UNIT 5 : Digital Filter Design

TABLE 8.1 Comparison of Digital and Analog Filters

Digital filter	*Analog filter*
1. It operates on digital samples (or sampled version) of the signal.	1. It operates on analog signals (or actual signals).
2. It is governed (or defined) by linear difference equations.	2. It is governed (or defined) by linear differential equations.
3. It consists of adders, multipliers, and delay elements implemented in digital logic (either in hardware or software or both).	3. It consists of electrical components like resistors, capacitors, and inductors.
4. In digital filters, the filter coefficients are designed to satisfy the desired frequency response.	4. In analog filters, the approximation problem is solved to satisfy the desired frequency response.

Advantages of digital filters

1. The values of resistors, capacitors and inductors used in analog filters change with temperature. Since the digital filters do not have these components, they have high thermal stability.
2. In digital filters, the precision of the filter depends on the length (or size) of the registers used to store the filter coefficients. Hence by increasing the register bit length (in hardware) the performance characteristics of the filter like accuracy, dynamic range, stability and frequency response tolerance, can be enhanced.
3. The digital filters are programmable. Hence the filter coefficients can be changed any time to implement adaptive features.
4. A single filter can be used to process multiple signals by using the techniques of multiplexing.

Disadvantages of digital filters

1. The bandwidth of the discrete signal is limited by the sampling frequency. The bandwidth of real discrete signal is half the sampling frequency.
2. The performance of the digital filter depends on the hardware (i.e., depends on the bit length of the registers in the hardware) used to implement the filter.

Important features of IIR filters

1. The physically realizable IIR filters do not have linear phase.
2. The IIR filter specifications include the desired characteristics for the magnitude response only.

Design of IIR filter by impulse invariant Transformation:

In this technique, the desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter. The main idea behind this is to preserve the frequency response characteristics of the analog filter. For the digital filter to possess the frequency response characteristics of the corresponding analog filter, the sampling period *T* should be sufficiently small (or the sampling frequency should be sufficiently high) to minimize (or completely avoid) the effects of aliasing.

Let $h_a(t)$ = Impulse response of analog filter

T = Sampling period

$h(n)$ = Impulse response of digital filter

For impulse invariant transformation,

$$h(n) = h_a(t)|_{t=nT} = h_a(nT)$$

The Laplace transform of the analog filter impulse response $h_a(t)$ gives the transfer function of analog filter.

$$\therefore \qquad L[h_a(t)] = H_a(s)$$

The transformation technique can be well understood by first considering a simple distinct poles case for the analog filter's system function as shown below.

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}$$

The impulse response $h_a(t)$ of the analog filter is obtained by taking the inverse Laplace transform of the system function $H_a(s)$.

$$\therefore \qquad h_a(t) = \mathrm{L}^{-1}[H_a(s)] = \sum_{i=1}^{N} A_i e^{p_i t} u_a(t)$$

where $u_a(t)$ is the unit step function in the continuous-time case.

The impulse response $h(n)$ of the equivalent digital filter is obtained by uniformly sampling $h_a(t)$, i.e.,

$$h(n) = h_a(nT) = \sum_{i=1}^{N} A_i e^{p_i nT} u_a(nT)$$

The system function of the digital system of above expression can be obtained by taking z-transform, i.e.

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

Using the above equation for $h(n)$, we have

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{N} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

Interchanging the order of summation, we have

$$H(z) = \sum_{i=1}^{N} \left[\sum_{n=0}^{\infty} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

$$= \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

Comparing the above expressions for $H_a(s)$ and $H(z)$, we can say that the impulse invariant transformation is accomplished by the mapping.

$$\frac{1}{s - p_i} \xrightarrow{\text{(is tranformed to)}} \frac{1}{1 - e^{p_i T} z^{-1}}$$

Relation between analog and digital poles

The above mapping shows that the analog pole at $s = p_i$ is mapped into a digital pole at $z = e^{p_c T}$. Therefore, the analog poles and the digital poles are related by the relation.

$$z = e^{sT}$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in polar form as $z = re^{j\omega}$.

$$\therefore \qquad re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$$

That means

$$|z| = r = e^{\sigma T}$$

and

$$\angle z = \omega = \Omega T$$

So the relationship between analog frequency Ω and digital frequency ω is $\omega = \Omega T$ or $\Omega = \dfrac{\omega}{T}$.

As a result of this, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$ and $\sigma = 0$ implies that $r = 1$. Therefore, the left half of s-plane is mapped into the interior of the unit circle in the z-plane. The right half of the s-plane is mapped into the exterior of the unit circle in the z-plane. This is one of the desirable properties for stability. The $j\Omega$-axis is mapped into the unit circle in z-plane. However, the mapping of $j\Omega$-axis is not one-to-one.

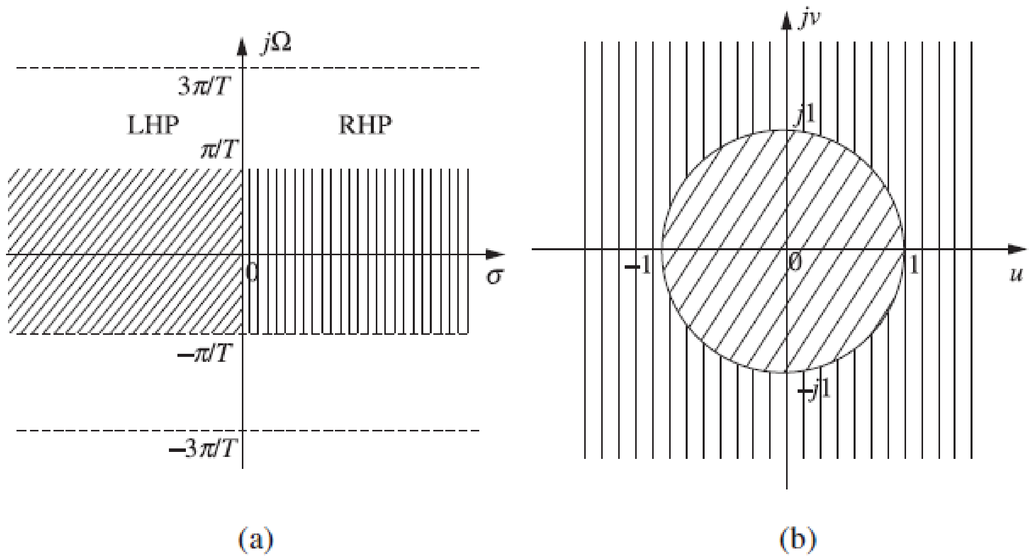

Figure 8.3 Mapping of (a) s-plane into (b) z-plane by impulse invariant transformation.

EXAMPLE 8.4 For the analog transfer function

$$H_a(s) = \frac{2}{(s+1)(s+3)}$$

determine $H(z)$ if (a) $T = 1$ s and (b) $T = 0.5$ s using impulse invariant method.

Solution: Given, $H_a(s) = \dfrac{2}{(s+1)(s+3)}$

Using partial fractions, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1)\,H_a(s)\big|_{s=-1} = \left.\frac{2}{s+3}\right|_{s=-1} = 1$$

$$B = (s+3)\,H_a(s)\big|_{s=-3} = \left.\frac{2}{s+1}\right|_{s=-3} = -1$$

$$\therefore \qquad H_a(s) = \frac{1}{s+1} - \frac{1}{s+3} = \frac{1}{s-(-1)} - \frac{1}{s-(-3)}$$

By impulse invariant transformation, we know that

$$\frac{A_i}{s-p_i} \xrightarrow{\text{(is transformed to)}} \frac{A_i}{1-e^{p_iT}z^{-1}}$$

Here $H_a(s)$ has two poles and $p_1 = -1$ and $p_2 = -3$.

Therefore, the system function of the digital filter is:

$$H(z) = \frac{1}{1-e^{p_1T}z^{-1}} - \frac{1}{1-e^{p_2T}z^{-1}}$$

$$= \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-3T}z^{-1}}$$

(a) When $T = 1$ s

$$H(z) = \frac{1}{1 - e^{-1}z^{-1}} - \frac{1}{1 - e^{-3}z^{-1}}$$

$$= \frac{1}{1 - 0.3678z^{-1}} - \frac{1}{1 - 0.0497z^{-1}}$$

$$= \frac{(1 - 0.0497z^{-1}) - (1 - 0.3678z^{-1})}{(1 - 0.3678z^{-1})(1 - 0.0497z^{-1})}$$

$$= \frac{0.3181z^{-1}}{1 - 0.4175z^{-1} + 0.0182z^{-2}}$$

(b) When $T = 0.5$ s

$$H(z) = \frac{1}{1 - e^{-0.5}z^{-1}} - \frac{1}{1 - e^{-3\times 0.5}z^{-1}}$$

$$= \frac{1}{1 - 0.606z^{-1}} - \frac{1}{1 - 0.223z^{-1}}$$

$$= \frac{(1 - 0.223z^{-1}) - (1 - 0.606z^{-1})}{(1 - 0.606z^{-1})(1 - 0.223z^{-1})}$$

$$= \frac{0.383z^{-1}}{1 - 0.829z^{-1} + 0.135z^{-2}}$$

EXAMPLE 8.7 The system function of an analog filter is expressed as:

$$H_a(s) = \frac{2}{s(s+2)}$$

Find the corresponding $H(z)$ using the impulse invariant method for a sampling frequency of 4 samples per second.

Solution: Given sampling rate = 4 samples/second

$\therefore$ Sampling period $T = \frac{1}{4} = 0.25$ s

Expressing the given $H_a(s)$ in terms of partial fractions, we have

$$H_a(s) = \frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2} = \frac{1}{s-(0)} - \frac{1}{s-(-2)}$$

By the impulse invariant transformation, we know that

$$\frac{A}{s-p_i} \xrightarrow{\text{(is transformed to)}} \frac{A}{1-e^{p_iT}z^{-1}}$$

Here $H_a(s)$ has two poles and $p_1 = 0$ and $p_2 = -2$.

Therefore, the system function of the digital filter is:

$$\begin{aligned} H(z) &= \frac{1}{1-e^{p_1T}z^{-1}} - \frac{1}{1-e^{p_2T}z^{-1}} \\ &= \frac{1}{1-e^{(0)T}z^{-1}} - \frac{1}{1-e^{(-2)T}z^{-1}} \\ &= \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-2(0.25)}z^{-1}} \\ &= \frac{1}{1-z^{-1}} - \frac{1}{1-0.606\,z^{-1}} \\ &= \frac{(1-0.606z^{-1})-(1-z^{-1})}{(1-z^{-1})(1-0.606z^{-1})} \\ &= \frac{0.394z^{-1}}{1-1.606z^{-1}+0.606z^{-2}} \end{aligned}$$

IIR filter by Bilinear Transformation Method:

To convert an analog filter function into an equivalent digital filter function, just put

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \text{ in } H_a(s)$$

EXAMPLE 8.12 Apply the bilinear transformation to

$$H_a(s) = \frac{4}{(s+3)(s+4)}$$

with $T = 0.5$ s and find $H(z)$.

Solution: Given that $H_a(s) = \dfrac{4}{(s+3)(s+4)}$ and $T = 0.5$ s.

To obtain $H(z)$ using the bilinear transformation, replace s by $\dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$ in $H_a(s)$

$$\therefore \qquad H(z) = \left.\frac{4}{(s+3)\,(s+4)}\right|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \left.\frac{4}{(s+3)\,(s+4)}\right|_{s=4\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{4}{\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+3\right]\left[4\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+4\right]}$$

$$= \frac{4}{\left[\dfrac{4-4z^{-1}+3+3z^{-1}}{1+z^{-1}}\right]\left[\dfrac{4-4z^{-1}+4+4z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{4(1+z^{-1})^2}{(7-z^{-1})\,8}$$

$$= \frac{1}{2}\,\frac{(1+z^{-1})^2}{(7-z^{-1})}$$

Design procedure for low pass digital Butterworth Low Pass Filter:

The low-pass digital Butterworth filter is designed as per the following steps:

Let A_1 = Gain at a passband frequency ω_1

A_2 = Gain at a stopband frequency ω_2

Ω_1 = Analog frequency corresponding to ω_1

Ω_2 = Analog frequency corresponding to ω_2

Step 1 Choose the type of transformation, i.e., either bilinear or impulse invariant transformation.

Step 2 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.

For bilinear transformation

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2},\ \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} \quad \therefore\ \frac{\Omega_2}{\Omega_1} = \frac{\tan\,\omega_2/2}{\tan\,\omega_1/2}$$

For impulse invariant transformation,

$$\Omega_1 = \frac{\omega_1}{T},\quad \Omega_2 = \frac{\omega_2}{T} \quad \therefore\ \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$$

Step 3 Decide the order N of the filter. The order N should be such that

$$N \geq \frac{1}{2}\,\frac{\log\left\{\left[\dfrac{1}{A_2^2}-1\right]\Big/\left[\dfrac{1}{A_1^2}-1\right]\right\}}{\log\dfrac{\Omega_2}{\Omega_1}}$$

Choose N such that it is an integer just greater than or equal to the value obtained above.

Step 4 Calculate the analog cutoff frequency $\Omega_c = \dfrac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

For bilinear transformation $\Omega_c = \dfrac{\dfrac{2}{T}\tan \omega_1/2}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

For impulse invariant transformation $\Omega_c = \dfrac{\omega_1/T}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

Step 5 Determine the transfer function of the analog filter.
Let $H_a(s)$ be the transfer function of the analog filter. When the order N is even, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

When the order N is odd, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

The coefficient b_k is given by

$$b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

For normalized case, $\Omega_c = 1$ rad/s

Step 6 Using the chosen transformation, transform the analog filter transfer function $H_a(s)$ to digital filter transfer function $H(z)$.

Step 7 Realize the digital filter transfer function $H(z)$ by a suitable structure.

Properties of Butterworth filters

1. The Butterworth filters are all pole designs (i.e. the zeros of the filters exist at ∞).
2. The filter order N completely specifies the filter.

3. The magnitude response approaches the ideal response as the value of N increases.
4. The magnitude is maximally flat at the origin.
5. The magnitude is monotonically decreasing function of Ω.
6. At the cutoff frequency Ω_c, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$. Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value.

EXAMPLE 8.17 Design a Butterworth digital filter using the bilinear transformation. The specifications of the desired low-pass filter are:

$$0.9 \le |H(\omega)| \le 1; \quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2; \quad \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ s

Solution: The Butterworth digital filter is designed as per the following steps.

From the given specification, we have

$$A_1 = 0.9 \text{ and } \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.2 \text{ and } \omega_2 = \frac{3\pi}{4} \qquad \text{and } T = 1 \text{ s}$$

Step 1 Choice of the type of transformation

Here the bilinear transformation is already specified.

Step 2 Determination of the ratio of the analog filter's edge frequencies, Ω_2/Ω_1

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{1}\tan\left[\frac{(3\pi/4)}{2}\right] = 2\tan\frac{3\pi}{8} = 4.828$$

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\left[\frac{(\pi/2)}{2}\right] = 2\tan\frac{\pi}{4} = 2$$

$$\therefore \qquad \frac{\Omega_2}{\Omega_1} = \frac{4.828}{2} = 2.414$$

Step 3 Determination of the order of the filter N

$$N \ge \frac{1}{2}\,\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

$$\geq \frac{1}{2} \frac{\log \left\{ \left[\dfrac{1}{(0.2)^2} - 1 \right] \Big/ \left[\dfrac{1}{(0.9)^2} - 1 \right] \right\}}{\log 1.207}$$

$$\geq \frac{1}{2} \frac{\log \{24/0.2345\}}{\log 2.414} \geq 2.626$$

Since $N \geq 2.626$, choose $N = 3$.

Step 4 Determination of the analog cutoff frequency Ω_c (i.e., –3 dB frequency)

$$\Omega_c = \frac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1 \right]^{1/2N}} = \frac{2}{\left[\dfrac{1}{0.9^2} - 1 \right]^{1/2 \times 3}} = 2.5467$$

Step 5 Determination of the transfer function of the analog Butterworth filter $H_a(s)$

For odd N, we have $H_a(s) = \dfrac{\Omega_c}{s + \Omega_c} \displaystyle\prod_{k=1}^{\frac{N-1}{2}} \dfrac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$

where $b_k = 2 \sin \left[\dfrac{(2k-1)\pi}{2N} \right]$

For $N = 3$, we have

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$$

where $b_1 = 2 \sin \left[\dfrac{(2 \times 1 - 1)\pi}{2 \times 3} \right] = 2 \sin \dfrac{\pi}{6} = 1$

Therefore, $H_a(s) = \left(\dfrac{2.5467}{s + 2.5467} \right) \left(\dfrac{(2.5467)^2}{s^2 + 1(2.5467)\, s + (2.5467)^2} \right)$

Step 6 Conversion of $H_a(s)$ into $H(z)$

Since bilinear transformation is to be used, the digital filter transfer function is:

$$H(z) = H_a(s) \Big|_{s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)} = H_a(s) \Big|_{s = 2\left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)}$$

$$H(z) = \left(\frac{2.5467}{2\left(\dfrac{1 - z^{-1}}{1 + z^{-1}} \right) + 2.5467} \right) \left[\frac{(2.5467)^2}{\left[2\left(\dfrac{1 - z^{-1}}{1 + z^{-1}} \right) \right]^2 + 2.5467 \left[2 \dfrac{1 - z^{-1}}{1 + z^{-1}} \right] + (2.5467)^2} \right]$$

$$= \frac{0.2332(1 + z^{-1})^3}{1 + 0.4394 z^{-1} + 0.3845 z^{-2} + 0.0416 z^{-3}}$$