}{4}$ d $\frac{2}{5} : \frac{3}{4}$ f $\frac{8}{11} : \frac{6}{5}$

X Quadrat MATHEMATIK 6 BADEN-BÜRTTEMBERG, 2015, p. 32

Although a holistic approach is not sufficiently applied, a holistic perspective is partially introduced, especially with some very creative and practical questions:

3 How big is portion?

a $\frac{7}{4}$ kg of wood chips is divided into 7 equal parts.

b $\frac{3}{4}$ kg of gypsum is put into 6 bags of the same size.

c $12\frac{1}{2}$ kg of sand are packaged into five equal portions.

X Quadrat MAT.HEMATIK 6 BADEN-BÜRTTEMBERG, 2015, p. 32

To divide a fraction with a whole number is not so difficult because 4 is the same thing as $\frac{4}{1}$.

Also: $\frac{1}{3} : 4 = \frac{1}{3} : \frac{4}{1} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

X Quadrat MAT.HEMATIK 6 BADEN-BÜRTTEMBERG, 2015, p. 32

The approach of this book is not holistic enough because the implied logic of the formula is not explained properly. On the one hand, some exercises are very well illustrated and practiced in a holistic, praxis-oriented format and on the other hand, the classic algorithm is introduced without any concrete connection to exercises that were practiced holistically. A statement such as:

4.5.2 **To divide a fraction with a whole number is not so difficult because 4 is the same thing as $\frac{4}{1}$.**

has no connection to reality; the formula is compelled and forced on the students instead of orienting itself to the praxis and examples given on 3a, 3b and 3c. The teacher should first and foremost invite the students to think and discuss the meaning and implications of these examples:

$$\frac{1}{3} : 4$$

Can one, for example, ask how often does 4 in $\frac{1}{3}$? This question makes no sense; but the teacher should guide the students through dialogue in order to postulate at a holistic perspective between the following two divisions:

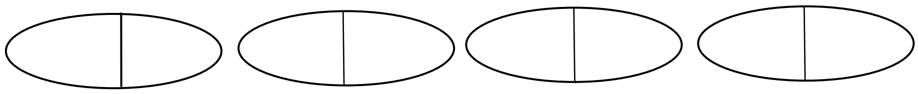
Teacher: What is the difference between $\frac{1}{2} : 4$ and $4 : \frac{1}{2}$?

Student: With $4 : \frac{1}{2}$ one can ask, how often does $\frac{1}{2}$ fit into 4?

but with $\frac{1}{2} : 4$ I have no idea, what should I ask.

Teacher: Can one ask, how often does 4 fit into $\frac{1}{2}$? Yes, one can ask that question, but that makes no sense because the 4 is just bigger than $\frac{1}{2}$. A 4 does not fit into $\frac{1}{2}$ With $4 : \frac{1}{2}$ one can logically ask, how often does $\frac{1}{2}$ fit into 4. Eight times as a matter of fact! $\frac{1}{2}$ fits exactly eight times into 4 as follows:

Teacher: Can one ask, how often fits a 4 into $\frac{1}{2}$? Yes, one can ask this question, but it doesn´t make sense, because the 4 is just bigger than $\frac{1}{2}$. A 4
does not fit into $\frac{1}{2}$. However, with $4 : \frac{1}{2}$ one can ask logically, how often does $\frac{1}{2}$ fit in a 4? Eight times as a matter of fact! One $\frac{1}{2}$ fits exactly eight times in 4 as follows (where t = times):



one time : two t three t : four t five t : six t seven t : eight t

One can even depict all of this as a long addition:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

1time : 2 times : 3 times : 4 times : 5 times : 6 times : 7 times : 8 times

At this point, the teacher may very well introduce the mechanical, senseless algorithm as a way to find the answer quickly:

Teacher: There is a trick, a formula, in order to solve the divisions of fractions quickly and that is by using the reciprocal value. For example, one can depict 4 as a fraction: $\frac{4}{1}$ and this fraction can be multiplied by the reciprocal value of $\frac{1}{2}$, that is, $\frac{2}{1}$ and voilà, one gets a quick answer:

$$4 : \frac{1}{2} = \frac{4}{1} : \frac{1}{2} = \frac{4}{1} . \frac{2}{1} = \frac{4 \,.\, 2}{1 \,.\, 1} = \frac{8}{1} = 8$$

With $\frac{1}{2} : 4$ one can just divide $\frac{1}{2}$ four times, as in a true division and thus we get four times $\frac{1}{8}$ and therefore $\frac{1}{2} : 4 = \frac{1}{8}$ because:

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{2}{4} = \frac{1}{2}$$

1 time : 2 times : 3 times : 4 times

Generally, I find the approach on pages 31-33 in the book X Quadrat MATHEMATIK 6 BADEN-BÜRTTEMBERG holistically deficient, but much better than **Mathe-Stars 6**. The book X Quadrat MATHEMATIK 6 should always substantiate the holistic nature from beginning to end so that the students may internalize and incarnate the whole logic of the algorithmic formulas. Therefore, the students may be empowered to understand not just the concept of the reciprocal value, but above all the logic of the algorithmic formula. The formula would then become an ingenious machinery, which one can apply, knowing that it´s artificial without any connection to real life. Students must understand that the formula does not correspond to reality and that it´s only an auxiliary device in order to quickly get to the answer in order to save time. The approach is nevertheless interesting from a holistic perspective, especially sections 2a, 2b and 2c on page 31 as well as sections 3a, 3b and 3c on page 32 with a follow up on p. 33.

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[26] These songs and melodies are to be edited with a bilingual format:

Hands Up- Singen, Tanzen und Bewegen || Kinderlieder
https://www.youtube.com/watch?v=qOfyTT_a8fc

Head, Shoulders, Knees and Toes- Singen, Tanzen und Bewegen || Kinderlieder
https://www.youtube.com/watch?v=SZRCAvMlyKY

Jarabe Tapatío: https://www.youtube.com/watch?v=tDeifn9lPE4&list=RDb9LRaPXEsbc&index=8

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Appendix A

An Email to Gustavo Vieyra from the Lecturer of Children´s and Youth Media at the Public Libraries of Dresden, Germany

Sehr geehrter Herr Vieyra,

vielen Dank für Ihre Mail vom 26.1.2012.
Wie Sie wissen fördern und betreiben wir als Städtische Bibliotheken Dresden zahlreiche Projekte für unterschiedliche Zielgruppen.
Die von Ihnen angeregte Veranstaltungsreihe **"Sprachförderung durch Rhythmik, Bewegung und Tanz" ist eine von jenen Reihen, die sich sehr erfolgreich etabliert haben und sich großer Beliebtheit erfreuen.**

Es ist uns bewusst, dass über die von Ihnen angestrebte Plattform "friends of library" noch weitere wichtige Initiativen und Angebote entwickelt werden könnten, leider können wir als Städtische Bibliotheken aus verschiedenen Gründen dabei aber zur Zeit nicht behilflich sein.

Ich schlage daher vor, dass wir unsere Zusammenarbeit auf dem Gebiet der o.g. Veranstaltungsreihe weiter pflegen und entwickeln und das Vorhaben zurückstellen.

Mit freundlichen Grüßen auch im Namen von Professor Flemming

Sonhild Menzel
Lektorin Kinder- und Jugendmedien

Dear Mr. Vieyra,

Thank you for your Email on 26.1.2012.
As you know, we promote and operate as City Libraries of Dresden numerous projects for our different target groups.
Your series of events that you´ve inspired **"Language Promotion via rhythmics, movement and dance" is one of those series that have become very successfully established and enjoy enormous popularity.**

We are aware that via your proposed platform "friends of library" there are still other important initiatives and offers to be developed, but unfortunately we as City Libraries, because of different reasons at this time, cannot be helpful.

Therefore, I propose that we foster and develop our cooperation in the area of the aformentioned series of events and that we postpone the proposal.

With cordial greetings and on behalf of Professor Flemming

Sonhild Menzel
Lecturer of Children´s and Youth Media

This Email is the only evidence we have in regards to a very successful holistic music-language program, **enjoying enormous popularity**, that took place at several public libraries in the city of Dresden, Germany for about two years up to 2012.

Appendix B **Curriculum Vitae**

Gustavo Vieyra, Gründer der Gestalt-Dialektik (GD)
Hindenburgstr. 17, 85072 Eichstätt
Tel. +49 1525 1331244 E-Mail: gestaltdialektik@azul-celeste.com
Website as *Work-in-progress*: https://azul-celeste.com/

Abstract of Azul Celeste
A holistic philosophy for princes and princesses ages 2 to 10 years old based on the spoken word as the alpha and omega of all learning

The objective of *Azul Celeste* is to develop *the best methods in the world* inspired by the love and spirit of the *spoken word* as the *Alpha and Omega*, the beginning and the end of all learning. The spoken word goes hand in hand with a dialogue, a recital, a song, or story, etc. within a Socratic process in order to learn in a natural way reading, writing math, Spanish, English, German, etc.

For children ages 2 to 5, who speak very well according to their age, learning may then reach mastery levels that are highly significant. With great success, we expect that our pupils will be able to go through primary, secondary and high school education, whether from Mexico, the USA or Germany according to *Gestalt-Dialektik*(GD), a philosophy based on inspiration and logic.

GD´s main goal is to follow a holistic approach to teaching and learning as exemplified by the great thinkers of humanity such as Alexander von Humboldt, Nikola Tesla, Friedrich Gauss, **The Prince of Mathematics**, Albert Einstein, etc., in an autodidactic process in order to master the corresponding academic areas.
Consequently, our celestial philosophy is to transform the program into an Academy of the Arts where each pupil becomes a princess or prince of love, hope and grace, fighting for the truth in light of his/her spiritual development.

Academic Background

Year		Institution	Academic Subject	Degrees
1973	1974	John Kelley Elementary Thermal, CA	Primary education	
1974	1978	Coachella Valley High School Coachella, CA	H.S. education	H.S. Diploma
Fall 1978	Fall 1981	College of the Desert Palm Desert, CA	Business Administration Foreign Languages Liberal Arts	A.A. Degree A.A. Degree A.A. Degree

Jan. 1982	Sept. 1982	UC, Riverside	Administrative Studies	
Jan 1983	Sept. 1984	UCLA	Spanish Literature	B.A. Degree
Aug. 1986	Aug. 1987	Universität Göttingen Göttingen, Germany	Spanish Literature German Language	
Fall 1989	Spring 1994	California State University at Los Angeles	Teaching Credentials Bilingual Education	M.A. Degree
Fall 1996	2000	Universität Passau, Passau, Germany	Doctoral Studies in Initial Reading and Writing	

Teaching Experience		School	Assignment	Principal
Sept. 1989	Jan. 1991	Will Rogers Elementary Lynwood Unified S.D.	1st Bilingual	Dr. Washington
Sept. 1991	June 1992	New Lexington Elem. El Monte City S.D.	4th, 5th, 6th Combination Bilingual	Ms. Yarbray
Sept. 1992	June 1993	Wilkerson Elementary El Monte City S.D.	K Bilingual	Ms. Nicholson
Sept. 1993	June 1994	Maxwell Elementary El Monte City S.D.	1st Bilingual	Mr. Rodriguez
Sept. 1994	June 1995	Utah Street Elementary Los Angeles Unified S.D.	2nd Bilingual	Ms. Mynatt
July 1996	June 2000	92nd Street School Elem. Los Angeles Unified S.D.	K & 1st Bilingual	Ms. Morris
July 2001	November 2017	75th Street Elementary Los Angeles Unified S.D.	K Bilingual	Mr. Campa
Dec. 2018	Present	Various projects in Mexico and the USA teaching children how to read and in their native language; also, teaching children in Dresden, Germany how to read and write in German as their native language as well as Spanish as a second or third language.		

Appendix C: Reference from Dr. Joshua Smith, Loving to Learn and Word of God International University, about my work with children in terms of my holistic methods

Loving to Learn Clinics and Centers

Health, Education and Social Services Association

June 6, 2016

Los Angeles Unified School District

To Whom It May Concern:

As a career educator and founder of Loving to Learn Association and The Word of God International University and Holistic Wellness Institute for many years along with a Master in Education, School Management and Administration from Pepperdine University, I started a successful alternative school, which mainstreamed children into public school who were having difficulty in school, home, and life. I've known Dr. Gustavo Vieyra for eight or nine years. I met him through Dr. W. John Martin, MI Hope Foundation, because of his oral language literacy program for young children. An avid supporter of our nonprofit, Dr. Martin highly recommended Dr. Vieyra to us because we advocate support for child, family, and community crisis intervention or life empowerment as a prolific nonprofit organization.

For as long as I've known Gustavo, he has been highly motivated to work with children and their parents, encouraging reading readiness and early childhood education. In my professional opinion, Dr. Vieyra is highly qualified to teach in a bilingual preschool and kindergarten setting as evidenced by his references in various school districts throughout the Los Angeles County since 1987 and in Germany since 2006. He has acquired a high level of expertise in initial reading and writing as well as bilingual education based on his holistic philosophy of education.

Dr. Vieyra has assisted us with our after school enrichment academy and literacy program, working with children at risk. Observing him with children, he is very focused in teaching them how to read through utilizing a curriculum of music and dramatic role play. They are effectually moved by his methodology and approach. Parents have shared how much they appreciate him for the progress that their children made because of the influence of his instruction. He has been very cooperative and supportive of our community values.

As a community support program, Gustavo also attends our university, completing his post-graduate requirements, and has been resident with us as a student, working on his doctoral degree for over five or six years. We appreciate his contributions. Please consider the attached multi-lingual magnet preschool proposal by Dr. Vieyra. As his Doctoral supervisor, I'm willing to oversee the development, progress, and success of this program.

For further questions, contact me at (323) 73-LIGHT (5-4448)

Respectfully yours,

Joshua Smith, Ph.D.
Overseer and President

O.U.R. Place
Opportunity-Unity-Restoration Center
(323) 282-9328 • 3650 S. Western Avenue, Los Angeles, CA 90018 • (323) 73-LIGHT
learn@lovingtolearn.org • Fax (323) 402-0422 • www.lovingtolearn.biz
P.O. Box 180254, Los Angeles, CA 90018-0726

Appendix D Reference from Barbara Seid-Grier, Consultant to *Los Angeles Unified School District* (LAUSD) in regards to my work as a teacher at *75th Street Elementary School* (*LAUSD*)

1130 S. Plymouth Boulevard,
Los Angeles, California 90019
United States of America
April 5, 2010

Kultusminister Prof. Dr. Roland Wöller
Sächsisches Staatsministerium fur Kultus
Carolaplatz 1
01097 Dresden
Tel.: + 49-351 564-0

Dear Prof. Dr. Wöller:

I observed Mr. Vieyra's teaching practices at 75th Street Elementary School, which is part of the Los Angeles Unified School District (LAUSD), between July, 2008 and January, 2009.

The Pedagogy that Mr. Vieyra employs is based upon the teachings of Lev Vigotsky, the great Russian Psychologist and Educator, and the "Zones of Proximal Development". Using this method (ZPD), the student receives individual instruction from a Tutor, i.e., a Teacher , or a Parent, with the objective being for the student to complete various targeted activities (objectives) independently, in due time. Thus promoting self-directed, reflective learning for all students.

I also observed that Mr. Vieyra's Pedagogy strongly parallels the California Standards for the Teaching Profession (CSTP), in its approach to the Kindergarten program, as presented.

During the time I observed in Mr. Vieyra's classroom, I saw the use of many strategies to engage students' attention, such as music, dance and the use of realia, manipulatives, and "hand-on" experiences, and activities, which promoted "holistic learning", i.e., using all the senses. The students were also engaged in problem-solving and critical thinking. I also observed that Mr. Vieyra had created a "safe, nurturing environment", that was uplifting to the spiritual and psychological well-being of the students.

The classroom was "print-rich", with many colorful and student-friendly displays. There were many centers and activities geared toward the differentiation of learning for different levels of student ability. The classroom routines were structured so that students were able to operate independently and make decisions about their own learning. Mr. Vieyra's curriculum showed organization and planning, both short term and long term, with emphasis on step-by-step instructional strategies. There were also opportunities for group-assessment and self-assessment by the students.

Mr. Vieyra kept the parents well-informed as to the students' progress. He held his own Parent Meetings, in order to train the parents to be proactive in their own children's learning at home. Mr. Vieyra was also active in the community, rallying support for his student-programs.

Mr. Vieyra has a very unique Philosophy and Pedagogy that work. He also has a great passion and expertise for Teaching. These combined factors would make him a great asset to any program!

If you have any questions, pleases feel free to contact me:
bar.seidgrier@gmail.com
(310) 991-6555

Yours truly,

Barbara Seid-Grier

Barbara Seid-Grier
Consulting Teacher
LAUSD, Retired

Appendix E A testimonial letter by a school psychologist in regards to the pilot program in Kassel, Germany 1998

https://www.youtube.com/watch?v=gDeNQyVM5Dc

What follows is a letter written by Tobias Maxwell, (psychologist working for the Los Angeles Unified School District) to Prof. Hubert Buchinger, ex-chair of Elementary School Education at the University of Passau, Germany in regards to a video presentation from the pilot program that took place in Kassel, Germany in 1998 (see "A Testimonial of the Vieyra Reading and Writing Method" translated into English below in the next page). Mr. Tobias Maxwell wrote in his letter as follows:

> From a clinical perspective, I was impressed by the Gestalt experience I observed in the small group of students being filmed. The ability to transcend— for lack of a better word, and express the entire base schema in story-telling mode was fascinating.

****

Tobias Maxwell, M.S., M.F.C.C. Lic. # M.F.C. 31498

6240 Whitsett Avenue #104 ♦ North Hollywood, CA 91606 (818) 508-5189

July 11, 1998

Dr. Hubert Buchinger
Universitaet Passau
94030 Passau
Germany

Dear Prof. Buchinger:

My colleague, Mr. Gustavo E. Vieyra, suggested that I write to you regarding the video demonstration of the pedagogical technique he has been developing over the years.

From a clinical perspective, I was impressed by the Gestalt experience I observed in the small group of students being filmed. The ability to transcend— for lack of a better word, and express the entire base schema in story-telling mode was fascinating. I am looking forward to reading more about the theories behind the application of Mr. Vieyra's ideas.

As with any new approach in the art and science of instruction, time and its tested results will speak louder than any testimonial. I wish Mr. Vieyra ongoing success with his research.

Yours truly,

Tobias Maxwell

Tobias Maxwell

Appendix F

A testimonial of the Vieyra reading and writing method in Kassel, Germany 1998
(for the original version written in German please see scanned testimonial below)

What follows is a testimony from **Inge Jakob**, a Montessori educator in Kassel, Germany that describes some aspects of literacy development according to *Gestalt-Dialektik* (*hereunder described as the Vieyra Reading and Writing Method*) and also gives some transcendental answers to its pedagogical philosophy, especially in regards to a nurturing, loving and caring environment vis-à-vis an individualized curriculum that is so extremely needed by children in any social and pedagogical setting.

****

Inge Jakob
Schwengebergstr. 14
D-34132 Kassel
Germany
Tel. 01149-561-4000499

Nov. 9, 1998

Prof. Dr. phil. Hubert Buchinger
Universität Passau
Lehrstuhl für Grundschuldidaktik
D-94030 Passau
Germany

Testimonial regarding the Vieyra Reading and Writing Method

A Pilot Program: Montessori-Kinderhaus
Konrad-Adnauer-Str. 143
34132 Kassel, Germany
Tel. 01149-561-408702
End of May until end of June 1998

Very respected Herr Prof. Buchinger,

At the end of May 1998 Mr. Gustavo Vieyra presented his reading and writing Method in our *Kindergartenhaus*. The pilot program lasted for about five weeks.

During the first two weeks Mr. Vieyra took part of the daily Kindergarten curriculum. During this time, he became familiar with the children (a group of 20 children) and with the daily routine. Mr. Vieyra was very well accepted by the children so that he could start with his method from the very first day. With a narrative style he introduced his cartoon story.

On the second day he asked the children to retell the story, based on the pictures of the story (about 1\4 of the whole story); new pictures came into play. The narrative was followed by all participants with gestures. The story was narrated backwards at the end of the hour.

In the morning, during recess time, the children were encouraged to draw and retell the story; small books were "produced". In the third week the children received square-designed paper, each square corresponding with a picture of the story. The children filled the squares with the pictures. Thereafter the designed paper appeared with syllables and numbers that appeared in the squares which corresponded previously to the pictures of the story. During circle time the children were able to imagine the pictures based on the syllables. They named the syllables with their corresponding rhythms, "Banane-Banane-**ba**, Catze-Catze-**ca**, Dahlie-Dahlie-**da** (etc.)": to each syllabic appellation the children applauded or hopped. **They had great concentration and enthusiasm with the lessons.**

Mr. Gustavo Vieyra worked during four weeks with the whole group and then the so-called "Gustavo Preschool" was established. This group included a girl of 4 ½ years, a boy of 4 ¾, a boy of 5 years and four girls of 6 years old. Smaller children joined them happily on their own.

In the fifth week the children sat like "school children" on their places. On the wall separate square-designed pieces of papers were hanging, decorated with the numbers and syllables. The children learned the numbers with the corresponding syllables. Very soon they could name the syllables, when one would point to the numbers. The time factor was integrated. It came to their attention via the small and big hands of the clock: within ten minutes the children filled in, as much as they could, the respective syllables inside the numbered squares of the designed pieces of paper. At the end of the fifth, beginning of the sixth week the children could partially decorate blank pieces of paper with squares, numbers and syllables.

It is remarkable and significant in regards to initial reading and writing that a few children wrote words and also whole sentences, which could be read. I would like to mention, that the children during the morning hours very often moved rhythmically, and it gave them a lot of pleasure to retell the story in a short period of time with their eyes closed and using gestures.

My personal impression

As a prelude I would like to say, they were five moving and merry weeks that we spent with Mr. Vieyra. The learning enthusiasm was contagious. The children as well as the educators lived in the theme. So, it always happened, that one would hear syllables during the mornings, such as, "Anna-Anna-**an**; Kartoffel-Kartoffel-**ka** . . . (extension-transfer). The pictures that were illustrated during their free play time usually belonged to the theme.

Hands on mechanical experiences, including that of plastics became part of their thematic plays; there was even "**ra**cket-ice cream". The whole group worked together up to the fourth week, as long as the drama was the focus of attention. The above-mentioned children that belonged to the so-called "Gustavo-Preschool" were partially able at the end of the project to write words and small sentences.

Preconditions of the children

The children worked since their third year of life with Montessori materials; exercises of the "daily life" and the "sense materials" stay at the foreground vis-à-vis the children and the movable letters, math materials, geometric forms, etc. are also included. A few of the children could form, read and write words with the movable letters. However, with the help of the reading of syllables (reading and writing of the Vieyra method) the children were faster in their ability to form and read words. In the sound-method from Maria Montessori and in the Vieyra-Method is the motivation of the teacher as well as the consideration of "holistic learning" (with all the senses) important.

Within the groups the children sing, dance, play theatre and illustrate a lot. Notwithstanding the children have a lot of freedom in nature.

My final opinion

During the sensitive phase (0 to 6 years of age) in which the children are receptive via imitation and play, the educators take over the great responsibility for the post life of the children. We should guard ourselves from "over-sheltering" the children and neither should we hold them back or halt (hinder) their development. Here we find the parents as well as the educators in contradiction. On the one hand they want a protective environment and on the other one they want the best possible improvement to the point that they become overburden. Hereby a lot of work is needed in order to raise the consciousness that a child does not learn like another one, that here too the child brings forth his own character and that we as adults should be held accountable via observation to give the child the respect that he/she deserves.

Important for the child is the mastery of the mother language, the training of the senses in order to be open and secured in life. I have experienced in the kindergarten that the children who come from an orderly family (in the sense of a life's rhythm) in which the contact persons are clear and apparent bring forth security –moving freely in the group– openness, a good self-esteem, and concentration and clearly more than the children in which these preconditions are not given. The aforementioned children correspondingly even want to have more speedy experiences.

It was wonderful to observe how Mr. Vieyra "picked the children up" at the level of their development, in accordance to Maria Montessori. Thus, the so-called fast children could always play the teacher in order to reckon with their forward-looking desires. The children that needed more time in the learning task received opportunities to look, to hear, and to jump playfully in exact accordance to their needs. In this sense the "**we-togetherness**" even in the group of 20 children (ages 3-6 years old) was warranted.

What the children also need to learn is "**movement, movement, movement!!!"** This is lacking in the children today more than ever! I find it pitiful, that in the public schools not enough attention is paid to the individuality of the child, that the curricular guide, which must be fulfilled, appears too much on the foreground. What happy children we would have if the competitive stress did not exist along with the feeling of the "I am not OK!" How far would we as humans go forward if we were not become oppressed through the labeling that we received from our former (test) notes, when we could be free from the competition, when we could love the accomplishment in the sense of cordial reciprocity!? To this we could add that the children need more contact persons at their disposal (here we always experience a lot of limitations).

When the children are allowed to have such experiences, namely when they could develop themselves according to their potentialities, then they would, in my opinion, mature themselves into the adults, which as per the **"we-togetherness"** would construct the world.

That, which Mr. Vieyra presented in the five weeks with us at the kindergarten via his method of learning how to read and write complements itself beautifully with the Maria Montessori method and with her philosophy. The children were feeling great, were enthusiastic, happy and wanted to learn more and more:

a) drama
b) movement
c) creativity
d) syllabic learning
e) word-findings
f) writing of small sentences and their reproduction
g) indirect learning of the numbers 1-20
h) the sensitivity towards time
i) etc.

All of that took place in five weeks.

I wish Mr. Vieyra more success in his work and am thankful for the time we spent together in which we could accompany the children of our Kindergarten-house.

With best wishes,

(signed in the original testimonial written in German, see below)
Inge Jakob

Educator at the Montessori *Kindergartenhaus*.

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Scanned testimonial (*written in German*) of Inge Jakob
about the outstanding results of the *Vieyra-Method* in Kassel, Germany 1998
(for a translation written in English please see above)

Inge Jakob
Schwengebergstr. 14
D-34132 Kassel
Germany
Tel.: 0561-4000499

9.Nov.1998

Herrn Prof. Dr. phil. Hubert Buchinger
Universität Passau
Lehrstuhl für Grundschuldidaktik
D-94030 Passau
Germany

Gutachten über die „Vieyra-Lese- und Schreibmethode“

Ein Pilotprogramm: Montessori-Kinderhaus
Konrad-Adenauer-Str. 143
34132 Kassel
0561-408702
Ende Mai bis Ende Juni 1998

Sehr geehrter Herr Prof. Buchinger,

Ende Mai 1998 stellte Herr Gustavo Vieyra seine Lese- und Schreibmethode in unserem Kinderhaus vor. Das Pilotprojekt dauerte ca. fünf Wochen.

In den ersten beiden Wochen teilte Herr Vieyra mit uns den Kindergartenalltag. Er lernte in dieser Zeit die Kinder (die Gruppe besteht aus 20 Kd.) und den Tagesablauf kennen. Herr Vieyra wurde von den Kindern sehr gut aufgenommen, so daß er schon am ersten Tag mit der Einführung seiner Methode beginnen konnte. Er stellte erzählerisch seine Bildgeschichte vor.

Am zweiten Tag ließ er die Kinder anhand der Bilder die Geschichte (etwa ¼ der gesamten Geschichte) wiederholen, neue Bilder kamen hinzu. Die Erzählung wurde von allen Beteiligten mit Gesten begleitet. Am Schluß der Stunde wurde die Geschichte rückwärts erzählt.

Morgens, während der Freispielzeit, durften die Kinder die Geschichte malend nacherzählen -es entstanden kleine Büchlein. In der dritten Woche erhielten die Kinder Blätter, versehen mit leeren Feldern (jedes Feld entsprach einem Bild der Geschichte). Die Kinder füllten die Felder mit Bildern aus. Hinzu kamen dann Blätter mit Silben und Zahlen, die ebenfalls in den Feldern auftauchten, in denen vorher die Bilder der Geschichte zu sehen waren. Im Stuhlkreis konnten sich die Kinder inzwischen die einzelnen Bilder anhand der Silben vorstellen. Sie benannten die Silben mit ihren entsprechenden Rhythmen, „Banane-Banane-**ba**, Catze-Catze-**ca**, Dahlie-Dahlie-**da** (usw.)"; zu jeder Silbenbenennung klatschten oder hüpften die Kinder. **Sie waren mit großer Konzentration und Begeisterung bei der Sache.**

Vier Wochen lang arbeitete Herr Gustavo Vieyra mit der gesamten Gruppe, dann entstand die sog. „Gustavo-Vorschule". Hierzu gehörte ein Mädchen im Alter von 4 1/2 J., ein Junge im Alter von 4 ¾ J., ein Junge, 5 J. und vier Mädchen im Alter von 6 J. Kleinere Kinder gesellten sich gern dazu.

In der fünften Woche saßen die Kinder wie „Schulkinder" auf ihren Plätzen. An der Wand hingen einzelne Blätter, versehen mit Zahlen und Silben. Die Kinder lernten die Zahlen mit den entsprechenden Silben; sehr bald konnten sie die Silben benennen, wenn man auf eine Zahl tippte. Die Zeit wurde mit einbezogen. Es wurde auf den kleinen und großen Zeiger der Wanduhr aufmerksam gemacht: innerhalb von 10 Minuten füllten die Kinder -soweit sie es schaffen konnten- Blätter, die mit Zahlenfeldern versehen waren, mit den entsprechenden Silben aus. Am Ende der fünften/Anfang der sechsten Woche konnten die Kinder teilweise ein leeres Blatt mit Feldern, Zahlen und Silben versehen.

Es ist bemerkenswert und signifikant im Sinne vom Erstlesen und –schreiben, daß einige Kinder Wörter oder auch ganze Sätze schrieben, die gelesen werden konnten. Erwähnen möchte ich, daß die Kinder sich immer wieder während der Vorschulstunden rhythmisch bewegten, und es bereitete ihnen viel Freude, die Geschichte innerhalb kürzester Zeit mit geschlossenen Augen -verbunden mit Gesten- zu erzählen.

Mein persönlicher Eindruck

Einleitend möchte ich sagen, es waren fünf bewegte, fröhliche Wochen, die wir mit Herrn Vieyra verbracht haben. Die Lernbegeisterung war ansteckend. Kinder wie Erzieherinnen lebten im Thema. So geschah es immer wieder, daß man während des Vormittages Silben hörte, wie „Anna-Anna-**an**; Kartoffel-Kartoffel-**ka** . . . (Erweiterung-Übertragung). Bilder, die während des Freispiels gemalt wurden, gehörten oft zum Thema.

Es wurde auch themenbezogen gebastelt, plastiziert, und es gab sogar „Raketeneis". Die gesamte Gruppe arbeitete bis einschließlich der vierten Woche zusammen, solange das Drama in Vordergrund stand. Die obenerwähnten Kinder, die dann in die sog. „Gustavo-Vorschule" gingen, waren am Ende des Projektes teilweise fähig, Wörter und kleine Sätze zu schreiben.

Voraussetzungen der Kinder

Die Kinder arbeiten seit ihrem dritten Lebensjahr mit Montessori-Materialen, Übungen des tägliches Lebens, und Sinnesmaterialen stehen bei den Kleinen im Vordergrund, bewegliche Buchstaben, Mathematik-Materialen, Geometrische Formen usw. kommen hinzu. Das eine oder andere Kind konnte mit beweglichen Buchstaben schon Worte legen, lesen und schreiben. Doch mit Hilfe des Erlernens von Silben (Lesen und Schreiben der Vieyra-Methode) waren die Kinder schneller fähig, Wörter zu bilden und zu lesen. Bei der Lautmethode von Maria Montessori wie bei der Vieyra-Methode ist die Begeisterung des Lehrers sowie die Berücksichtigung des „ganzheitlichen Lernens" (mit allen Sinnen) wichtig.

In der Gruppe wird viel gesungen, getanzt, Theater gespielt und gemalt. Außerdem bewegen sich die Kinder viel in der Natur.

Meine abschließende Meinung

Während der sensitiven Phase (0 bis 6 Jahre), in der die Kinder nachahmend und spielerisch aufnehmen, übernehmen die Erzieher/innen die größte Verantwortung für das weitere Leben des Kindes. Wir sollten uns davor hüten, dem Kind in diesem Alter „etwas überzustülpen", sollten es jedoch auch nicht zurückhalten bzw. in seiner Entwicklung klein halten (hemmen). Hier befinden sich Eltern wie auch Erzieher/innen im Widerspruch. Einerseits wollen sie Geborgenheit für die Kinder, andererseits sollen die Kinder die bestmöglichste Förderung erhalten, so daß diese oft überfordert werden. Hier muß daran gearbeitet werden, Bewußtheit dafür zu schaffen, daß nicht ein Kind wie das andere lernen kann, daß auch hier das Kind seine eigene Art mitbringt und wir Erwachsene gehalten sind, ihm immer wieder durch Beobachtung und Achtung gerecht zu werden.

Wichtig für das Kind ist die Beherrschung der Muttersprache, die Schulung der Sinne, um offen und sicher ins Leben zu gehen. Ich habe im Kindergarten erfahren, daß die Kinder, die aus einem geordneten Elternhaus (im Sinne von Lebensrhythmus) kommen, wo die Bezugspersonen klar sind, Sicherheit mitbringen, sich in einer Gruppe zu bewegen, Offenheit, ein gutes Selbstwertgefühl und Konzentration und zwar deutlich mehr als die Kinder, bei denen diese Voraussetzungen nicht gegeben sind. Die erstgenannten Kinder wollen somit entsprechend schnell Lernerfahrungen machen.

Es war schön zu beobachten, wie Herr Vieyra die Kinder „dort abholte, wo sie sich befanden", ganz im Sinne von Maria Montessori. So konnten die sog. schnellen Kinder immer wieder Lehrer spielen, um ihrem Drang, nach vorne zu kommen, gerecht zu werden. Die Kinder, die mehr Zeit zum Lernen benötigten, erhielten Gelegenheit zu schauen, zu hören, spielerisch einzusteigen, ganz wie sie es brauchten. So war das **Miteinander** auch in der Gruppe mit 20 Kindern unterschiedlichen Alters (3-6 J.) gewährleistet.

Was die Kinder zum Lernen auch benötigen, ist **„Bewegung, Bewegung, Bewegung!!!"**. Das fehlt den Kinder heute mehr, denn je! Ich finde es schade, daß in den Staatsschulen noch nicht genügend auf das Individuum des Kindes eingegangen werden kann, daß der Lehrplan, der erfüllt werden muß, so sehr im Vordergrund steht. Was könnten wir für glückliche Schulkinder haben, wäre der Leistungs**druck** nicht da und somit das Gefühl, „ich bin nicht OK!" Wie weit könnten wir Menschen nach vorne gehen, würden wir nicht durch die Stempel, die wir durch die frühen Zensuren bekommen, gedrückt werden, wenn wir frei wären von Konkurrenz, wenn wir die Leistung im Sinne von sich gegenseitigem Ergänzen lieben könnten!? Dazu gehört auch, daß den Kindern mehr Begleitpersonen an die Seite gestellt werden (hier erfahren wir immer mehr Einschränkung).

Wenn dies die Kinder schon erleben dürften, daß sie sich **ihren Fähigkeiten** gemäß entwickeln könnten, dann würden sie m. E. zu Erwachsenen heranreifen, die **im Sinne des Miteinanders** an der Welt bauen würden.

Das, was uns Herr Vieyra in den fünf Wochen bei uns im Kindergarten mit seiner Methode des Lesen- und Schreibenlernens aufzeigte, ergänzt sich sehr schön mit der Maria-Montessori-Methode und mit ihrer Philosophie. **Die Kinder fühlten sich wohl, waren begeistert, glücklich und wollten weiter und weiter lernen**:

a) Drama (Theater)
b) Bewegung
c) Kreativität
d) Silbenerlernen
e) Wortfindungen
f) Schreiben von kleinen Sätzen und Wiedergabe
g) Indirektes Erlernen der Zahlen 1-20
h) Das Empfinden von Zeit
i) usw.

Das alles wurde in den fünf Wochen geschafft.

Ich wünsche Herrn Vieyra weiterhin viel Erfolg für seine Arbeit und bin dankbar für die gemeinsame Zeit, in der wir die Kinder unseres Kinderhauses begleiten durften.

Mit freundlichen Grüßen,

Inge Jakob

Inge Jakob
Erzieherin im Montessori-Kinderhaus

Appendix G

Results of the initial reading and writing method in Duarte, California

DUARTE UNIFIED SCHOOL DISTRICT

1620 HUNTINGTON DRIVE, DUARTE, CALIFORNIA 91010
TELEPHONE (818) 358-1191 • FAX (818) 358-4317

July 6, 1994

To Whom It May Concern,

I hereby submit this letter of reference on behalf of Gustavo Vieyra. Gustavo taught fulltime a First Grade Bilingual Class at Maxwell Elementary School from September 1993 through June 1994. Gustavo was employed to fill a one year tempoary teaching position at Maxwell. I am the principal at Maxwell.

During his teaching assignment I was able to observe first hand the approach Gustavo employed in teaching reading, writing, and math. He calls it a Gestaltic Approach (Vieyra 1994). In my professional opinion, the approach is wholistic in nature because it focuses in students being able to identify and acquire patterns and the interrelationships found therein. This newly acquired ability then allows them to predict results and to discover insights in learning how to read, write and do math at an accelerated pace.

It pleases me to say that towards the end of the school year his students' performance in reading, writing, and mathematics demonstrated exceptional results. Over 90% of the students assigned to his class read at or above grade level. In writing, students were able to express their thoughts clearly, correctly, and, often times, more in content than what is expected from a child at that grade level. In mathematics, besides the operations of addition and subtraction, students were able to demonstrate comprehension of multiplication facts. It is evident that his students demonstrated accelerated progress in reading, writing and mathematics.

Mr. Vieyra established a great rapport with his students. They responded to his classroom management system. Gustavo also established good relationships with the parents of his students. The parents were very supportive to his style of teaching. They were very pleased with the progress their children had made as students of his.

If I can be of any further assistance I may be contacted at (818) 358-1191, Ext. 254.

Respectfully Submitted,

Julian Rodriguez, Principal
Maxwell Elementary

ADMINISTRATION

Dr. Marcia McVey, Superintendent | Dr. Alan Johnson, Deputy Superintendent

DUARTE UNIFIED SCHOOL DISTRICT
1620 HUNTINGTON DRIVE, DUARTE, CALIFORNIA 91010
TELEPHONE (818) 358-1191 • FAX (818) 358-4317

August 30, 1994

To Whom It May Concern:

Gustavo Vieyra has taught a Bilingual First Grade at Maxwell Elementary School for the past year.

His classroom performance has been exemplary. He has used an accelerated method of teaching reading and writing in Spanish, which he has developed himself. The results have been enviable and have made a great impression on both parents and staff.

Mr. Vieyra takes a methodical and scholarly approach to instruction. I would recommend Mr. Vieyra without reservation for teaching positions at any level.

Sincerely,

Patricia A. Hicks

Pat Hicks
Bilingual Coordinator