Lesson 19 - MRS22- Mr. Pineda

# TRIGONOMETRIC IDENTITIES

# TRIGONOMETRIC IDENTITIES

A trigonometric identity is an equation involving trigonometric functions that hold for all values of the argument, typically chosen to be θ. In other words, an identity is an equation that is true for all of its domain values.

# FUNDAMENTAL TRIGONOMETRIC IDENTITIES

## Fundundamental Trigonometric Identities

| | | | |
|---|---|---|---|
| **Reciprocal identities** | $\sin x = \dfrac{1}{\csc x}$ | $\cos x = \dfrac{1}{\sec x}$ | $\tan x = \dfrac{1}{\cot x}$ |
| **Ratio identities** | $\tan x = \dfrac{\sin x}{\cos x}$ | $\cot x = \dfrac{\cos x}{\sin x}$ | |
| **Pythagorean identities** | $\sin^2 x + \cos^2 x = 1$ | $\tan^2 x + 1 = \sec^2 x$ | $1 + \cot^2 x = \csc^2 x$ |
| **Odd–even identities** | $\sin(-x) = -\sin x$<br>$\cos(-x) = \cos x$ | $\tan(-x) = -\tan x$<br>$\cot(-x) = -\cot x$ | $\sec(-x) = \sec x$<br>$\csc(-x) = -\csc x$ |

# VERIFICATION OF TRIGONOMETRIC IDENTITIES

To verify an identity, we show that one side of the identity can be rewritten in an equivalent form that is identical to the other side. There is no one method that can be used to verify every identity; however the following guidelines should prove useful.

## Guidelines for Verifying Trigonometric Identities

- If one side of the identity is more complex than the other, then it is generally best to try first to simplify the more complex side until it becomes identical to the other side.
- Perform indicated operations such as adding fractions or squaring a binomial. Also be aware of any factorization that may help you to achieve your goal of producing the expression on the other side.

- Use previously established identities that enable you to rewrite one side of the identity in an equivalent form.
- Rewrite one side of the identity so that it involves only sine and/or cosines.
- Rewrite one side of the identity in terms of a single trigonometric function.
- Multiplying both the numerator and the denominator of a fraction by the same factor (such as the conjugate of the denominator or the conjugate of the numerator) may get you closer to your goal.
- Keep your goal in mind. Does it involve products, quotients, sums, radicals, or powers? Knowing exactly what your goal is may provide the insight you need to verify the identity.

# EXAMPLES

Verify the following identities.

a) $\tan\theta\sec\theta\sin\theta = \tan^2\theta$

b) $\dfrac{\sin\theta}{1-\cos\theta} = \csc\theta + \cot\theta$

c) $\dfrac{1+\tan^3\theta}{1+\tan\theta} = 1-\tan\theta+\tan^2\theta$

d) $(\sin\theta-\cos\theta)(\sin\theta+\cos\theta) = 1-2\cos^2\theta$

e) $\dfrac{\cos x\tan x + 2\cos x - \tan x - 2}{\tan x + 2} = \cos x - 1$

f) $\dfrac{\sin^2 x\cos x + \cos^3 x - \sin^3 x\cos x - \sin x\cos^3 x}{1-\sin^2 x}$

g) $\dfrac{1-\sin x+\cos x}{1+\sin x+\cos x} = \dfrac{\cos x}{1+\sin x}$

h) $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

i) $\sec x = \dfrac{\cot x + \tan x}{\csc x}$

j) $\dfrac{1-\sin x}{1+\sin x} - \dfrac{1+\sin x}{1-\sin x} = -4\sec x\tan x$

# SUM AND DIFFERENCE IDENTITIES

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

# EXAMPLES

1. Find the exact value of each trigonometric expression.
   a) $\cos 75^0$ b) $\sin(\pi/12)$ c) $\tan 195^0$
2. Write each expression as a single trigonometric expression.
   a) $\sin(-x)\cos 3x - \cos(-x)\sin 3x$
   b) $\cos 4x \cos(-2x) - \sin 4x \sin(-2x)$
3. Find the exact value of a) $\cos(\beta - \alpha)$ and b) $\sin(\alpha + \beta)$, if $\sin\alpha = 24/25$, $\alpha$ in QII, and $\cos\beta = -(4/5)$, $\beta$ in QIII.
4. Verify the following identities:
   a) $\cos(\theta + \pi) = -\cos\theta$
   b) $\sin(\alpha - \beta) - \sin(\alpha + \beta) = -2\cos\alpha\sin\beta$

## DOUBLE – ANGLE IDENTITIES

| Sine | Cosine | Tangent |
|---|---|---|
| $\sin 2\alpha = 2\sin\alpha\cos\alpha$ | $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ | $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$ |
| | $\cos 2\alpha = 1 - 2\sin^2\alpha$ | |
| | $\cos 2\alpha = 2\cos^2\alpha - 1$ | |

## POWER–REDUCING IDENTITIES

| | | |
|---|---|---|
| $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$ | $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$ | $\tan^2\alpha = \dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}$ |

# EXAMPLES

1. Write each trigonometric expression in terms of a single trigonometric function .

   a) $2 \sin 3\theta \cos 3\theta$ b) $\cos^2(x + 2) - \sin^2(x + 2)$

2. If $\cos \alpha = 24/25$; $270^0 < \alpha < 360^0$, find a) $\sin 2\alpha$ b) $\cos 2\alpha$ and c) $\tan 2\alpha$

3. Use the power–reducing identities to write each trigonometric expression in terms of the first power of one or more cosine funtions.

   a) $\cos^4 \alpha$ b) $\sin^2 x \cos^4 x$

4. Verify :

   a) $\cos 8x = \cos^2 4x - \sin^2 4x$ b) $\dfrac{1}{1-\cos 2x} = \dfrac{1}{2}\csc^2 x$

# HALF – ANGLE IDENTITIES

| Sine | Cosine | Tangent |
|---|---|---|
| $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ | $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ | $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ |
| | | $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$ |
| | | $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$ |

# EXAMPLES

1. Use half – identities to find the exact value of each trigonometric expression.
   a) $\cos 105^0$ b) $\sin(3\pi/8)$ c) $\cot 67.5^0$
2. Find the exact value of the sine, cosine, and tangent of $(\alpha/2)$ given the following information.
   a) $\cos\alpha = 12/13, \quad 0^0 < \alpha < 90^0$
   b) $\csc\alpha = -(5/3)$, $\alpha$ is in Quadrant IV
3. Verify the following identities :
   a) $\cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos x$
   b) $2\csc x\cos^2\frac{x}{2} = \frac{\sin x}{1-\cos x}$

# PRODUCT – TO – SUM IDENTITIES

| |
|---|
| $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$ |
| $\cos\alpha \sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$ |
| $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$ |
| $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ |

# SUM – TO – PRODUCT IDENTITIES

| |
|---|
| $\sin x + \cos y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ |
| $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ |
| $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$ |
| $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$ |

# EXAMPLES

1. Write each expression as a sum or difference of sines and/or cosines.
   a) $2 \sin 4x \sin 2x$
   b) $4 \cos(-x) \cos 2x$
   c) $\sin(3x/2) \sin(5x/2)$
   d) $\sin(-\pi x/4) \cos(-\pi x/2)$
2. Write each expression as a product of sines and/or cosines.
   a) $\cos 5x - \cos 3x$
   b) $\sin(3x/4) + \sin(x/2)$
   c) $\sin(0.4x) - \sin(0.6x)$
   d) $\cos(-\pi x/4) + \cos(\pi x/6)$
3. Simplify the following trigonometric expressions.
   a) $\dfrac{\cos 5x - \cos 3x}{\sin 5x + \sin 3x}$
   b) $\sin(x+y) \sin(x-y)$