AP Calculus BC_Taylor Series

1. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0)=5, f'(0)=-3, f''(0)=1, \text{and } f'''(0)=4$. (AP 1998)
   (A) Write the third-degree Taylor polynomial for f about $x=0$ and use it to approximate $f(0.2)$.
   (B) Write the fourth-degree Taylor polynomial for g, where $g(x)=f(x^2)$, about $x=0$.
   (C) Write the third degree Taylor polynomial for h, where $h(x)=\int_0^x f(t)dt$, about $x=0$.
   (D) Let h be defined as in part (C). Given that $f(1)=3$, either find the exact value of $h(1)$ or explain why it cannot be determine.

2. The Taylor series about x = 5 for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at x =5 is given by
   $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n(n+2)}$, *and* $f(5)=\dfrac{1}{2}$. (AP 2000)
   (A) Write the third-degree Taylor polynomial for f about x = 5.
   (B) Find the radius of convergence of the Taylor series for f about x = 5.
   (C) Show that the sixth-degree Taylor polynomial for f about x = 5 approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

3. The function f has derivatives of all orders for all real number x. Assume $f(2)$= -3, $f'(2)$ = 5, $f''(2)$ = 3, and $f'''(2)$ = -8. (AP 1999)
   (A) Write the thud-degree Taylor polynomial for f about x = 2 and use it to approximate $f(1.5)$.
   (B) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right|\le 3$ for all x in the closed interval [1.5, 2]. Use the Lagrange error bound on the approximation to f(l. 5) found in part (a) to explain why f(1-5) ≠ -5.
   (C) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2+2)$ about x = 0. Use P to explain why g must have a relative minimum at x = 0.

4. The Maclaurin series for the function f is given by
$$f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\ldots+\frac{(2x)^{n+1}}{n+1}+\ldots$$
   on its interval of convergence. (AP 2002)
   (A) Find the interval of convergence of the Maclaurin series for f Justify your answer.
   (B) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.
   (C) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

5. A function f is defined by
$$f(x)=\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2+\ldots+\frac{n+1}{3^{n+1}}x^n+\ldots$$

For all x in the interval of convergence of the given power series. (AP 2001)

(A) Find the interval of convergence for this power series. Show the work that leads to your answer.

(B) Find $\lim_{x\to 0}\dfrac{f(x)-\frac{1}{3}}{x}$.

(C) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)dx$.

(D) Find the sum of the series determined in part (c).

6. Let f be the function given by $f(x) = \sin\left(5x+\dfrac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$ (AP 2004)

(A) Find $P(x)$.

(B) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(C) Use the Lagrange error bound to show that $\left|f\left(\dfrac{1}{10}\right)-P\left(\dfrac{1}{10}\right)\right|<\dfrac{1}{100}$.

(D) Let G be the function given by $G(x) = \int_0^x f(t)dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

7. The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\ldots+\frac{(-1)^n x^{2n}}{(2n+1)!}+\ldots$$

for all real numbers x. (AP 2003)

(A) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(B) Show that $1-\dfrac{1}{3!}$ approximates $f(1)$ with error less than $\dfrac{1}{100}$.

(C) Show that $y = f(x)$ is a solution to the differential equation $xy'+y=\cos x$.

8. The function f is defined by the power series

$$f(x)=-\frac{x}{2}+\frac{2x^2}{3}-\frac{3x^3}{4}+\ldots+\frac{(-1)^n nx^n}{n+1}+\ldots$$

For all real numbers x for which the series converges. The function g is defined by the power series

$$g(x)=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\ldots+\frac{(-1)^n x^n}{(2n)!}+\ldots$$

For the real numbers x for which the series converges. (AP 2006)

(A) Find the interval of convergence of the power series for f, Justify your answer.

(B) The graph of $y=f(x)-g(x)$ passes though the point $(0,-1)$. Find $y'(0)$ and $y''(0)$.

Determine whether y has a relative minimum, a relative maximum, or neither at $x=0$. Given a reason for you answer.

9. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at x- 2 is 0. When n is even and $n \geq 2$, the nth derivative of f at x- 2 is given by $f^{(n)}(2)$ - $\frac{(n-1)!}{3^n}$. (AP 2005)

(A) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(B) In the Taylor series for f about $x = 2$, what is the coefficient of $(x - 2)^{2n}$

for $n \geq 1$?

(C) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

$$f(x) - \begin{cases} \dfrac{\cos x - 1}{x^2} & for\ \ x \neq 0 \\ -\dfrac{1}{2} & for\ \ x = 0 \end{cases}$$

10. The function f. defined above, has derivatives of all orders. Let g be the function defined by $g(x) - 1 + \int_0^x f(t)\,dt$. (AP 2010)

(A) Write the first three nonzero terms and the general term of the Taylor series for cos x about x = 0. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about x = 0.

(B) Use the Taylor series for f about x = 0 found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at x = 0. Give a reason for your answer.

(C) Write the fifth-degree Taylor polynomial for g about x = 0.

(D) The Taylor series for g about x = 0, evaluated at x = 1, is an alternating series with individual terms that decrease in absolute value to 0. Use the third- degree Taylor polynomial for g about x = 0 to estimate the value of g(1). Explain why this estimate differs from the actual value of g(1) by less than $\frac{1}{6!}$.

11. The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \ldots$ The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at x = 1. (AP 2009)

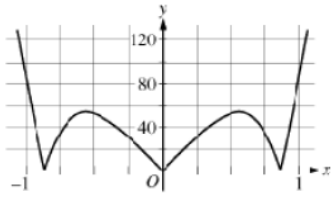
(A) Write the first four nonzero terms and the general term of the Taylor series $e^{(x-1)^2}$ about x = 1.

(B) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about x = 1.

(C) Use the ratio lest to find the interval of convergence for the Taylor series found in part (b).

(D) Use the Taylor scries for f about x = 1 to determine whether the graph of f has any points of inflection.

12. Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown in the figure. (AP 2011)



Graph of $y = \left|f^{(5)}(x)\right|$

(A) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$. and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(B) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(C) Find the value of $f^{(6)}(0)$.

(D) Let $P_4(x)$be the? fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown in fig., shown that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

13. The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$. (AP 2012)

(A) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(B) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g = \left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(C) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

14. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$. (AP 2013)

(A) It is known that $f(0) = -4$ and that $P\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(B) It is known that $f''(0) = -\frac{1}{2}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(C) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

15. The Taylor series for a function f about $x = 1$ is given by $\sum_{n-1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1|<R,$ where R is the radius of convergence of the Taylor series. (AP 2014)

(A) Find the value of R.

(B) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(C) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1|<R,$ Use this function to determine f for $|x-1|<R.$

16. The Maclaurin series for a function f is given by $\sum_{n-1}^{\infty}\frac{(-3)^{n-1}}{n}x^n = x-\frac{3}{2}x^2+3x^3-\cdots+\frac{(-3)^{n-1}}{n}x^n+\cdots$ and converges to $f(x)$ for $|x|<R,$ where R is the radius of convergence of the Maclaurin series. (AP 2015)

(A) Use the ration test to find R.

(B) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f'' as a rational function for $|x|<R,$

(C) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x)=e^x f(x)$ about $x=0$

17. The function f has a Taylor series about x=1 that converges to f(x) for al x in the interval of convergence. It is known that $f(1)-1, f'(1)=-\frac{1}{2}$, and the n^{th} derivative of f at x-1 is given by $f^n(1)=(-1)^n\frac{(n-1)!}{2^n}$ for $n\geq 2$. (AP 2016)

(A) Write the first four nonzero terms and the general term of the Taylor series for f about x -1.

(B) The Taylor series for f about x = 1 has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(C) The Taylor series for f about x = 1 can be used to represent f (1, 2) as an alternating series. Use the first three nonzero terms of the alternating series to approximate f(1, 2).

(D) Show that the approximation found in part (C) is within 0.001of the exact value of f (1.2).

18.

$$f(0)=0$$

$$f'(0)=1$$

$$f^{(n+1)}=-n\cdot f^{(n)}(0)\text{ for all } n\geq 1$$

A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x|<1$.

(AP 2017)

(A) Show that the first four nonzero terms of the Maclaurin series for *f* are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for *f*.

(B) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(C) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(D) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}$.

19.

The Maclaurin series for (1 + *x*) is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to in (1 + *x*). Let *f* be the function defined by $f(x) = x\ in\left(1 + \frac{x}{3}\right)$. (AP 2018)

(A) Write the first four nonzero terms and the general term of the Maclaurin series for *f*.

(B) Determine the interval of convergence of the Maclaurin series for *f*. Show the work that leads to your answer.

(C) Let $P_4(x)$ be the fourth-degree Taylor polynomial for *f* about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.