Chapter 4

Nitty-Gritty Limit Problems

In This Chapter

- Algebra, schmalgebra
- Calculators — taking the easy way out
- Making limit sandwiches
- Infinity — "Are we there yet?"
- Conjugate multiplication — sounds *R* rated, but it's strictly *PG*

In this chapter, you practice two very different methods for solving limit problems: using algebra and using your calculator. Learning the algebraic techniques are valuable for two reasons. The first, *incredibly* important reason is that the mathematics involved in the algebraic methods is beautiful, pure, and rigorous; and, second — something so trivial that perhaps I shouldn't mention it — you'll be tested on it. Do I have my priorities straight or what? The calculator techniques are useful for several reasons: 1) You can solve some limit problems on your calculator that are either impossible or just very difficult to do with algebra, 2) You can check your algebraic answers with your calculator, and 3) Limit problems can be solved with a calculator when you're not required to show your work — like maybe on a multiple choice test.

But before we get to these two major techniques, how about a little rote learning. A few limits are a bit tricky to justify or prove, so to make life easier, simply commit them to memory. Here they are:

REMEMBER

- $\lim_{x \to a} c = c$

  ($y = c$ is a horizontal line, so the limit equals c regardless of the *arrow-number* — the constant after the arrow.)
- $\lim_{x \to 0^+} \frac{1}{x} = \infty$
- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$
- $\lim_{x \to \infty} \frac{1}{x} = 0$
- $\lim_{x \to -\infty} \frac{1}{x} = 0$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
- $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

Solving Limits with Algebra

REMEMBER

You can solve limit problems with several algebraic techniques. But your first step should always be plugging the arrow-number into the limit expression. If you get a number, that's the answer. You're done. You're also done if plugging in the arrow-number gives you

- A number or infinity or negative infinity over zero, like $\frac{3}{0}$, or $\frac{\pm\infty}{0}$; in these cases the limit does not exist (DNE).
- Zero over infinity; the answer is zero.

When plugging in fails because it gives you $\frac{0}{0}$, you've got a real limit problem, and you have to convert the fraction into some expression where plugging in *does* work. Here are some algebraic methods you can try:

- FOILing
- Factoring
- Finding the least common denominator
- Canceling
- Simplification
- Conjugate multiplication

The following examples each use a different method to solve the limit.

Q. Evaluate $\lim_{x \to 16} \frac{16 - x}{4 - \sqrt{x}}$.

A. **The limit is 8.**

1. **Try plugging 16 into x — no good.**
2. **Multiply numerator and denominator by the conjugate of $4 - \sqrt{x}$, namely $4 + \sqrt{x}$.**

   The conjugate of a two-term expression has a plus sign instead of a minus sign — or vice-versa.

   $$\lim_{x \to 16} \frac{(16 - x)}{\left(4 - \sqrt{x}\right)} \cdot \frac{\left(4 + \sqrt{x}\right)}{\left(4 + \sqrt{x}\right)}$$

3. **FOIL the conjugates and simplify.**

   $$= \lim_{x \to 16} \frac{(16 - x)\left(4 + \sqrt{x}\right)}{\left(4^2 - \sqrt{x}^2\right)}$$

   Because, of course, $(a - b)(a + b) = a^2 - b^2$.

   $$= \lim_{x \to 16} \frac{(16 - x)\left(4 + \sqrt{x}\right)}{(16 - x)}$$

   $$= \lim_{x \to 16} \left(4 + \sqrt{x}\right)$$

4. **Now you can cancel and then plug in.**

   $$= 4 + \sqrt{16}$$
   $$= 8$$

Note that while plugging in did not work in Step 1, it did work in the final step. That's your goal: to change the original expression — usually by canceling — so that plugging in works.

Q. What's $\lim_{x \to -2} \frac{x^2 - x - 6}{x^2 + x - 2}$?

A. **The limit is $\frac{5}{3}$.**

1. **Try plugging –2 into x — that gives you $\frac{0}{0}$, so on to plan B.**
2. **Factor and cancel.**

   $$= \lim_{x \to -2} \frac{(x + 2)(x - 3)}{(x + 2)(x - 1)}$$

   $$= \lim_{x \to -2} \frac{(x - 3)}{(x - 1)}$$

3. **Cancel now and plug in.**

   $$= \frac{-2 - 3}{-2 - 1}$$
   $$= \frac{-5}{-3}$$
   $$= \frac{5}{3}$$

1. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Solve It

2. $\lim_{x \to 1} \frac{x - 1}{x^2 + x - 2}$

Solve It

3. $\lim_{x \to -2} \frac{x + 2}{x^3 + 8}$

Solve It

4. $\lim_{x \to 2} \frac{x^2 - 4}{4x^2 + 5x - 6}$

Solve It

5. $\lim_{x \to 9} \frac{x - 9}{3 - \sqrt{x}}$

Solve It

6. $\lim_{x \to 10} \frac{\sqrt{x - 5} - \sqrt{5}}{x - 10}$

Solve It

7. $\lim_{x \to 0} \frac{\cos x - 1}{x}$

Solve It

8. $\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

Solve It

9. $\lim_{x\to 0}\frac{x}{\frac{1}{6}+\frac{1}{x-6}}$

Solve It

10. $\lim_{x\to 0}\frac{\sin x}{x}$

Solve It

***11.** $\lim_{x\to 0}\frac{x}{\sin 3x}$

Solve It

***12.** $\lim_{x\to 0}\frac{x}{\tan x}$

Solve It