Discrete Signal and Discrete Time Signal

The ***discrete signal*** is a function of a discrete independent variable. The independent variable is divided into uniform intervals and each interval is represented by an integer. The letter "n" is used to denote the independent variable. The discrete or digital signal is denoted by x(n).

The discrete signal is defined for every integer value of the independent variable "n". The magnitude (or value) of discrete signal can take any discrete value in the specified range. Here both the value of the signal and the independent variable are discrete. The discrete signal can be represented by a one-dimensional array as shown in the following example.

Example :

$$x(n) = \{\,2, 4, -1, 3, 3, 4\,\}$$

Here the discrete signal x(n) is defined for, $n = 0, 1, 2, 3, 4, 5$

$$\backslash \quad x(0) = 2\,; \quad x(1) = 4\,; \quad x(2) = -1\,; \quad x(3) = 3\,; \quad x(4) = 3\,; \quad x(5) = 4\,.$$

When the independent variable is time t, the discrete signal is called ***discrete time signal***. In discrete time signal, the time is divided uniformly using the relation $t = nT$, where T is the sampling time period. (The sampling time period is the inverse of sampling frequency). The discrete time signal is denoted by x(n) or x(nT).

2.8 Classification of Discrete Time Systems

The discrete time systems are classified based on their characteristics. Some of the classifications of discrete time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. FIR and IIR systems
7. Recursive and nonrecursive systems

2.8.1 Static and Dynamic Systems

A discrete time system is called ***static*** or ***memoryless*** system if its output at any instant n depends at most on the input sample at the same time but not on the past or future samples of the input. In any other case, the system is said to be ***dynamic*** or to have memory.

Example :

$$\left.\begin{aligned} y(n) &= a\,x(n)\\ y(n) &= n\,x(n) + 6\,x^3(n)\end{aligned}\right\}\text{Static systems}$$

$$\left.\begin{aligned}&\left.\begin{aligned} y(n) &= x(n) + 3\,x(n-1)\\ y(n) &= \sum_{m=0}^{N} x(n-m)\end{aligned}\right\}\text{Finite memory is required}\\ &\left. y(n) = \sum_{m=0}^{\infty} x(n-m)\quad\right\}\text{Infinite memory is required}\end{aligned}\right\}\text{Dynamic systems}$$

Digital Signal

The ***digital signal*** is same as discrete signal except that the magnitude of the signal is quantized. The magnitude of the signal can take one of the values in a set of quantized values. Here quantization is necessary to represent the signal in binary codes.

The generation of a discrete time signal by sampling a continuous time signal and then quantizing the samples in order to convert the signal to digital signal is shown in the following example.

Let, x(t) = Continuous time signal

T = Sampling time

A typical continuous time signal and the sampling of this continuous time signal at uniform interval are shown in fig 2.1a and fig 2.1b respectively. The samples of the continuous time signal as a function of sampling time instants are shown in fig 2.1c. (In fig 2.1c, 1T, 2T, 3T, ...etc., represents sampling time instants and the value of the samples are functions of this sampling time instants).

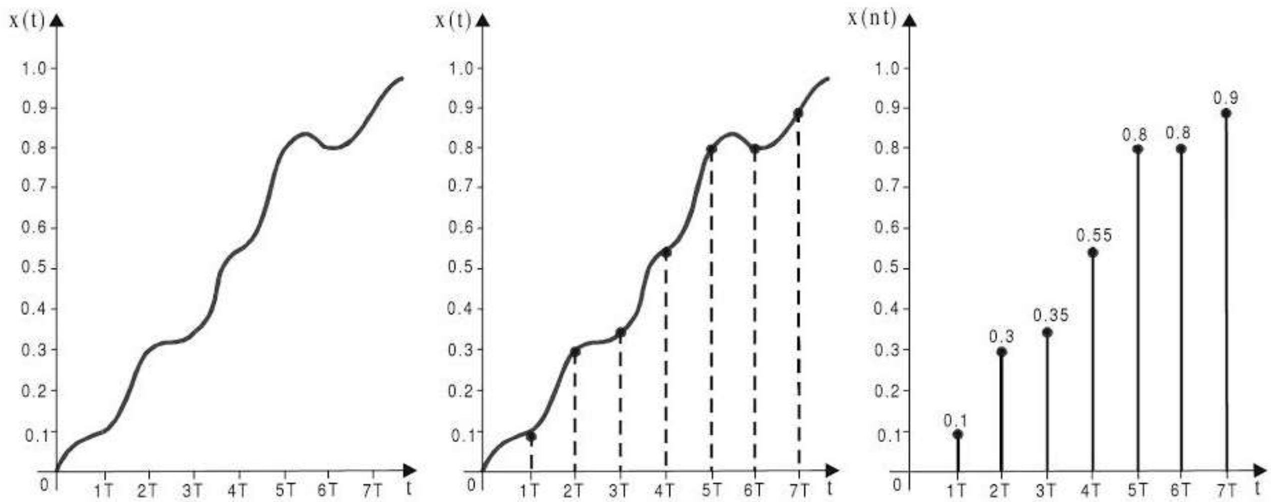

Fig 2.1a. *Fig 2.1b.* *Fig 2.1c.*

Fig 2.1 : *Sampling a continuous time signal to generate discrete time signal.*

2.2.2 Representation of Discrete Time Signals

The discrete time signal can be represented by the following methods.

1. Functional representation

In functional representation, the signal is represented as a mathematical equation, as shown in the following example.

$$
\begin{aligned}
x(n) &= -\ 0.5 \quad ; \quad n = -2\\
&= 1.0 \quad ; \quad n = -1\\
&= -\ 1.0 \quad ; \quad n = 0\\
&= 0.6 \quad ; \quad n = 1\\
&= 1.2 \quad ; \quad n = 2\\
&= 1.5 \quad ; \quad n = 3\\
&= 0 \quad ; \quad \text{other } n
\end{aligned}
$$

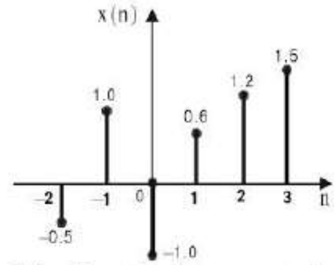

Fig 2.2 : Graphical representation of a discrete time signal.

2. Graphical representation

In graphical representation, the signal is represented in a two-dimensional plane. The independent variable is represented in the horizontal axis and the value of the signal is represented in the vertical axis as shown in fig 2.2.

3. Tabular representation

In tabular representation, two rows of a table are used to represent a discrete time signal. In the first row, the independent variable "n" is tabulated and in the second row the value of the signal for each value of "n" are tabulated as shown in the following table.

n	....	–2	-1	0	1	2	3	....
x(n)	....	–0.5	1.0	–1.0	0.6	1.2	1.5	....

4. Sequence representation

In sequence representation, the discrete time signal is represented as a one-dimensional array as shown in the following examples.

An infinite duration discrete time signal with the time origin, n = 0, indicated by the symbol – is represented as,

$x(n) = \{ \ldots\ -0.5,\ 1.0,\ -1.0,\ 0.6,\ 1.2,\ 1.5,\ \ldots \}$

An infinite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

$x(n) = \{ -1.0,\ 0.6,\ 1.2,\ 1.5,\ \ldots \}$ or $x(n) = \{-1.0,\ 0.6,\ 1.2,\ 1.5,\ \ldots \}$

A finite duration discrete time signal with the time origin, n = 0, indicated by the symbol – is represented as,

$x(n) = \{ -0.5,\ 1.0,\ -1.0,\ 0.6,\ 1.2,\ 1.5 \}$

A finite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

$x(n) = \{ -1.0,\ -0.6,\ 1.2,\ 1.5 \}$ or $x(n) = \{ -1.0,\ 0.6,\ 1.2,\ 1.5\}$

2.2.3 Standard Discrete Time Signals

1. Digital impulse signal or unit sample sequence

Impulse signal, $\delta(n) = 1 \;\; ; \;\; n = 0$

$\qquad\qquad\qquad\quad = 0 \;\; ; \;\; n \neq 0$

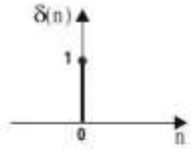

***Fig 2.3** : Digital impulse signal.*

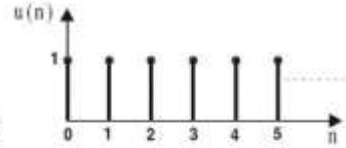

***Fig 2.4** : Unit step signal.*

2. Unit step signal

Unit step signal, $u(n) = 1 \; ; \; n \geq 0$

$\qquad\qquad\qquad\quad = 0 \; ; \; n < 0$

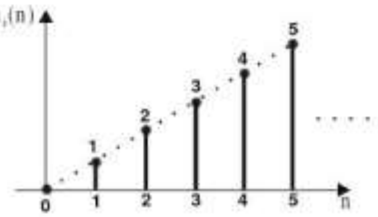

***Fig 2.5** : Ramp signal.*

3. Ramp signal

Ramp signal, $u_r(n) = n \;\; ; \; n \geq 0$

$\qquad\qquad\qquad\quad = 0 \; ; \; n < 0$

4. Exponential signal

Exponential signal, $g(n) = a^n \;\; ; \; n \geq 0$

$\qquad\qquad\qquad\qquad\quad = 0 \;\; ; \; n < 0$

***Fig 2.6a** : Decreasing exponential signal.*

***Fig 2.6b** : Increasing exponential signal.*

***Fig 2.6** : Exponential signal.*

PROPERTIES OF Z TRANSFORM (ZT)

1) Linearity

The linearity property states that if

$$x1(n) \xleftrightarrow{z} X1(z) \quad \text{And}$$

$$x2(n) \xleftrightarrow{z} X2(z) \quad \text{Then}$$

Then

$$a1\ x1(n) + a2\ x2(n) \xleftrightarrow{z} a1\ X1(z) + a2\ X2(z)$$

z Transform of linear combination of two or more signals is equal to the same linear combination of z transform of individual signals.

2) Time shifting

The Time shifting property states that if

$$x(n) \xleftrightarrow{z} X(z) \quad \text{And}$$

Then $\quad x(n-k) \xleftrightarrow{z} X(z)\ z^{-k}$

Thus shifting the sequence circularly by _k' samples is equivalent to multiplying its z transform by z^{-k}

3) Scaling in z domain

This property states that if

$$x(n) \xleftrightarrow{z} X(z) \quad \text{And}$$

Then $\quad a^n\ x(n) \xleftrightarrow{z} x(z/a)$

Thus scaling in z transform is equivalent to multiplying by a^n in time domain.

4) Time reversal Property

The Time reversal property states that if

$$x(n) \xleftrightarrow{z} X(z) \quad \text{And}$$

Then $\quad x(-n) \xleftrightarrow{z} x(z^{-1})$

It means that if the sequence is folded it is equivalent to replacing z by z^{-1} in z domain.

5) Differentiation in z domain

The Differentiation property states that if

$$x(n) \xleftrightarrow{z} X(z) \quad \text{And}$$

Then $\quad n\ x(n) \xleftrightarrow{z} -z\ d/dz\ (X(z))$

6) Convolution Theorem

The Circular property states that if

$$x1(n) \xleftrightarrow{z} X1(z) \text{ And}$$

$$x2(n) \xleftrightarrow{z} X2(z) \text{ Then}$$

$$\text{Then } x1(n) * x2(n) \xleftrightarrow{z} X1(z)\,X2$$

Convolution of two sequences in time domain corresponds to multiplication of its Z transform sequence in frequency domain.

7) Correlation Property

The Correlation of two sequences states that if

$$x1(n) \xleftrightarrow{z} X1(z) \text{ And}$$

$$x2(n) \xleftrightarrow{z} X2(z) \text{ Then}$$

$$\text{then} \quad \sum_{n=-\infty}^{\infty} x1(l)\; x2(-l) \xleftrightarrow{z} X1(z)\, x2(z^{-1})$$

8) Initial value Theorem

Initial value theorem states that if

$$x(n) \longleftrightarrow X(z) \text{ And}$$

then

$$x(0) = \lim_{z \to \infty} X(Z)$$

9) Final value Theorem

Final value theorem states that if

$$x(n) \xleftrightarrow{z} X(z) \text{ And}$$

then

$$\lim_{z \to \infty} x(n) = \lim_{z \to 1} (z-1)\, X(z)$$

Find the Z-transform of the following sequence

$$x(n) = u(n) - u(n-4)$$

The given sequence is:

$$x(n) = u(n) - u(n-4)$$

From Figure 3.4, we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } 0 \le n \le 3$$
$$= 0, \quad \text{otherwise}$$

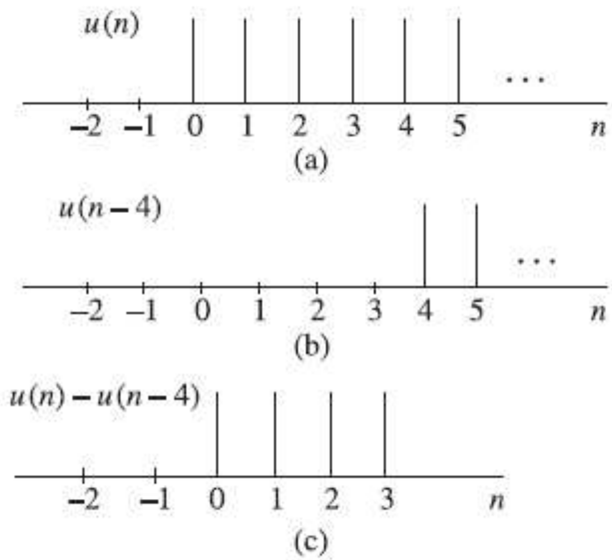

Figure 3.4 Sequences (a) $u(n)$, (b) $u(n-4)$ and (c) $u(n) - u(n-4)$.

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$$

The ROC is entire z-plane except at $z = 0$.

EXAMPLE 3.10 Find the Z-transform of the following sequences:

(a) $u(n) - u(n-4)$ (b) $u(-n) - u(-n-3)$ (c) $u(2-n) - u(-2-n)$

Solution:

(a) The given sequence is:

$$x(n) = u(n) - u(n-4)$$

From Figure 3.4, we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } 0 \le n \le 3$$
$$= 0, \quad \text{otherwise}$$

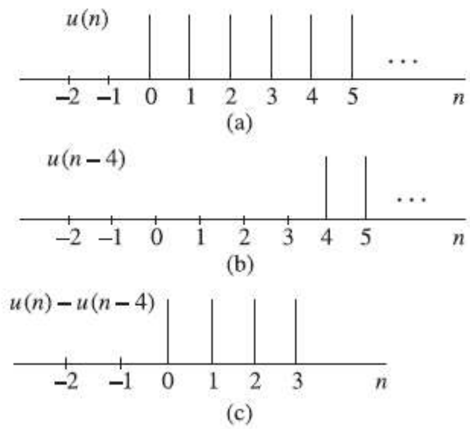

Figure 3.4 Sequences (a) $u(n)$, (b) $u(n - 4)$ and (c) $u(n) - u(n - 4)$.

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$$

The ROC is entire z-plane except at $z = 0$.

(b) The given sequence is:

$$x(n) = u(-n) - u(-n-3)$$

From Figure 3.5, we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } -2 \le n \le 0$$
$$= 0, \quad \text{otherwise}$$

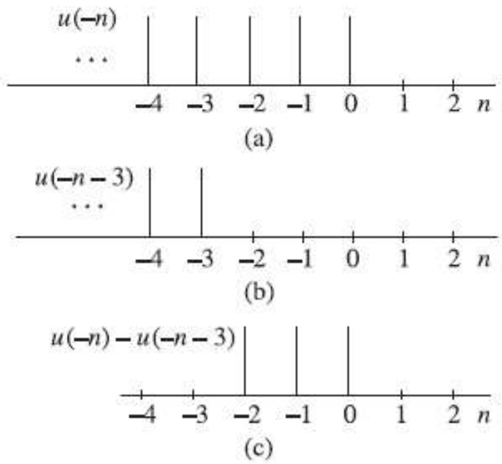

Figure 3.5 Sequences (a) $u(-n)$, (b) $u(-n - 3)$ and (c) $u(-n) - u(-n - 3)$.

We know that $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z + z^2$$

The ROC is entire z-plane except at $z = \infty$.

(c) The given sequence is:

$$x(n) = u(2-n) - u(-2-n)$$

From Figure 3.6, we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } -1 \le n \le 2$$
$$= 0, \quad \text{otherwise}$$

Substituting the sequence values, we get

$$X(z) = z + 1 + z^{-1} + z^{-2}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

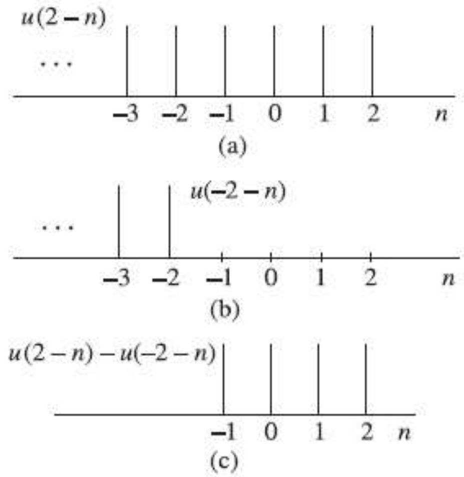

Figure 3.6 Sequences (a) $u(2 - n)$, (b) $u(-2 - n)$ and (c) $u(2 - n) - u(-2 - n)$.

Inverse z-Transform

The z-transform of the sequence $x(n)$ and the inverse z-transform of the function $X(z)$ are defined as, respectively,

$$X(z) = Z(x(n)) \tag{5.7}$$

$$\text{and } x(n) = Z^{-1}(X(z)), \tag{5.8}$$

where $Z(\ )$ is the z-transform operator, while $Z^{-1}(\ )$ is the inverse z-transform operator.

The inverse z-transform may be obtained by at least three methods:

1. Partial fraction expansion and look-up table
2. Power series expansion
3. Residue method.

TABLE 5.3 Partial fraction(s) and formulas for constant(s).

Partial fraction with the first-order real pole:	
$\dfrac{R}{z-p}$	$R = (z-p)\dfrac{X(z)}{z}\Big\vert_{z=p}$
Partial fraction with mth-order real poles:	
$\dfrac{R_m}{(z-p)} + \dfrac{R_{m-1}}{(z-p)^2} + \cdots + \dfrac{R_1}{(z-p)^m}$	$R_k = \dfrac{1}{(k-1)!}\dfrac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \dfrac{X(z)}{z}\right)\Big\vert_{z=p}$

Example 5.9.

a. Find the inverse of the following z-transform:

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

Solution:

a. Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})}.$$

$$= \frac{z^2}{(z - 1)(z - 0.5)}$$

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z - 1)(z - 0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z - 1)} + \frac{B}{(z - 0.5)}.$$

Then A and B are constants found using the formula in Table 5.3, that is,

$$A = (z - 1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z - 0.5)}\bigg|_{z=1} = 2,$$

$$B = (z - 0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z - 1)}\bigg|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z - 1)} + \frac{-1}{(z - 0.5)}.$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z - 1)} + \frac{-z}{(z - 0.5)}.$$

TABLE 5.4 Determined sequence in Example 5.9.

n	0	1	2	3	4	...	∞
$x(n)$	1.0	1.5	1.75	1.875	1.9375	...	2.0

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Tabulating this solution in terms of integer values of n, we obtain the results in Table 5.4.

The situation dealing with the real repeated poles is presented in Example 5.11.

Example 5.11.

a. Find $x(n)$ if $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2}$.

Solution:

a. Dividing both sides of the previous z-transform by z yields

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)^2} = \frac{A}{z-1} + \frac{B}{z-0.5} + \frac{C}{(z-0.5)^2},$$

where $A = (z-1)\dfrac{X(z)}{z}\bigg|_{z=1} = \dfrac{z}{(z-0.5)^2}\bigg|_{z=1} = 4.$

Using the formulas for mth-order real poles in Table 5.3, where $m = 2$ and $p = 0.5$, to determine B and C yields

$$B = R_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{d}{dz}\left(\frac{z}{z-1}\right)\bigg|_{z=0.5} = \frac{-1}{(z-1)^2}\bigg|_{z=0.5} = -4$$

$$C = R_1 = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{z}{z-1}\bigg|_{z=0.5} = -1.$$

Then $$X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2}. \tag{5.9}$$

The inverse z-transform for each term on the right-hand side of Equation (5.9) can be achieved by the result listed in Table 5.1, that is,

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n),$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n),$$

$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n).$$

From these results, it follows that

$$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n).$$

3.4.1 Inverse z-transform by Contour integration

Cauchy residue Theorem:

Let $f(z)$ be a function of the complex variable *z* and *C* be a closed path in the z-plane.

If the derivative $df(z)/dz$ exists on and inside the contour *C*

and if $f(z)$ has no poles at $z = z_0$ then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{z - z_0}\,dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases}$$

More generally, if the (k+1)-order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{(z - z_0)^k}\,dz = \begin{cases} \frac{1}{(k-1)!}\frac{d^{k-1} f(z_0)}{dz^{k-1}}\Big|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases}$$

In more generalized form, the integrand of the contour integral is $P(z) = f(z)/g(z)$ where $f(z)$ has no poles inside the contour *C* and $g(z)$ is a polynomial with distinct (simple) roots $z_1, z_2, \cdots, z_n$ inside *C*. Then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{g(z)}dz = \frac{1}{2\pi j}\oint_C\left[\sum_{i=1}^{n}\frac{A_i(z)}{z-z_i}\right]dz \qquad \text{where} \qquad A_i(z) = (z-z_i)P(z) = (z-z_i)\frac{f(z)}{g(z)}$$

$$= \sum_{i=1}^{n}\frac{1}{2\pi j}\oint_C \frac{A_i(z)}{z-z_i}dz$$

$$= \sum_{i=1}^{n} A_i(z_i)$$

The values $\{A_i(z_i)\}$ are residues of the corresponding poles at $z = z_i, i = 1, 2, \cdots, n.$

The values of the contour integral is equal to the sum of the residues of all poles inside the contour *C*.

In case that the poles $\{z_i\}$ are simples,

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

$$= \sum_{\substack{all\ poles \\ \{z_i\}\ inside C}}\left[residue\ of\ X(z)z^{n-1}\ at\ z = z_i\right]$$

$$= \sum_i (z-z_i)X(z)z^{n-1}\Big|_{z=z_i}$$

If $X(z)z^{n-1}$ has no poles inside the contour *C* for one or more values of n, then $x(n) = 0$ for these values.

EXAMPLE 3.37 Determine the inverse Z-transform using the complex integral

$$X(z) = \frac{3z^{-1}}{[1-(1/2)\,z^{-1}]^2};\ \text{ROC; } |z| > \frac{1}{4}$$

Solution: We know that the inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

at the poles inside c where c is a circle in the z-plane in the ROC of $X(z)$.

This can be evaluated by finding the sum of all residues of the poles that are inside the circle c. Therefore, the above equation can be written as:

$$x(n) = \sum \text{Residues of } X(z) z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1}\Big|_{z=z_i}$$

If there is a pole of multiplicity k, then the residue at that pole is:

$$\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} [(z-z_i)^k\ X(z)\, z^{n-1}] \text{ at the pole } z = z_i$$

Given $$X(z) = \frac{3z^{-1}}{[1-(1/2)z^{-1}]^2} = \frac{3z}{[z-(1/2)]^2}$$

The given $X(z)$ has a pole of order 2 at $z = 1/2$.

$$x(n) = \sum \text{Residues of } X(z) z^{n-1} \text{ at its poles}$$

$$= \sum \text{Residue of } 3z^n/[z-(1/2)]^2 \text{ at the pole } z = (1/2) \text{ of multiplicity 2}$$

$$\therefore \quad x(n) = \frac{1}{1!} \frac{d}{dz} \left[\left(z - \frac{1}{2}\right)^2 \frac{3z^n}{[z-(1/2)]^2} \right]\Bigg|_{z=1/2} = 3nz^{n-1}\Big|_{z=1/2} = 3n\left(\frac{1}{2}\right)^{n-1} u(n)$$

" 3 "

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DSP

Nr. 1) A discrete-time signal $x(n)$ is defined as

$$x(n) = \begin{cases} 1 + n/3 & ; \; -3 \le n \le -1 \\ 1 & ; \; 0 \le n \le 3 \\ 0 & ; \; \text{else where} \end{cases}$$

Determine the values of $x(n)$ and sketch the signal $x(n)$.

Sol^n :-

$n \rightarrow -3 \; -2 \; -1 \; 0, 1, 2, 3$ --- Given

$\therefore$ $x(-3) = 0$ $\quad$ $x(0) = 1$

$x(-2) = 0.33$ $\quad$ $x(1) = 1$

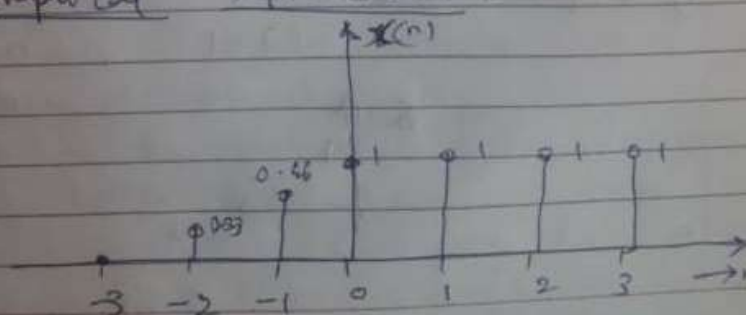
$x(-1) = 0.66$ $\quad$ $x(2) = 1$

$x(3) = 1$

→ Tabular representation of $x(n)$:

n	-3	-2	-1	0	1	2	3
$x(n)$	0	0.33	0.66	1	1	1	1

→ Graphical representation :-



→ Sequence representation :-

$$x(n) = \{0,\ 0.33,\ 0.66,\ \underset{\uparrow}{1},\ 1,\ 1,\ 1\}$$

ii) Delay the sequence by 4 samples:

$$y(n) = x(n-4)$$

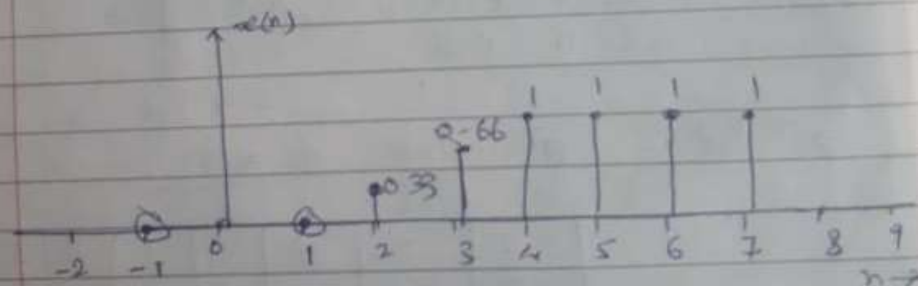
$y(0) \longrightarrow$ $n-4=0$, $n=4$

$y(1) \longrightarrow$ $n-4=1$, $n=5$

$y(2) \longrightarrow$ $n-4=2$, $n=6$

$y(3) \longrightarrow$ $n-4=3$, $n=7$

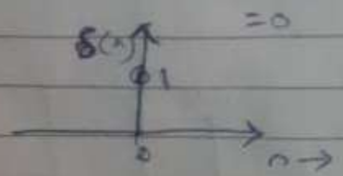
$y(4) \longrightarrow$ $n-4=4$ $n=8$



Standard Discrete time Signals :-

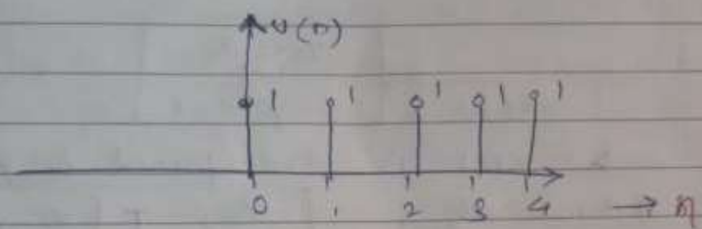
1) Impulse signal

$$\delta(n) = 1 \quad ; \quad n = 0$$
$$= 0 \quad ; \quad n \neq 0$$



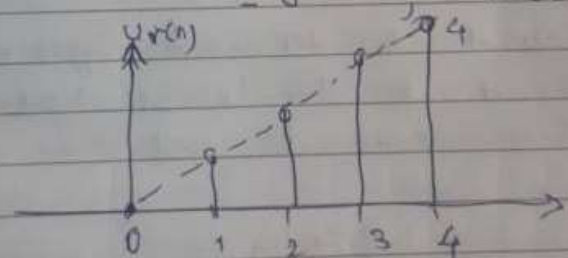
2) Unit step signal ;

$$u(n) = 1 \quad ; \quad n \geq 0$$
$$= 0 \quad ; \quad n < 0$$



3) Ramp signal ;

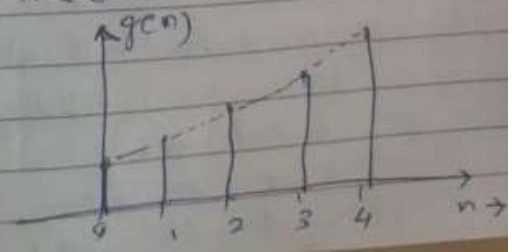
$$u_r(n) = n \quad ; \quad n \geq 0$$
$$= 0 \quad ; \quad n < 0$$



4) Exponential signal ;

$$g(n) = a^n \quad ; \quad n \geq 0$$
$$= 0 \quad ; \quad n < 0$$

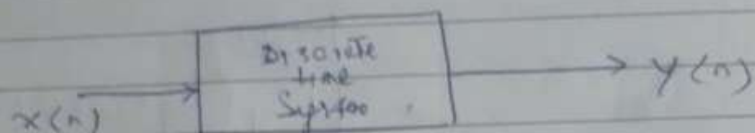
$\boxed{a > 1}$ for increasing exponential signal



$a < 1$ — .. for decreasing exponential signal

④ classification of Discrete time System:

Block Diagram :



→ $x(n)$ is transformed to $y(n)$

Mathematically , $\boxed{y(n) = T[x(n)]}$

Types of Discrete System :

1) Static (memoryless) and dynamic (Memory) System :
2) Causal & Noncausal System.
3) Linear & Non-Linear System
4) Time invariant and time varying System.

Properties of DSP System :

→ Linearity :—, Time invariance, etc.

1) Static and Dynamic System :—

A system is said to be static if the response is due to present input alone. i.e. output $y(n)$ at any instant n depends only on input $x(n)$ applied at that instant n.

(not on future values of i/p or past values of the o/p)

Z - Transform

Z.T. methods for analysis of discrete time system in frequency domain.

$x(n)$ = Discrete time signal

$X(z)$ = Z. Transform of $x(n)$

1) Bilateral or Two sided Z-transform of discrete time signal or a sequence $x(n)$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

where, Z - complex variable

symbolically, $Z\{x(n)\}$

$$\therefore \boxed{X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}} \quad \text{(I)}$$

2) One sided or unilateral Z-transform of DT sequence $x(n)$:-

$$\boxed{X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}} \quad \text{(II)}$$

As we can see in both formulas:

Computation of $X(z)$ involves summation of infinite terms which are functions of z.

ROC (Region of Convergence)

→ The set of values of z for which $X(z)$ converges is called ROC of $X(z)$

→ The set of points in z-plane, for which $X(z)$ converges is called ROC of $X(z)$.

→ If there is no value of z for which $X(z)$ converges, then the sequence $x(n)$ is said to be having no Z-transform.

N — Determine the z-transform & their ROC of the following DT signal

$$x(n) = \{ \underset{\uparrow}{3}, 4, 2, 7 \}$$

here, $x(0) = 3$, $x(1) = 4$, $x(2) = 2$, $x(3) = 7$

$n \to 0, 1, 2, 3$

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$X(z) = x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$\therefore X(z) = 3z^{-0} + 4z^{-1} + 2z^{-2} + 7z^{-3} \quad \text{— (I)}$$

$$\therefore X(z) = 3 + \frac{4}{z} + \frac{2}{z^2} + \frac{7}{z^3}$$

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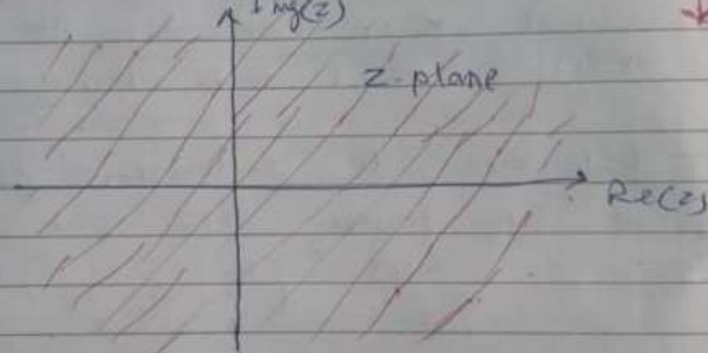
Date / /

when $z = 0$, (putting $z = 0$ in eqn (I))
then $X(Z) = \infty$ (infinite)
For other values of z:
$X(z)$ - is finite
Therefore $X(Z)$ is finite for all values of z except $z = 0$.

$\therefore$ ROC $\rightarrow$ entire z-plane except $z = 0$.



- Determine the z-transform of the following finite duration signal

(1) $x_1(n) = \{2, 5, \underset{\uparrow}{4}, 6, 0, 7\}$

ROC - entire z plane except $z = 0$ & $z = \infty$

(2) $x_2(n) = \{0, 0, \underset{\uparrow}{0}, 5, 7, 9, 2\}$

$n = 0, 1, 2, 3, 4$

ROC — entire z plane except $Z = 0$

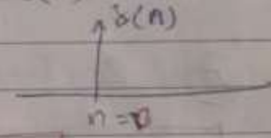
③ ~~$\delta(n)$~~

$x_4(n) = \delta(n)$

$$Z\{x_4(n)\} = \sum_{n=-\infty}^{\infty} x_4(n) z^{-n}$$

$$= \delta(-\infty) z^{\infty} + \cdots + \delta(-1) z^{1} + \delta(0) z^{0} + \delta(1) z^{-1} + \delta(2) z^{-2} + \cdots + \delta(\infty) z^{-\infty}$$

$$\boxed{X_4(z) = 1}$$



$\boxed{\text{ROC} \rightarrow \text{entire z-plane}}$

④ $x_5(n) = \delta(n-k)$; $k > 0$

$$Z\{x_5(n)\} = \sum_{n=-\infty}^{\infty} x_5(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-k) z^{-n}$$

$$X(z) = \delta(-\infty-k) z^{\infty} + \cdots + \delta(-1-k) z^{1} + \delta(-k) z^{0} + \cdots + \delta(0) z^{-k} + \cdots + \delta(\infty - k) z^{-\infty}$$

k^{th} sample

$$\boxed{X_5(z) = z^{-k}}$$

ROC → entire z plane except $z = 0$

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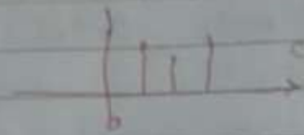
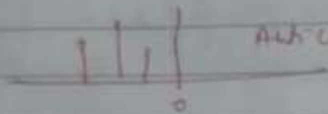
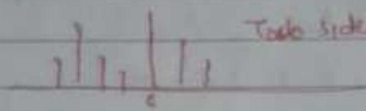
Note:

(i)  Causal $\Rightarrow$ entire z-plane except $z = 0$

(2)  Anti-causal or Noncausal $\Rightarrow$ entire z-plane except $z = \infty$

(3)  Two sided $\Rightarrow$ entire z-plane except $z = 0$ & $z = \infty$

Z-Transform of infinite duration Signal

formulae for Geometric series:

(i) $$\sum_{n=n_1}^{n_2} a^n = \begin{cases} \dfrac{a^{n_1} - a^{n_2+1}}{1-a} & ; \quad a \neq 1 \\ n_2 - n_1 + 1 & ; \quad a = 1 \end{cases}$$

(ii) $$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad ; \quad |a| < 1$$

(iii) $$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \quad ; \quad |a| < 1$$

(iv) $$\sum_{n=n_1}^{\infty} a^n = \frac{a^{n_1}}{1-a} \quad ; \quad |a| < 1$$

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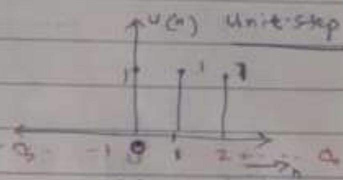
(V) $\displaystyle\sum_{n=0}^{\infty} n\,a^n = \frac{a}{(1-a)^2}\ , \quad |a| < 1$

Prob :- Determine Z-transform of the following infinite duration signal. Assume $k > 0$.

1) $u(n)$ 2) $u(n-k)$ 3) $u(n+k)$
4) $a^n u(n)$ 5) $a^n u(n+k)$ 6) $n a^n u(n)$
7) $n a^n u(n+k)$

1) $x(n) = u(n)$



$u(n) = 1 \ ; \ n \geq 0$
$\quad\ = 0 \ ; \ n < 0$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \underbrace{\sum_{n=-\infty}^{-1} u(n) z^{-n}}_{0} + \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} \qquad \because u(n) = 1$$

$$= \sum_{n=0}^{\infty} \left(z^{-1}\right)^n \qquad \boxed{\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

a

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$\therefore \boxed{X(z) = \frac{z}{z-1}}$$

$|z^{-1}| < 1$

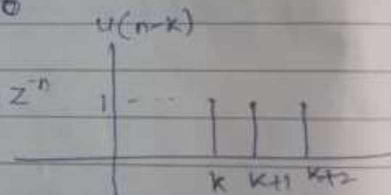
$\therefore \left|\frac{1}{z}\right| < 1$

$\boxed{|z| > 1}$ ✓ → exterior z-plane

∴ ROC – entire z-plane except $|z| < 1$

Therefore ROC is exterior Z-plane

② $u(n-k)$, $k > 0$



$Z\{u(n-k)\} = \sum_{n=-\infty}^{\infty} u(n-k)\, z^{-n}$

$= \sum_{n=k}^{\infty} z^{-n}$

$= \sum_{n=k}^{\infty} (\underbrace{z^{-1}}_{a})^{n}$ $\because n_1 = k$

$u(n-k) = 0 \quad n < k$

$u(n-k) = 1 \quad n \geq k$

$= \frac{(z^{-1})^{k}}{1 - z^{-1}}$ $\boxed{\because \sum_{n=n_1}^{\infty} a^n = \frac{a^{n_1}}{1-a}}$

$= \frac{z^{-k}}{1 - z^{-1}}$ → ROC $|z^{-1}| < 1$

$= \frac{z^{-k}}{1 - \frac{1}{z}} = \frac{z^{-k}}{1} \times \frac{z}{z-1} = \frac{z^{1-k}}{z-1}$

$\boxed{x(z) = \frac{z^{1-k}}{z-1}}$

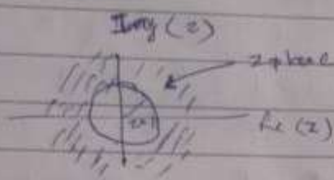
ROC :-

$|z^{-1}| < 1$

ROC : exterior z-plane except $z < 1$.

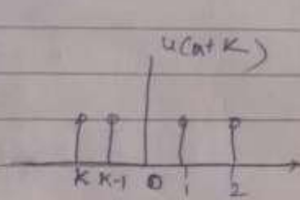
$\left|\frac{1}{z}\right| < 1$

$\boxed{|z| > 1}$



⇐ exterior except $z < 1$

(3) <u>u(n+k)</u> $x(n) = u(n+k)$



$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-k}^{\infty} u(n+k)\, z^{-n}$$

$u(n+k) = 1 \; ; n \geq k$

$= 0 \; ; n < k$

$$= \sum_{n=-k}^{\infty} z^{-n}$$

$$= \sum_{n=-k}^{\infty} (z^{-1})^{n}$$

$$= \frac{(z^{-1})^{-k}}{1 - z^{-1}}$$

$|z^{-1}| < 1$

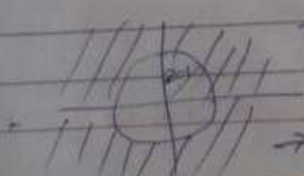
$$= \frac{z^{k}}{1 - z^{-1}}$$

$\cancel{\left|\frac{1}{z}\right| > 1}$

$\left|\frac{1}{z}\right| < 1$

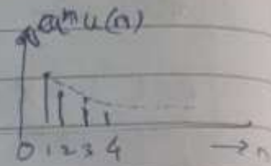
$\boxed{|z| > 1}$

ROC ; exterior z plane



→ entire z plane except $z < 1$

(4) $x(n) = a^n u(n)$



$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) Z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n Z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1 - a z^{-1}} \qquad |a z^{-1}| < 1$$

$$|a \tfrac{1}{z}| < 1$$

$$= \frac{z}{z-a} \qquad \boxed{|z| > a}$$

$\Rightarrow$ ROC:

entire z-plane

except $z < a$

$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

Prob. $x(n) = \{0, 0, 0, 5, 7, 9, 2\}$

~~n → 0 1 2 3 4 5 6~~

~~n = 0 1 2 3~~

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore X(z) = \sum_{n=0}^{3} x(n) z^{-n}$$

$$\therefore X(z) = \cancel{x(0)} z^{0} + \cancel{x(1)} z^{-1} + \cancel{x(2)} z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5} + x(6) z^{-6}$$

~~$\therefore X(z) = 5z$~~

$$\therefore X(z) = 5z^{-3} + 7z^{-4} + 9z^{-5} + 2z^{-6}$$

$$\therefore X(z) = \frac{5}{z^3} + \frac{7}{z^4} + \frac{9}{z^5} + \frac{2}{z^6}$$

when $z = 0$ ... $X(z) = \infty$

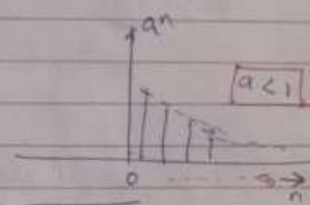
$\therefore$ $\boxed{\text{ROC} \Rightarrow \text{entire z-plane except } z = 0}$

Ques. Determine the z-transform including the region of convergence of:

$$x(n) = \begin{cases} a^n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

Sol^n: The z-transform of $x(n)$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} a^n z^{-n}$$



$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \underbrace{(a z^{-1})}_{a}{}^{n}$$

$$\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\boxed{X(z) = \frac{1}{1 - a z^{-1}}}$$

$\rightarrow$ ROC: $|a z^{-1}| < 1$

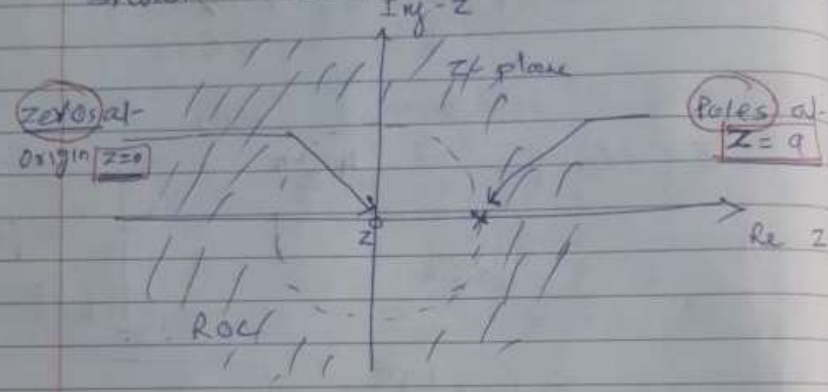
$\Rightarrow \left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$

$|a| < 1$

$$\boxed{X(z) = \frac{z}{z-a}}$$

Imp Note: This converges when $|a z^{-1}| < 1$ or $|z| > |a|$.
Values of z for which $X(z) = 0$ are called $\boxed{\text{zeros}}$ of $X(z)$ and values of z for which $X(z) \rightarrow \infty$ are called $\boxed{\text{poles}}$ of $X(z)$.

Here the Poles are at $Z = a$ and zeros at $Z = 0$. The ROC is as shown below.



Properties of Z-transform :-

⑥ Convolution :

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(Z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$

then $x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow}$

$$X(z) = X_1(z)\, X_2(z)$$

* Properties of Z-transform :-

1) **Linearity :**

If $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$ then

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} a_1 X_1(z) + a_2 X_2(z)$$

2) **Time Reversal :**

If $x(n) \xleftrightarrow{z} X(z)$; ROC $r_1 < z < r_2$

then $x(-n) \xleftrightarrow{z} X(z^{-1})$; ROC $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

eg. find the z-transform of signal $x(n) = u(-n)$

soln :- we know,

$u(n) \xleftrightarrow{z} \frac{z}{z-1}$, ROC $|z| > 1$

By using time reversal property,

$$u(-n) = \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z} \quad ; \text{ROC } |z| < 1$$

3) **Time Shifting :**

If $x(n) \xleftrightarrow{z} X(z)$ then

$x(n-k) \xleftrightarrow{z} z^{-k} X(z)$

4) **Scaling :-**

If $x(n) \xleftrightarrow{z} X(z)$, ROC $r_1 < |z| < r_2$

then $a^n x(n) \xleftrightarrow{z} X(a^{-1} z)$;

ROC $|a| r_1 < |z| < |a| r_2$

5) <u>Differentiation</u>:

If $x(n) \xleftrightarrow{z} X(z)$ then

$$n\,x(n) \xleftrightarrow{z} -z\frac{d}{dz}X(z) \quad \text{or}$$

$$z^{-1}\frac{d\,X(z)}{dz^{-1}}$$

→ Q. find the z-transform of $x(n) = n^2 u(n)$

$$x(n) = n^2 u(n)$$

$$X(z) = Z\{[n^2 u(n)]\}$$

$$= Z\{n\,\underbrace{(n\,u(n))}_{n\,x(n)}\}$$

$$= z^{-1}\frac{d}{dz^{-1}}\frac{z^{-1}}{(1-z^{-1})^2}$$

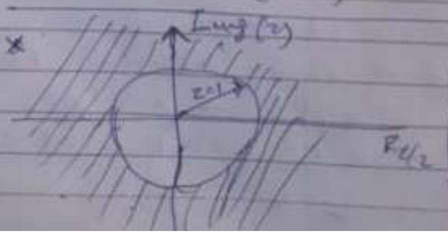
$\because n\,u(n) \xleftrightarrow{z} \frac{z^{-1}}{(1-z^{-1})^2}$ or $\because n\,u(n) \xleftrightarrow{z} \frac{z}{(z-1)^2}$

$$= z^{-1}\left[\frac{(1-z^{-1})^2 \cdot 1 - z^{-1}[2(1-z^{-1})(-1)]}{(1-z^{-1})^4}\right]$$

$$= z^{-1}\frac{\cancel{(1-z^{-1})}[(1-z^{-1})+2z^{-1}]}{(1-\cancel{z^{-1}})^4}$$

$$\boxed{X(z) = z^{-1}\frac{[(1+z^{-1})]}{(1-z^{-1})^3}}$$

ROC ×



ROC : $|z^{-1}| < 1$

$\therefore |\frac{1}{z}| < 1$

$\therefore \boxed{|z| > 1}$

$= 0$

Thus $\boxed{x(n) = 0 \quad ; \quad n < 0}$

$\therefore \boxed{x(n) = a^n u(n)}$

Problem using differentiation Property :-

(1) Determine Z- transform of

$x(n) = n\, a^n u(n)$ ...... $K \geq 0$

using differentiation Property

$z \to \infty$

Soln :-

$\boxed{\begin{array}{l} \text{Differentiation Property -} \\ \text{if } x(n) \xleftrightarrow{z} X(z) \text{ then} \\ n x(n) \xleftrightarrow{z} -z \dfrac{d}{dz} X(z) \end{array}}$

Z-transform of $\boxed{a^n u(n) \xleftrightarrow{z} \underbrace{\dfrac{z}{(z-a)}}_{X(z)} \; ; \quad |z| > a}$

$\therefore \; n\, a^n u(n) \xleftrightarrow{z} -z \dfrac{d}{dz} X(z)$

$\xleftrightarrow{z} -z \dfrac{d}{dz} \dfrac{z}{(z-a)}$.

$\xleftrightarrow{z} -z \left\{ \dfrac{(z-a) \cdot 1 - z \cdot (1)}{(z-a)^2} \right\}$

$\xleftrightarrow{z} -z \left\{ \dfrac{\cancel{z} - a - \cancel{z}}{(z-a)^2} \right\}$

$$\xleftrightarrow{z} \frac{az}{(z-a)^2}$$

$$\xleftrightarrow{z} \frac{az}{z^2(1-az^{-1})^2}$$

$$\xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2} \quad ; \quad |z| > a$$

OR

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad ; n \geq$$

$$= \sum_{n=0}^{\infty} n\, a^n\, u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} n\, (\underbrace{az^{-1}}_{a})^n \qquad \because u(n) = 1$$

Using formula:- $\boxed{\sum_{n=0}^{\infty} n\, a^n = \frac{a}{(1-a)^2}}$

$$\boxed{X(z) = \frac{az^{-1}}{(1-az^{-1})^2}} \Rightarrow$$

ROC

$|az^{-1}| < 1$

$\left|\frac{a}{z}\right| < 1$

$|a| < |z|$

$\therefore \boxed{|z| > |a|}$

ROC :

$\boxed{z > a}$

ROC :- Exterior z-plane except z < a

(a) Definition of inverse z-Transform

The inverse z-transform is computed or desired to recover the original signal time domain discrete signals sequence x(n) from its frequency domain signal X(z). The operation can be expressed mathematically as

$$\boxed{x(n) = z^{-1}[X(z)]}$$

(1) Power series expansion Method :-

Prob. Determine the inverse z-transform of

$$X(z) = \frac{z^{-2}}{1 - z^{-1}} \; ; \; |z| > 1$$

Soln :- Since ROC is the exterior of circle, x(n) is causal signal (i.e. power series expansion in negative power of z)

formula : $X(z) = \frac{N}{D} = \ldots$ Given

z → Quotient

x | u

↓ ↳ dividend

Divisor

- Exterior ROC, $|z| > 1$
- causal
- Consist of -ve Power of z

ROC :- Exterior z-plane except z<a

(a) Definition of inverse Z-Transform

The inverse z-transform is computed or desired to recover the original signal time domain discrete signals sequence x(n) from its frequency domain signal X(z). The operation can be expressed mathematically as

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Soln :- Since ROC is the exterior of circle, x(n) is causal signal (i.e. power series expansion in negative power of z)

formula : $X(z) = \frac{N}{D} = \ldots$ Given

z → Quotient
x | u → dividend
↓
Divisor

- Exterior ROC, $|z| > 1$
- causal
- Consist of -ve power of z

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$$z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

$$1 - z^{-1} \,\big)\, z^{-2}$$

$$\underline{\begin{array}{l} z^{-2} - z^{-3} \\ - \quad + \end{array}}$$

$$z^{-3}$$

$$\underline{\begin{array}{l} z^{-3} - z^{-4} \\ - \quad + \end{array}}$$

$$z^{-4}$$

$$\underline{\begin{array}{l} z^{-4} - z^{-5} \\ - \quad + \end{array}}$$

$$z^{-5}$$

$$\underline{\begin{array}{l} z^{-5} - z^{-6} \\ - \quad + \end{array}}$$

$$z^{-6}$$

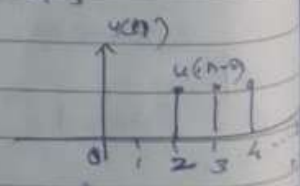
$$\therefore X(z) = \frac{z^{-2}}{(1 - z^{-1})} = z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

$$= \underline{0 z^{0} + 0 z^{-1}} + z^{-2} + z^{-3} + z^{-4} + z^{-5} \cdots$$

$$x(n) = \{ \underset{\uparrow}{0}, 0, 1, 1, 1, 1 \cdots \}$$

$n \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$$\therefore \boxed{x(n) = u(n-2)}$$



$u(n-2) = 1 \quad n \geq$

$= 0 \quad n <$

$$z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

$$1 - z^{-1} \,\big)\, z^{-2}$$

$$\begin{array}{l}
z^{-2} - z^{-3} \\
- \quad + \\
\hline
z^{-3} \\
z^{-3} - z^{-4} \\
- \quad + \\
\hline
z^{-4} \\
z^{-4} - z^{-5} \\
- \quad + \\
\hline
z^{-5} \\
z^{-5} - z^{-6} \\
- \quad + \\
\hline
z^{-6}
\end{array}$$

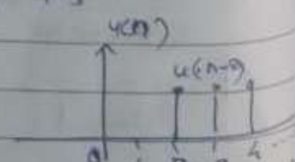
$$\therefore X(z) = \frac{z^{-2}}{(1 - z^{-1})} = z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

$$= \underline{0z^{0} + 0z^{-1}} + z^{-2} + z^{-3} + z^{-4} + z^{-5} \cdots$$

$$x(n) = \{ \underset{\uparrow}{0}, 0, 1, 1, 1, 1 \cdots \}$$

$$n \quad 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5$$

$$\therefore \boxed{x(n) = u(n-2)}$$



$u(n-2) = 1 \quad n \geq 2$

$= 0 \quad n < 2$

(2) Determine the inverse Z-transform of

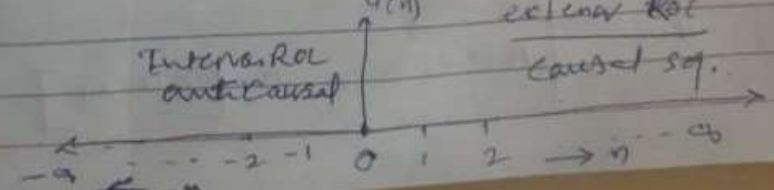
$$X(z) = \frac{z^3}{1-z} \quad ; \quad |z| < 1$$

Sol:- since the ROC is interior of circle $x(n)$ should be anticausal (i.e. power series expression must contain the positive power of z)

$$1 - z \,\overline{)\, z^3 \,}\quad z^3 + z^4 + z^5 + z^6 + \cdots$$

$$z^3 - z^4$$
$$- \quad +$$
$$z^4$$
$$z^4 - z^5$$
$$- \quad +$$
$$z^5$$
$$z^5 - z^6$$
$$- \quad +$$
$$z^6$$
$$z^6 - z^7$$
$$- \quad +$$
$$z^7$$

$$\frac{z^3}{1-z} = z^3 + z^4 + z^5 + z^6 + z^7 + \cdots\cdots$$



$\therefore X(z) = \underline{0z^0 + 0z^1 + 0z^2} + z^3 + z^4 + z^5 + z \; + \cdots$

$\therefore x(n) = \{ \cdots 1\,1\,1\,1\,0\,0\,0 \}$ with $-3\;-2\;-1\;\underset{\uparrow}{0}$

$\therefore \boxed{x(n) = u(-n-3)}$

$-n-3 = 0$

$-n = 3$

$\therefore \boxed{n = -3}$

Using Properties of z-transform :-

$\boxed{x(n) = n^2 u(n)}$

Using differentiation Property

$$\boxed{\begin{array}{l} \text{If} \quad x(n) \xleftrightarrow{z} X(z) \quad \text{then} \\ n\,x(n) \longleftrightarrow -z \dfrac{d}{dz} X(z) \end{array}}$$

$\therefore n\,u(n) \longleftrightarrow -z \dfrac{d}{dz}\left(\dfrac{z}{z-1}\right)$ $\quad \therefore \cancel{x(z)}$ $\boxed{u(n) \longleftrightarrow \dfrac{z}{z-1}}$

$\therefore \longleftrightarrow -z \left\{ \dfrac{(z-1) \times 1 - z(1-0)}{(z-1)^2} \right\}$

$\xleftrightarrow{z} -z \left\{ \dfrac{\cancel{z}-1 - \cancel{z}}{(z-1)^2} \right\}$

$n\,u(n) \longleftrightarrow \dfrac{z}{(z-1)^2}$

<u>Now</u> $\boxed{n^2 u(n) = n\,[n\,u(n)]}$

form:

$n^2 u(n) \xleftrightarrow{z} -z \frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$

$\therefore n^2 u(n) \longleftrightarrow -z\left\{\frac{(z-1)^2 \times 1 - z \times 2(z-1)\cdot 1}{(z-1)^4}\right\}$

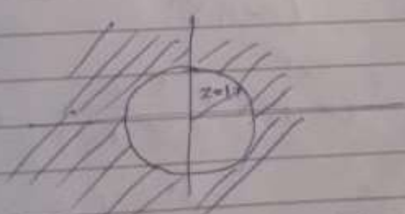
$\therefore n^2 u(n) \longleftrightarrow -z\left\{\frac{z^2 - \cancel{2z} + 1 - 2z^2 + \cancel{2z}}{(z-1)^4}\right\}$

$\longleftrightarrow -z\left\{\frac{-z^2+1}{(z-1)^4}\right\}$

$\longleftrightarrow -z\left\{\frac{1-z^2}{(z-1)^4}\right\}$

$\longleftrightarrow \frac{z^3 - z}{(z-1)^4}$ $\boxed{|z| > 1}$

ROC



exterior z-plane

ii) Partial fraction Method :-

if $X(z)$ is of form

$$\boxed{X(z) = \frac{N(z)}{D(z)}}$$

$$n^2 u(n) \xleftrightarrow{z} -z \frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$$

$$\therefore n^2u(n) \longleftrightarrow -z\left\{\frac{(z-1)^2\times 1 - z\times 2(z-1)\cdot 1}{(z-1)^4}\right\}$$

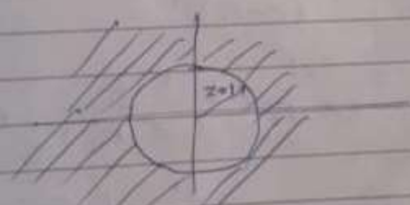
$$\therefore n^2u(n) \longleftrightarrow -z\left\{\frac{z^2-\cancel{2z}+1-2z^2+\cancel{2z}}{(z-1)^4}\right\}$$

$$\longleftrightarrow -z\left\{\frac{-z^2+1}{(z-1)^4}\right\}$$

$$\longleftrightarrow -z\left\{\frac{1-z^2}{(z-1)^4}\right\}$$

$$\longleftrightarrow \frac{z^3-z}{(z-1)^4} \qquad \boxed{|z|>1}$$

ROC



external z-plane

ii) Partial fraction Method :-

If $X(z)$ is of form

$$\boxed{X(z)=\frac{N(z)}{D(z)}}$$

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Inverse

Prob. Determine the inverse Z-transform of

$$x(z) = \frac{1}{1-4z^{-1}+3z^{-1}}$$

if ROC $|z|>1$

→ multiply ~~RHS~~ and divide RHS by z^2

$$\boxed{X(z) = \frac{z^2}{z^2-4z+3}}$$

∴ Dividing both side by z.

$$\therefore \frac{X(z)}{z} = \frac{z}{z^2-4z+3}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{z^2-z-3z+3}$$

$$\therefore = \frac{z}{z(z-1)-3(z-1)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-3)}$$

$$\text{for } c_1 = (z-1)\frac{z}{(z-1)(z-3)}\Big|_{z=1} \qquad \because c_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1}$$

$$\therefore c_1 = -\tfrac{1}{2}$$

$$\text{for } c_2 = (z-3)\frac{z}{(z-1)(z-3)}\Big|_{z=3} \qquad \because c_2 = (z-3)\frac{X(z)}{z}\Big|_{z=3}$$

$\therefore \boxed{c_2 = 3/2}$

$$\therefore \frac{X(z)}{z} = \frac{-1/2}{(z-1)} + \frac{3/2}{(z-3)}$$

$$\therefore X(z) = \frac{3}{2}\left(\frac{z}{z-3}\right) - \frac{1}{2}\left(\frac{z}{z-1}\right)$$

But, $\left(\frac{z}{z-a}\right) \longleftrightarrow a^n u(n) \; ; |z| >$

if ROC $\boxed{|z| > 1}$ (causal)

$$\therefore \boxed{x(n) = \frac{3}{2}(3)^n u(n) - \frac{1}{2} \, 1 \, u(n)}$$

$$\therefore \boxed{x(n) = \frac{1}{2}\left\{(3)^{n+1} - 1\right\} u(n)}$$

$\boxed{\text{if ROC } |z| < 1}$ (anticausal)

then $\left(\frac{z}{z-a}\right) \longleftrightarrow -a^n u(-n-1)$

$$\therefore \boxed{x(n) = -\frac{3}{2}(3)^n u(-n-1) + \frac{1}{2}(1)^n u(-n-1)}$$

(3) Partial fractions and formulas for constants:

1] Partial fraction with first order real pole:

$$\frac{C}{(z-p)} \quad \text{where} \quad C = (z-p)\frac{X(z)}{z}\Big|_{z=p}$$

2) Partial fraction with mth order real poles:

$$\frac{C_m}{(z-p)} + \frac{C_{m-1}}{(z-p)^2} + \cdots + \frac{C_1}{(z-p)^m} \quad \text{where}$$

k from m to 1 — order first

$$C_K = \frac{1}{(K-1)!}\,\frac{d^{K-1}}{dz^{K-1}}\left[(z-p)^m\frac{X(z)}{z}\right]\Bigg|_{z=p}$$

Example 4.5 Find the z-transform of $x(n) = \cos \omega_0 n$ for $n \geq 0$

Solution $x(n) = \cos \omega_0 n = \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}]$

Using the transform, for $n \geq 0$,

$$Z[a^n] = \frac{1}{1 - az^{-1}}, |z| > a$$

Therefore, for $n \geq 0$, $Z[(e^{j\omega_0})^n] = \frac{1}{1 - e^{j\omega_0} z^{-1}}, \; |z| > 1$

Similarly for $n \geq 0$, $Z[(e^{-j\omega_0})^n] = \frac{1}{1 - e^{-j\omega_0} z^{-1}}, |z| > 1$

Therefore,

$$X(z) = Z[\cos \omega_0 n] = Z\left[\frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)\right]$$

$$= \frac{\frac{1}{2}}{1 - e^{j\omega_0} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-j\omega_0} z^{-1}}$$

$$= \frac{1 - \frac{1}{2}\left[e^{j\omega_0} + e^{-j\omega_0}\right] z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}$$

$$= \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}$$

$$= \frac{z(z - \cos \omega_0)}{z^2 - 2z \cos \omega_0 + 1}, |z| > 1$$

Similarly, we can find $Z[\sin \omega_0 n]$ using the property of linearity,

$$Z[\sin \omega_0 n] = Z\left[\frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)\right]$$

$$= \frac{z^{-1} \sin \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} = \frac{z \sin \omega_0}{z^2 - 2z \cos \omega_0 + 1}, \; |z| > 1$$

4.3.2 Time Reversal

If $x(n) \xleftrightarrow{z}$ ...