Given a sequence $x(n)$ for $0 \le n \le 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$,

a. Evaluate its DFT $X(k)$.

Solution:

$$N = 4 \text{ and } W_4 = e^{-j\frac{\pi}{2}} \quad \Rightarrow \quad X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi k n}{2}}$$

Thus, for $k = 0$

$$\begin{aligned} X(0) &= \sum_{n=0}^{3} x(n)e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0} \\ &= x(0) + x(1) + x(2) + x(3) \\ &= 1 + 2 + 3 + 4 = 10 \end{aligned}$$

$$\begin{aligned} X(1) &= \sum_{n=0}^{3} x(n)e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}} \\ &= x(0) - jx(1) - x(2) + jx(3) \\ &= 1 - j2 - 3 + j4 = -2 + j2 \end{aligned}$$

$$\begin{aligned} X(2) &= \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} \\ &= x(0) - x(1) + x(2) - x(3) \\ &= 1 - 2 + 3 - 4 = -2 \end{aligned}$$

$$\begin{aligned} X(3) &= \sum_{n=0}^{3} x(n)e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}} \\ &= x(0) + jx(1) - x(2) - jx(3) \\ &= 1 + j2 - 3 - j4 = -2 - j2 \end{aligned}$$

Using MATLAB,

```
>> X = fft([1 2 3 4])
   X = 10.0000   -2.0000 + 2.0000i   -2.0000   -2.0000 - 2.0000i
```

Inverse DFT of the previous example.

$N = 4$ and $W_4^{-1} = e^{j\frac{\pi}{2}}$, $\longrightarrow$ $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi k n}{2}}$.

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j0} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}\right)$$
$$= \frac{1}{4}(10 + (-2 + j2) - 2 + (-2 - j2)) = 1$$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) + jX(1) - X(2) - jX(3))$$
$$= \frac{1}{4}(10 + j(-2 + j2) - (-2) - j(-2 - j2)) = 2$$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\pi} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right)$$
$$= \frac{1}{4}(X(0) - X(1) + X(2) - X(3))$$
$$= \frac{1}{4}(10 - (-2 + j2) + (-2) - (-2 - j2)) = 3$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi 3}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3))$$
$$= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4$$

Using MATLAB,

$$\gg \mathrm{x} = \mathrm{ifft}([10 \quad -2 + 2j \quad -2 \quad -2 - 2j])$$
$$\mathrm{x} = 1 \qquad 2 \qquad 3 \qquad 4.$$

DFT as a Linear Transformation

The Discrete Fourier Transform can be calculated using matrix notation.

$$X(k) = \sum_{n=0}^{N-1} x[n]W_N^{nk} \text{ , } k = 0,1,\cdots,N-1$$

$$\text{Where } W_N = e^{-j\frac{2\pi}{N}}$$

Expanding the above equation

$$X(0) = x[0] + x[1] + x[2] + \cdots\cdots + x[N-1]$$

$$X(1) = x[0] + x[1]W_N + x[2]W_N^2 + \cdots\cdots + x[N-1]W_N^{(N-1)}$$

$$\vdots$$

$$X(N-1) = x[0] + x[1]W_N^{(N-1)} + x[2]W_N^{2(N-1)} + \cdots\cdots + x[N-1]W_N^{(N-1)(N-1)}$$

Expressing the above set of equations in matrix notation, we obtain

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$or, \overline{\boldsymbol{X}} = \overline{\boldsymbol{W_N}}\ \overline{\boldsymbol{x}} \dots\dots\dots Eqn.A$$

Similarly, for the IDFT equation, $x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X(k)W_N^{-nk}$, $n = 0, 1, \cdots, N-1$, the matrix notation would be

$$\overline{\boldsymbol{x}} = \frac{1}{N}\overline{\boldsymbol{W_N}}^{*}\ \overline{\boldsymbol{X}} \dots\dots\dots Eqn\ B$$

But, from equation A,

$$\overline{\boldsymbol{x}} = \overline{\boldsymbol{W_N}}^{-1}\ \overline{\boldsymbol{X}} \dots\dots\dots Eqn.C$$

From Equations B & C, we have

$$\overline{\boldsymbol{W_N}}^{-1} = \frac{1}{N}\overline{\boldsymbol{W_N}}^{*}$$

Example $x[n] = [0\ 1\ 2\ 3]$

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Only additions are needed to compute this specific transform.

(This is a well-known *radix-4 FFT*)

Thus, the DFT of $x[n]$ is $\mathbf{X}_4 = \mathbf{W}_4 \mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$

Example Determine the DFT of a length-4 sequence given by $x(n) = \{(1,1,1,1)\}$. and go further to obtain the IDFT also using matrices. The sequence used is a length-4 sequence given by $x(n) = \{(1\ 1\ 1\ 1)\}$.

Solution

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}.$$

To simplify we must use the property of the twiddle factor that for $n > N, W_N^n = \langle W_N^n \rangle_N$, where the operator $\langle \rangle_N$ implies modulo N operation.

Therefore substituting $W_4^4 = W_4^0, W_4^6 = W_4^2, W_4^9 = W_4^1$ in the matrix equation we obtain

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Evaluating $W_4^0 = 1, W_4^1 = e^{-i2\pi/4} = -j, W_4^2 = (-j)^2 = -1, W_4^3 = (-j)(-1) = j$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Therefore $X(k) = [4, 0, 0, 0]$,

In order to find the IDFT we need to find $\overline{W_N}^{-1} = \overline{(W_N)}^*$. This is obtained by conjugating each element of $\overline{W_N}$ as follows:

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Properties:

Property 01: Linearity:

Statement: if $x_1(n) \xrightarrow{DFT} X_1(K)$

$x_2(n) \xrightarrow{DFT} X_2(K)$

then:

$$a x_1(n) + b x_2(n) \xrightarrow{DFT} a X_1(K) + b X_2(K)$$

Proof: $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi Kn}{N}}$

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi Kn}{N}}$$

$$DFT\{a x_1(n) + b x_2(n)\} = \sum_{n=0}^{N-1} [a x_1(n) + b x_2(n)] e^{-j\frac{2\pi Kn}{N}}$$

$$= a \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi Kn}{N}} + b \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi Kn}{N}}$$

$$\boxed{DFT\{a x_1(n) + b x_2(n)\} = a X_1(K) + b X_2(K)}$$

Property 02: periodicity.

if $x(n) \xrightarrow{DFT} X(K)$ → statement

a) $x(n+N) = x(n)$

b) $X(K+N) = X(K)$

a) $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi Kn}{N}}$

$$x(n+N) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi K(n+N)}{N}}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi KN}{N}} e^{j\frac{2\pi Kn}{N}}$$

$$\boxed{x(n+N) = x(n)}$$

b) $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi Kn}{N}}$

$$X(K+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi (K+N)n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi Kn}{N}} \cdot e^{-j\frac{2\pi Nn}{N}}$$

$$\boxed{X(K+N) = X(K)}$$

Property 3: Circular shift in time domain

statement: if $x(n) \xrightarrow{DFT} X(K)$

$$x(n-m)_N \longrightarrow X(K)\, e^{-j\frac{2\pi Km}{N}}$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi Kn}{N}}$$

$$IDFT\{X(K)\} = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi Kn}{N}}$$

$$IDFT\{X(K)\, e^{-j\frac{2\pi Km}{N}}\} = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{-j\frac{2\pi Km}{N}} \cdot e^{j\frac{2\pi Kn}{N}} \cdot 1$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K) \cdot e^{-j\frac{2\pi Km}{N}} \cdot e^{j\frac{2\pi Kn}{N}} \cdot e^{j\frac{2\pi KN}{N}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi K}{N}(-m+n+N)}$$

$$= x(n-m+N)$$

$$\boxed{X(K) \cdot e^{-j\frac{2\pi Km}{N}} = x(n-m)_N}$$

Property 4: Circular shift in frequency domain.

statement if $x(n) \xrightarrow{DFT} X(K)$

$$x(n)\, e^{j\frac{2\pi nm}{N}} \longrightarrow X(K-m)_N$$

W.K.T $X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi Kn}{N}}$

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi Kn}{N}}$$

$$DFT\{x(n)\, e^{j\frac{2\pi nm}{N}}\} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi Kn}{N}} \cdot e^{j\frac{2\pi nm}{N}} \cdot 1$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi Kn}{N}} \cdot e^{j\frac{2\pi nm}{N}} \cdot e^{-j\frac{2\pi nN}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n}{N}(K-m+N)}$$

$$X(K-m+N)$$

$$\boxed{x(n) \cdot e^{j\frac{2\pi nm}{N}} = X(K-m)_N}$$

Property 5: Time Reversal (or) Reflection property

if $x(n) \xrightarrow{DFT} X(K)$

$$x(-n)_N = X(-K)_N.$$

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi kn}{N}}$$

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}$$

$$DFT\{x(-n)_N\} = \sum_{n=0}^{N-1} x(-n)_N\, e^{-j\frac{2\pi k(n)}{N}}$$

Take $N-n=l$

$n = N-l$

put $n=0$; $l=N$

$n=N-1$; $l=1$

$$= \sum_{n=0}^{N-1} x(N-n)\, e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{l=N}^{1} x(l)\, e^{-j\frac{2\pi k(N-l)}{N}} \quad \text{interchanging the sum}$$

$$= \sum_{l=1}^{N} x(l)\, e^{-j2\pi k} \cdot e^{+j\frac{2\pi kl}{N}} \quad (1)$$

$$= \sum_{l=1}^{N} x(l)\, e^{+j\frac{2\pi kl}{N}} \cdot 1$$

$$= \sum_{l=1}^{N} x(l)\, e^{-j\frac{2\pi l}{N}(-K+N)}$$

$$= X(-K+N)$$

$$\boxed{x(-n)_N = X(-K)_N}$$

Problem (1) if $x(n) = \{1, 2, 3/4, 1/4\}$ find $y(n)$ if their DFTs are related as $Y(K) = W_4^{3K} \cdot X(K)$

soln

$$Y(K) = W_4^{3K}\, X(K)$$

$$= e^{-j\frac{2\pi 3k}{4}}\, X(K)$$

$$y(n) = x(n-3)_4$$

$$= \{x(0-3)_4, x(1-3)_4, x(2-3)_4, x(3-3)_4\}$$

$$= \{x(-3)_4, x(-2)_4, x(-1)_4, x(0)_4\}$$

$$= \{x(4-3), x(4-2), x(4-1), x(4-0)\}$$

$$= \{x(1), x(2), x(3), x(4)\} \rightarrow x(4-4) \rightarrow x(0)$$

$$\boxed{y(n) = \{2, \tfrac{3}{4}, \tfrac{1}{4}, 1\}}$$

FFT(Fast Fourier Transform):

Divide and conquer approach:

This approach is based on decomposition of an N-point DFT into successively smaller DFT's. This approach leads to an efficient algorithm called Fast Fourier transform.

Let $x(n) \rightarrow 0 \le n \le N-1$.

$N = 8$.

$\therefore x(n) \rightarrow 0 \le n \le 7$.

$\therefore x(n) = \{ x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7) \}$

0 1 2 3 4 5 6 7

Radix-2 algorithm:

The general form is, $N = r^m$

where m = no. of stages and

if $r = 2$, then it is radix-2 algorithm.

if $r = 4$, then it is radix-4 algorithm.

* Decimation-in-time Fast Fourier transform algorithm:

(Radix-2, DIT-FFT).

Divide and conquer approach results in splitting of N-point data sequence to 2 N/2 point data sequence $x_e[n]$ and $x_o[n]$ corresponding to the even numbered and odd numbered samples of $x[n]$.

3.5.1 The Decimation-in-Time FFT Algorithm

The Fast Fourier Transform (FFT) is used to compute an N-Point DFT by computing smaller-size DFTs and taking advantage of the periodicity and symmetry properties of the complex function W_N^{kn}. Consider a sequence $x(n)$ of length N, where N is a power of 2 (i.e., $N = 2^\mu$). In Equation (3.3) we have defined the DFT of a sequence $x(n)$ as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \text{for } 0 \le k \le N-1, \tag{3.17}$$

where $W_N = e^{-j2\pi/N}$.

The computation of $X(k)$ requires N^2 complex multiplications and $N(N-1)$ complex additions.

The process of decimation-in-time

(i) Computation of $X(k)$

If N is a power of 2 it is possible to decimate $x(n)$ into $N/2$-point sequences such that one has samples which are even numbered and the other has samples that are odd numbered.

$$X(k) = \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn}$$

or

$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{k2r} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{k(2r+1)}. \tag{3.18}$$

Applying the properties of the complex function W_N^{kn} we can write

$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) {W_{N/2}}^{kr} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) {W_{N/2}}^{kr} \tag{3.19}$$

or

$$X(k) = X_o(k) + W_N^k X_1(k), \tag{3.20}$$

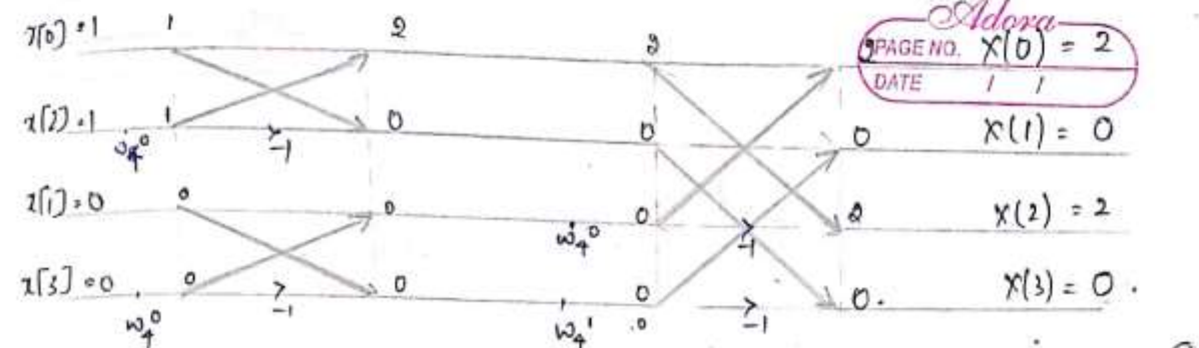
where $X_0(k)$ and $X_1(k)$ are $N/2$-point DFTs and $x_0(n) = x(2r)$ and $x_1(n) = x(2r+1)$. For $N = 8$ eight equations can be written relating N-point DFT $X(k)$ to $N/2$-point DFTs $X_0(k)$ and $X_1(k)$.

$$\left.\begin{aligned}
X(0) &= X_0(0) + W_N^0 X_1(0) \\
X(1) &= X_0(1) + W_N^1 X_1(1) \\
X(2) &= X_0(2) + W_N^2 X_1(2) \\
X(3) &= X_0(3) + W_N^3 X_1(3) \\
\\
X(4) &= X_0(0) + W_N^4 X_1(0) \\
X(5) &= X_0(1) + W_N^5 X_1(1) \\
X(6) &= X_0(2) + W_N^6 X_1(2) \\
X(7) &= X_0(3) + W_N^7 X_1(3)
\end{aligned}\right\} \tag{3.21}$$

A flow graph representation of Equation (3.21) is shown in Figure 3.8.

The computation of the N-point DFT using the modified scheme requires two $N/2$-point DFTs which are combined with N complex multiplications and N complex additions. The total number of complex multiplications is $2\left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$ and the total number of complex additions is $2\left(\frac{N}{2} - 1\right)\left(\frac{N}{2}\right) + N = \frac{N^2}{2}$. Compared to the original computation using Equation (3.7) the percentage reduction for both complex multiplication and additions is close to 50% for large values of N.

(7). Find 4 point DFT of sequence $x[n] = \{1, 0, 1, 0\}$ using DIT FFT.



$W_4^0 = 1$, $W_4^1 = -j$ $\quad$ $N = 4$, $N = 2^m \Rightarrow 4 = 2^m$, $m = 2$

$X(K) = \{2, 0, 2, 0\}$

(8). Find 4 point DFT of the sequence $x[n] = \{0, 1, 2, 3\}$ using DIF-FFT algorithm.

Example Given $x(n) = \{0, 1, 2, 3\}$, find $X(k)$ using DIT FFT algorithm.

Solution Given $N = 4$

$$W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$$

$$W_4^0 = 1 \text{ and } W_4^1 = e^{-j\pi/2} = -j$$

Using DIT FFT algorithm, we can find $X(k)$ from the given sequence $x(n)$ as shown in Fig.

Therefore, $X(k) = \{6, -2 + j2, -2, -2 - j2\}$

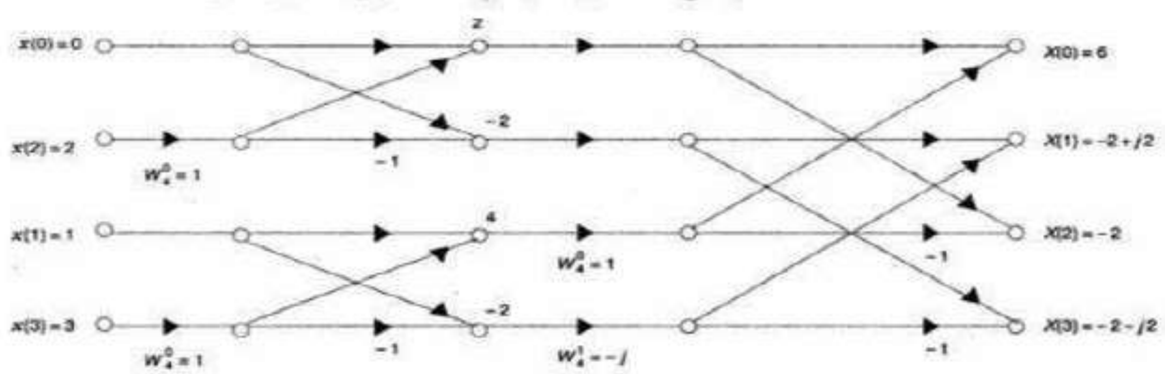


Fig.

THE 8-POINT DFT USIKG RADIX-2 DIT FFT

The computation of 8-point DFT using radix-2 FFT involves three stages of computation. Here $N = 8 = 2^3$, therefore, $r = 2$ and $m = 3$. The given 8-point sequence is decimated into four 2-point sequences. For each 2-point sequence, the two point DFT is computed. From the results of four 2-point DFTs, two 4-point DFTs are obtained and from the results of two 4-point DFTs, the 8-point DFT is obtained.

Let the given 8-sample sequence $x(n)$ be $\{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$. The 8-samples should be decimated into sequences of two samples. Before decimation they are arranged in bit reversed order as shown in Table 2.1.

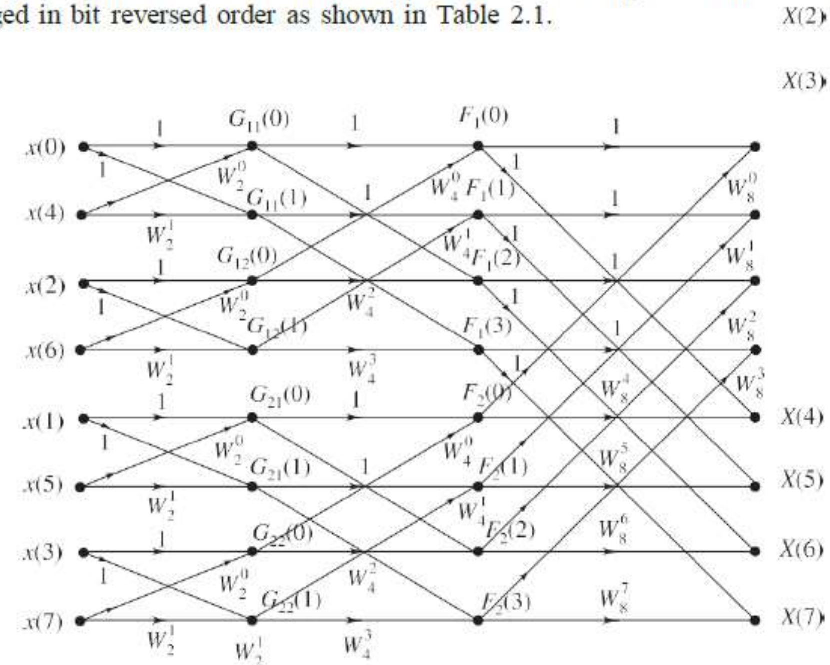


Figure 2.4 Illustration of complete flow graph obtained by combining all the three stages for $N = 8$.

TABLE 2.1 Normal and bit reversed order for $N = 8$.

Normal order		*Bit reversed order*	
$x(0)$	$x(000)$	$x(0)$	$x(000)$
$x(1)$	$x(001)$	$x(4)$	$x(100)$
$x(2)$	$x(010)$	$x(2)$	$x(010)$
$x(3)$	$x(011)$	$x(6)$	$x(110)$
$x(4)$	$x(100)$	$x(1)$	$x(001)$
$x(5)$	$x(101)$	$x(5)$	$x(101)$
$x(6)$	$x(110)$	$x(3)$	$x(011)$
$x(7)$	$x(111)$	$x(7)$	$x(111)$

The $x(n)$ in bit reversed order is decimated into 4 numbers of 2-point sequences as shown below.

(i) $x(0)$ and $x(4)$
(ii) $x(2)$ and $x(6)$
(iii) $x(1)$ and $x(5)$
(iv) $x(3)$ and $x(7)$

Using the decimated sequences as input, the 8-point DFT is computed. Figure 7.5 shows the three stages of computation of an 8-point DFT.

The computation of 8-point DFT of an 8-point sequence in detail is given below. The 8-point sequence is decimated into 4-point sequences and 2-point sequences as shown below.

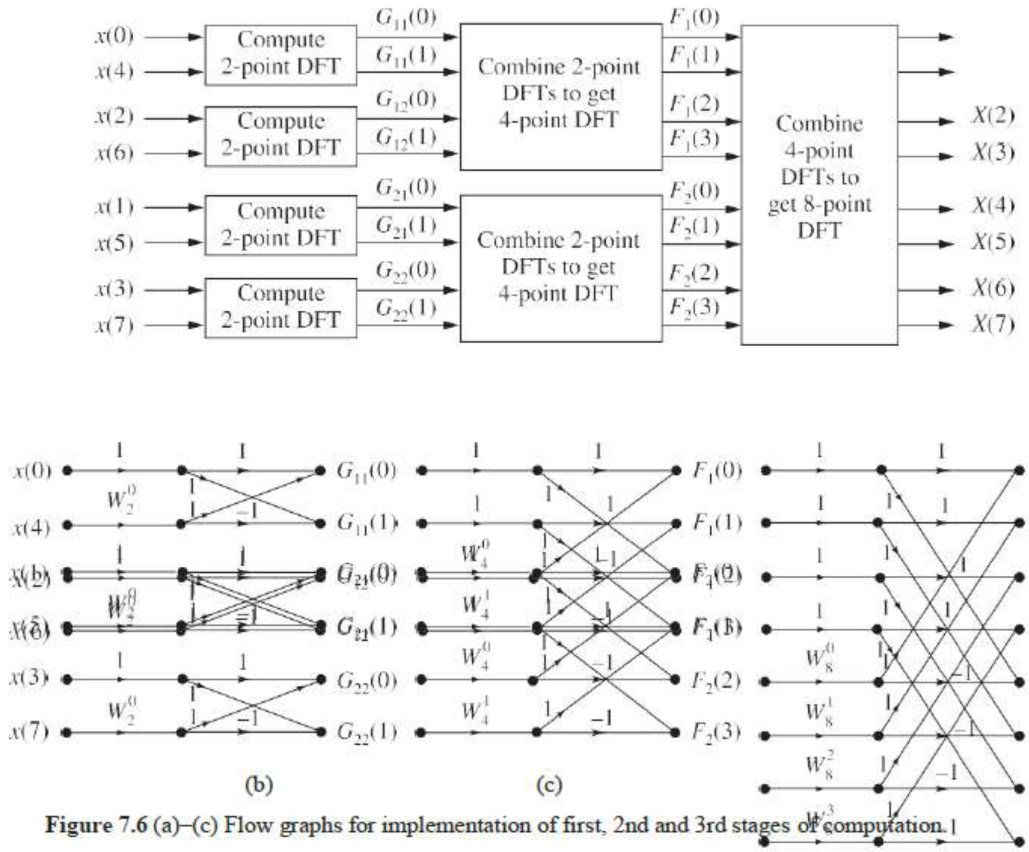


Figure 7.6 (a)–(c) Flow graphs for implementation of first, 2nd and 3rd stages of computation.

Butterfly Diagram

Observing the basic computations performed at each stage, we can arrive at the following conclusions:

- (i) In each computation, two complex numbers a and b are considered.
- (ii) The complex number b is multiplied by a phase factor W_N^k
- (iii) The product bW_N^k is added to the complex number a to form a new complex number A.
- (i) The product bW_N^k is subtracted from complex number a to form new complex number B.

The above basic computation can be expressed by a signal flow graph shown in Figure 7.7. The signal flow graph is also called butterfly diagram since it resembles a butterfly.

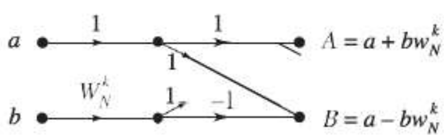


Figure 7.7 Basic butterfFy diagram or fFow graph of radix–2 DI† FF†.

The complete flow graph for 8-point DIT FFT considering periodicity drawn in a way to remember easily is shown in Figure 7.8. In radix-2 FFT, $N/2$ butterflies per stage are required to represent the computational process. In the butterfly diagram for 8-point DFT shown in Figure 7.8, for symmetry, W_2^0, W_4^0 and W_8^0 are shown on the graph eventhough they are unity. The subscript 2 indicates that it is the first stage of computation. Similarly, subscripts 4 and 8 indicate the second and third stages of computation.

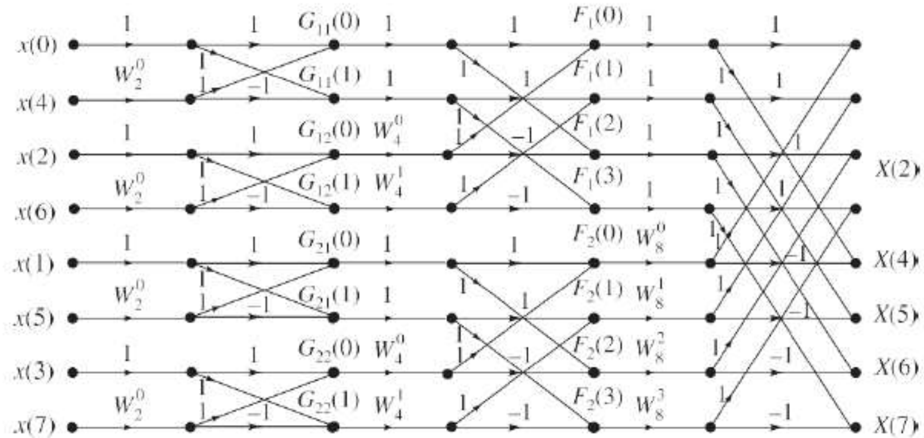


Figure 7.8 †he signal flow graph or butterfly diagram for 8–point radix–2 DIT FFT.

EXAMPLE 1 Draw the butterfly line diagram for 8-point FFT calculation and briefly explain. Use decimation-in-time algorithm.

Solution: The butterfly line diagram for 8-point DIT FFT algorithm is shown in following Figure

Solution: For 8-point DIT FFT

1. The input sequence $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$,
2. bit reversed order, of input as i.e. as $x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$. Since $N = 2^m = 2^3$, the 8-point DFT computation
3. Radix-2 FFT involves 3 stages of computation, each stage involving 4 butterflies. The output $X(k)$ will be in normal order.
4. In the first stage, four 2-point DFTs are computed. In the second stage they are combined into two 4-point DFTs. In the third stage, the two 4-point DFTs are combined into one 8-point DFT.
5. The 8-point FFT calculation requires $8 \log_2 8 = 24$ complex additions and $(8/2) \log_2 8 = 12$ complex multiplications.

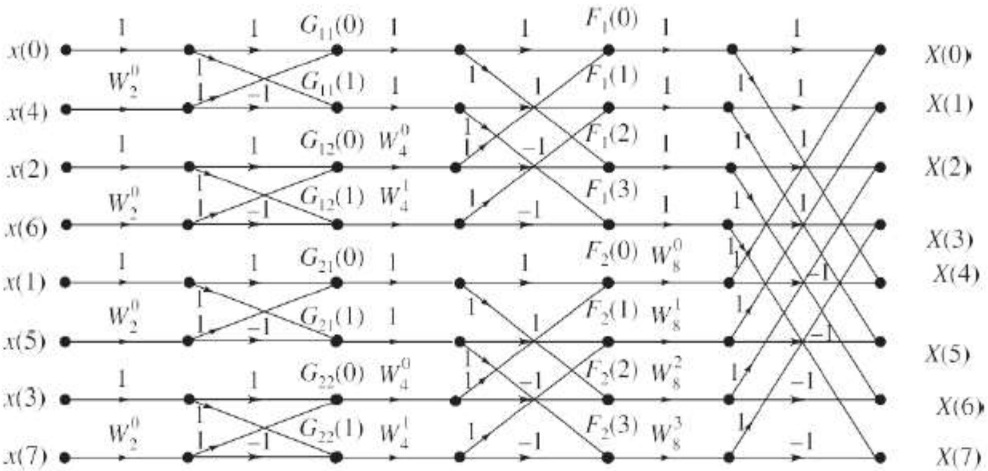


Figure : Butterfly Fine diagram for 8–point DIT FFT algorithm for $N = 8$.

EXAMPLE 7.13 Find the 8-point DFT by radix-2 DIT FFT algorithm.

$$x(n) = \{2, 1, 2, 1, 2, 1, 2, 1\}$$

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$k$$

$$= \{2, 1, 2, 1, 2, 1, 2, 1\}$$

For DIT FFT computation, the input sequence must be in bit reversed order and the output sequence will be in normal order.

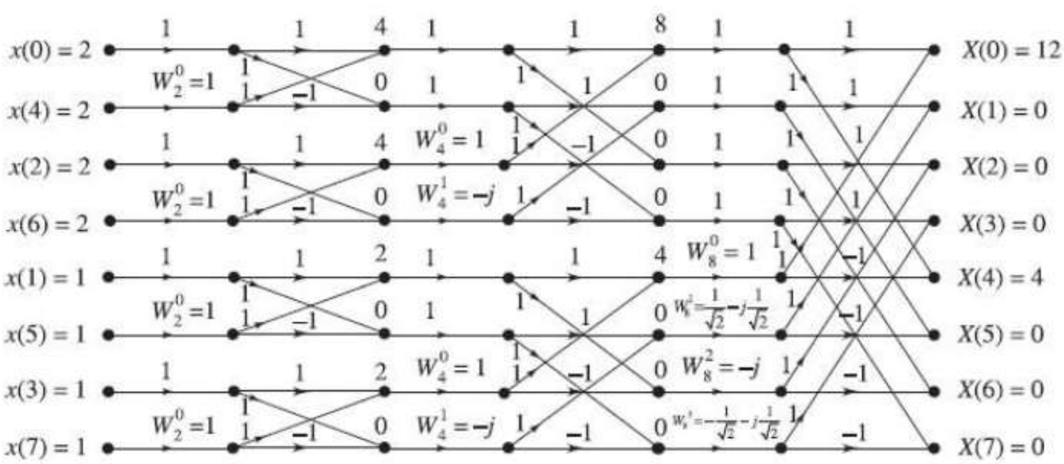
$x(n)$ in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{2, 2, 2, 2, 1, 1, 1, 1\}$$

The computation of 8-point DFT of $x(n)$ by radix-2 DIT FFT algorithm is shown in Figure 7.35.

From Figure 7.35, we get the 8-point DFT of $x(n)$ as $X(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$



EXAMPLE 7.14 Compute the DFT for the sequence $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$.

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{1, 1, 1, 1, 1, 1, 1, 1\}$$

The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by radix-2, DIT FFT algorithm is shown in Figure 7.36.

The given sequence in bit reversed order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$
$$= \{1, 1, 1, 1, 1, 1, 1, 1\}$$

For DIT FFT, the input is in bit reversed order and output is in normal order.

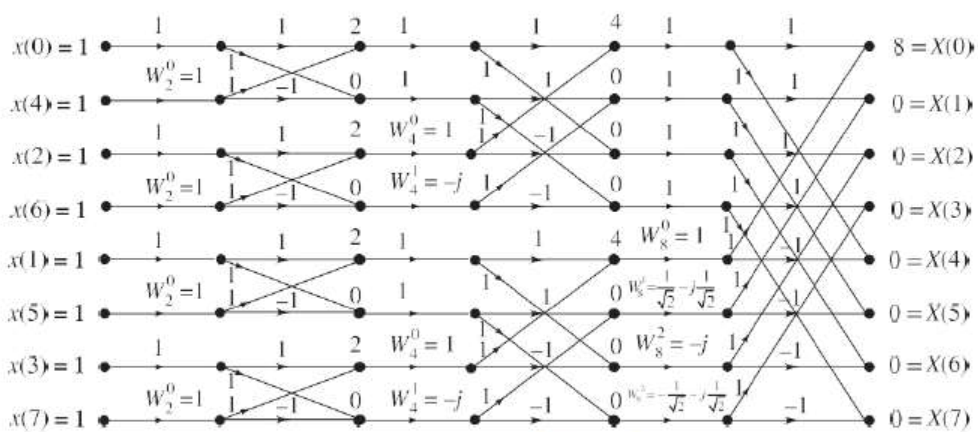


Figure 7.36 Computation of 8–point DF† of $x(n)$ by radix–2, DI† FF†.

From Figure 7.36, we get the 8-point DFT of $x(n)$ as $X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$.

→ 4 point DFT using DIF FFT :