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3D modeling of fractional diffusion to describe electromagnetic induction in fractured geological media

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SUMMARY

The fractional diffusion of the electromagnetic (EM) induction in fractured media is modelled using the 3D finite difference (FD) method in frequency domain. The EM fractional diffusion is one subset of anomalous transport processes, which is usually induced by the heterogeneities with various scales in the media. The governing equation of this process is based on the classical Maxwell equation and modified according to the continuous time random walk (CTRW) theory. The roughness parameter β in the new equation is directly related to the degree of heterogeneity of formations. To access the performance of this theory, the 3D FD modelling experiments are developed. The calculated EM response of a 1D layered model is compare with the analytical solution to validate the accuracy of the code. Then the FD solution to a homogenous model containing a 3D rough faulting zone is calculated to compare with responses to several classical diffusion model (with β=0). The comparable classical models are expected to obtain to explore the relationship between the roughness β with the heterogeneities of the media.

Keywords: 4-6 keywords, forward modeling, inversion, finite element, marine CSEM, case study

INTRODUCTION

Anomalous transport processes arise in complex systems containing spatial heterogeneities of all length scales. Geometrically these systems often generate a fractal structure over the space. The anomalous transport was first described by Scher and Lax (1973a), concerning the diffusion of electric charges in amorphous semiconductors. Similar behaviour has been reported to emerge in many physical situations, such as fluid transport and heat conduction (Geiger and Emmanuel 2010). Anomalous diffusion is distinct from classical diffusion in that the particle transport usually occurs along preferential paths spatially and, furthermore, multi-scale transport rates are achieved that can not be explained by Fick's second law. Subdiffusion is a category of non-Fickian transport, in which the mean square particle displacement scales as a fractional order power law in time. A heavy-tailed breakthrough curve at late time is characteristic of a subdiffusion process.

EM induction due to excitation by a controlled source of a spatially rough subsurface can be anomalous, when the geoelectrical structure exhibits hierarchical or fractal heterogeneity (Everett and Weiss, 2002; Weiss and Everett, 2007; Ge et. al. 2012). To account for this process, Everett (2009) derived the fractional Maxwell equation according to the theory of continuous time random walk (CTRW) (Metzler and Klafter, 2000):

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \sigma_\beta \, {}_0D_t^{1-\beta} \mathbf{E}(t) - \mu_0 \frac{\partial}{\partial t} \mathbf{J}_s , \qquad (1)$$

where, σ_β is the generalized electrical conductivity; and

$${}_0D_t^{1-\beta} \mathbf{E}(t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{\mathbf{E}(t')dt'}{(t-t')^{1-\beta}} , \qquad (2)$$

is the fractional derivative or Riemann-Louiville (R-L) operator (Oldham and Spanier, 1974). The parameter β, which varies from 0 to 1, describes the roughness of geoelectrical structures in formations. The value β=0 denotes the medium is homogeneous and diffusing particles obey classical laws. With β approaching 1, the medium becomes rougher and anomalous diffusion is expected. To determine the value of β of a geological medium, several aspects must be considered, with the fracture density believed to be a major factor. To quantitatively explore possible relationships between fracture density and the roughness parameter β is a long-term goal of our research.

We herein develop a 3D finite difference code to calculate the EM fields in rough formations based on the fractional Maxwell equation (1) in the frequency domain. The EM response of a 1D layered model is first compared with the analytical solution to verify the accuracy of the code. Then we develop a halfspace model containing a 3D rough fault zone, with fractures randomly distributed within it. The solutions to the E_x E_y secondary fields, based on classical Maxwell equation (β=0), are calculated for this model. We also calculate the E_x E_y secondary fields for the same model based on fractional Maxwell equation with different roughness β values assigned to the fault zone.

CODE DEVELOPMENT

To formulate the forward modeling problem, the Fourier transform of time is applied to equation (1), assuming $e^{i\omega t}$ time dependence, with ω the angular frequency. The Fourier transform of the fractional derivative in equation (2) is analytic (Metzler and Klafter 2000):

$$\mathcal{F}\left\{ {}_0D_t^{1-\beta}\mathbf{E}(t) \right\} = (i\omega)^{1-\beta}\tilde{\mathbf{E}}(\omega), \tag{3}$$

which renders equation (1) to be conveniently solved in frequency domain. To obtain a non-singular solution at the location of the physical source and at the primary boundary discontinuities, we choose to work with the secondary electric field $\mathbf{E}^s$ defined by:

$$\mathbf{E}^s = \mathbf{E}^t - \mathbf{E}^p, \tag{4}$$

where $\mathbf{E}^t$ is the total field generated by excitation of the full 3-D model while $\mathbf{E}^p$ is the primary field generated by the simpler background model. After the substitution og (4), the basic equation (1) for the modelling problem is obtained:

$$\tilde{\nabla} \times \tilde{\nabla} \times \tilde{\mathbf{E}}^s + \mu_0 \sigma_\beta (i\omega)^{1-\beta} \tilde{\mathbf{E}}^s = -i\omega\mu_0(\sigma_\beta - \sigma^p)\tilde{\mathbf{E}}^p, \tag{5}$$

where μ_0 is the free space magnetic permeability, and σ^p is the conductivity of the background model.

To discretize equation (5), the face-centered staggered grid is adopted (Smith, 1996b; Streich, 2009), where the electric components E_x, E_y and E_z are evaluated at the corresponding face centers of unit cells. A homogeneous Dirichlet boundary condition is applied at the model outer boundaries. The number of cells is $N_xN_yN_z$; this results in a total number of unknowns 3 $N_xN_yN_z$ - N_xN_y- N_xN_z - N_yN_z, not including nodes on the boundary. Since there are two model parameters in the equation, σ_β and β, we apply a volume-weighted averaging scheme seperately on each parameter at interface discontinuities. Namely,

$$\sigma_{i+\frac{1}{2},j,k} = \frac{\Delta x_i \sigma_{i,j,k} + \Delta x_{i+1}\sigma_{i+1,j,k}}{\Delta x_i + \Delta x_{i+1}}, \tag{6}$$

where $\sigma_{i+\frac{1}{2},j,k}$ is the conductivity at the shared face center of i[th] and (i+1)[th] cells along x- direction, and Δx is the grid width of each cell along x- direction. A similar averaging scheme is adopted for the β parameter.

This aforementioned discretization scheme results in a linear equation system that is solved using the iterative quasi-minimal residual (QMR) method (Freund and Nachtigal, 1994). The system matrix is preconditioned by Jacobi scaling. For low frequency problems (~1 Hz), a divergence correction (Smith, 1996) must be applied.

ALGORITHM VERIFICATION

A 1D layered model (Fig. 1) is developed to verify the accuracy of the algorithm and our implementation of it. The validation model consists of a halfspace structure, with air filling the upper space. The lower basement is homogeneous, of conductivity 0.003 S/m. A rough, resistive layer of conductivity 0.0001 S/m and β=0.2 lies between depths z=50-90 m. The nonzero β indicates that fractional diffusion is associated with this layer. The size of the model is (1.8 km)^3, which is discretized on a 70x70x70 grid. The minimum cell width is 12 m in all directions.The loop source is of radius 3 m and operates at frequency 1 kHz and lies directly on the surface of the lower halfspace. Receivers are located along the x-direction, on the surface at y=5 m. The comparision between the FD and the analytical solution for the E_x and E_y components, in terms of the total field, at the various receiver locations is shown in Fig. 2. Both the real and imaginary parts of the solutions agree very well. A slight discrepancy arises at distal receiver locations. The overall good agreement confirms that the averaging scheme for both conductivity and roughness is acceptable.

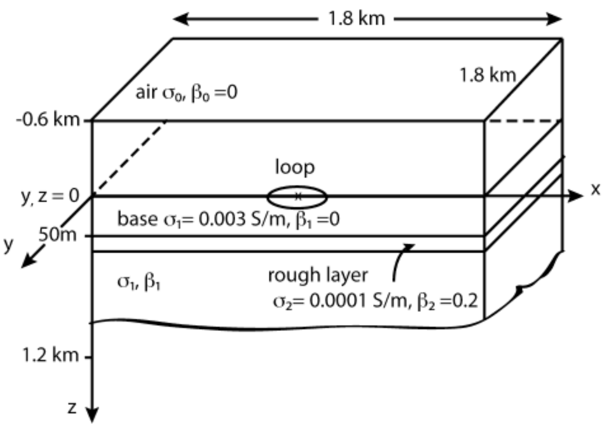

Figure 1. The layout of 1D conductivity model, with a thin resistive layer lying between z=50-90 m.

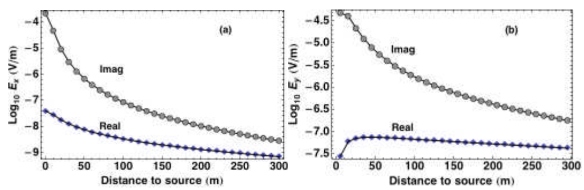

Figure 2. A comparison between FD (marker) and analytical (line) solutions to the E_x (a) and E_y (b) total field components at the receiver locations. The real and imaginary parts are displayed using different symbols.

3D FAULT ZONE MODEL INVESTIGATION

In this section, we further investigate fractional EM diffusion by considering a more realistic model (Fig. 3). The model is composed of a background halfspace that includes a rough fault zone. The zone is discretized on a subregion incorporating 10 × 20 × 20 cells.

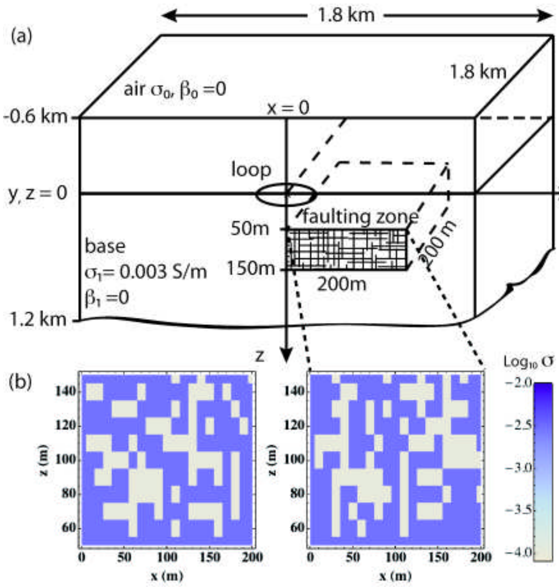

Figure 3. (a) The layout of the 3D fault zone model. (b) The vertical intersection of the fault zone in the x-z plane at y = 20m (left panel), 100m (right panel). The fracture density is 0.28 and the fracture conductivity is 0.0001 S/m, modeling open, or air-filled, fractures. The matrix is denoted by the purple regions and the fractures are denoted by the grey regions.

The matrix conductivity in the fault zone is the same as the lower half conductivity 0.003 S/m. The fracture conductivity 0.0001 S/m is randomly assigned to cells within the fault zone. In Fig. 3b, we display the vertical section of the fracture zone in the x-z plane at two different y-locations. The other elements of the model are the same as the previous layered background model. The fracture density is calculated as the ratio of the number of fracture conductivity cells to the total number of cells within the fault zone. Here we fix the fault zone fracture density to be 0.28. We first calculate the E_x^s, E_y^s fields based on the classical Maxwell equation for this model (β=0 for whole space). Then we calculate the fields for the same model but based on the fractional EM equation (1), which means that the conductivity in the fault zone is identical, but now the roughness β is nonzero. In the fractional-diffusion model, the generalized conductivity of the zone is 0.0002 S/m and β=0.5. In addition, the $\mathbf{E}^s$ fields for a homogeneous faulting zone (β=0) with conductivity 0.0001 S/m are also calculated for the reference.

The comparison is displayed in Figure 4. We observe that generally the classical and fractional diffusion solutions are very close. However for the E_x component, the discrepancy increases after the sign of the data flips, that is, around x = 170m. For the E_y component, at x=210 m, a noticeable discrepancy is also observed. This may partially result from the intrinsic errors due to finite difference solutions, since finer grids are essential to properly discretize the fault zone. We also notice that even for the classical fractured model, with the matrix conductivity being one order higher than the fracture conductivity and the local fracture density only 28%, the model response is still

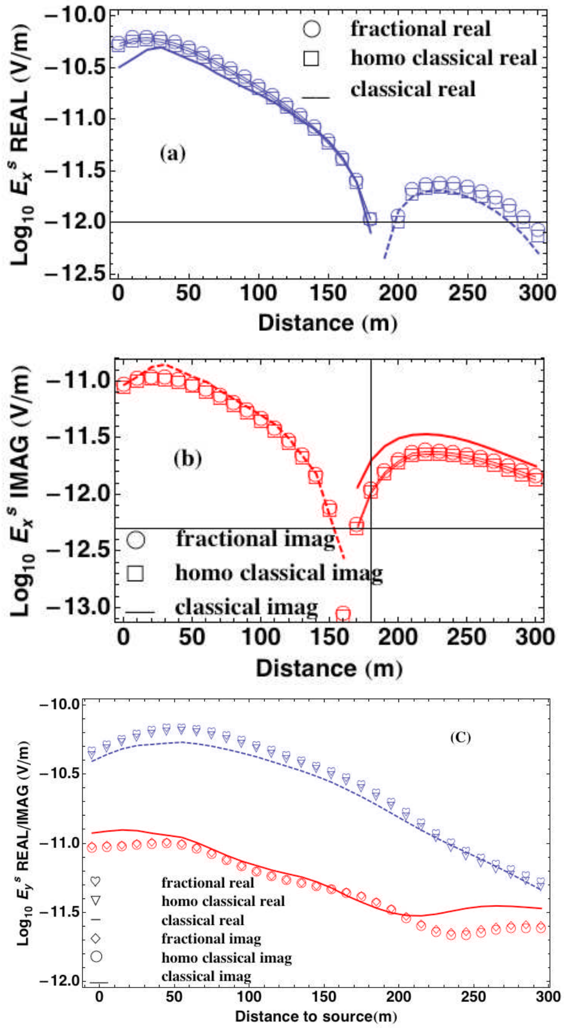

Figure 4. Comparison of $\mathbf{E}^s$ field calculation for the fault zone model using different methods. The "fractional" curve indicates solutions based on the fractional diffusion theory, with fault-zone generalized conductivity 0.0002 S/m and β=0.5; "classical" indicates solutions based on the classical Maxwell equation; "homo classical" indicates the fault zone in the model is homogeneous with conductivity 0.0001 S/m. In (a) and (b), the dashed line denotes positive response and the joined line between symbols indicates negative response.

comparable with a fault zone that has single conductivity of 0.0001 S/m. The early suggestion of this work is that the fractional diffusion solution does not agree well with the classical fractured zone model response, indicating that there is much additional research required to understand whether subdiffusing electromagnetic fields do indeed have a classical counterpart. We have long suspected that the subdiffusion of EM fields into hierarchical or fractured media may have no simple classical analog.

CONCLUSION

We discretize the fractional Maxwell equation in frequency domain using the 3D finite difference scheme on staggered grids. A halfspace model with a fractured fault zone is generated. The classical diffusion solutions for this model show that multi-scale diffusion dominates the response. The lack of agreement between the subdiffusion and the classical diffusion hints that, as long expected, electromagnetic induction into a hierarchical or fractal structure might have no precise classical analog.

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