Taylor Series Approximation for the Ratio of Two Estimates

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A common problem in statistics is approximating the mean and variance of a ratio of two random variables *X*/*Y*. This is essential for understanding the uncertainty when one measured quantity is divided by another. The technique again relies on a Taylor series expansion and is a direct application of the **Delta Method**.

The General Approach

The method expands the function $f(X, Y)$ as a Taylor series around the means of X and Y (μ_x, μ_y). Assume $\mu_y \neq 0$. The general second-order Taylor expansion of a function $f(X, Y)$ around (μ_x, μ_y) is:

$$f(X,Y) \approx f(\mu_x,\mu_y) + (X-\mu_x)f_x + (Y-\mu_y)f_y + \\ + \frac{1}{2}\left[(X-\mu_x)^2 f_{xx} + 2(X-\mu_x)(Y-\mu_y)f_{xy} + (Y-\mu_y)^2 f_{yy}\right]$$

where the partial derivatives are evaluated at (μ_x, μ_y).

For function, $f(X, Y) = X/Y$, the partial derivatives are:

- $f_x = \frac{\partial}{\partial X}\left(\frac{X}{Y}\right) = \frac{1}{Y}$, which evaluates to $\frac{1}{\mu_y}$.
- $f_y = \frac{\partial}{\partial Y}\left(\frac{X}{Y}\right) = -\frac{X}{Y^2}$, which evaluates to $-\frac{\mu_x}{\mu_y^2}$.
- $f_{xx} = \frac{\partial^2}{\partial X^2}\left(\frac{X}{Y}\right) = 0$
- $f_{yy} = \frac{\partial^2}{\partial Y^2}\left(\frac{X}{Y}\right) = \frac{2X}{Y^3}$, which evaluates to $\frac{2\mu_x}{\mu_y^3}$.
- $f_{xy} = \frac{\partial^2}{\partial X \partial Y}\left(\frac{X}{Y}\right) = -\frac{1}{Y^2}$, which evaluates to $-\frac{1}{\mu_y^2}$.

Approximation for the Mean (Expected Value)

Using the second-order expansion gives a more robust approximation for the mean. Take the expectation of the expansion, which eliminates terms involving $E[X-\mu_x]$ and $E[Y-\mu_y]$. The approximation becomes:

$$E\left[\frac{X}{Y}\right] \approx \frac{\mu_x}{\mu_y} + \frac{1}{2}\left(E[(Y-\mu_y)^2]f_{yy} + 2E[(X-\mu_x)(Y-\mu_y)]f_{xy}\right)$$

Substituting the derivatives and the definitions of variance σ_y^2 and covariance σ_{xy}:

$$E\left[\frac{X}{Y}\right] \approx \frac{\mu_x}{\mu_y} + \frac{1}{2}\left(\sigma_y^2 \frac{2\mu_x}{\mu_y^3} + 2\sigma_{xy}\frac{-1}{\mu_y^2}\right)$$

This simplifies the second-order approximation:

$$E\left[\frac{X}{Y}\right] \approx \frac{\mu_x}{\mu_y} + \frac{\mu_x}{\mu_y^3}\sigma_y^2 - \frac{\sigma_{xy}}{\mu_y^2}$$

Approximation for the Variance

The variance is typically approximated using the first-order terms of the Taylor expansion. The general formula is:

$$\mathrm{Var}(f(X,Y)) \approx (f_x)^2\,\mathrm{Var}(X) + (f_y)^2\,\mathrm{Var}(Y) + 2f_xf_y\,\mathrm{Cov}(X,Y)$$

Substituting first partial derivatives $f_x = 1/\mu_y$ and $f_y = -\mu_x/\mu_y^2$:

$$\mathrm{Var}\left(\frac{X}{Y}\right) \approx \left(\frac{1}{\mu_y}\right)^2\sigma_x^2 + \left(-\frac{\mu_x}{\mu_y^2}\right)^2\sigma_y^2 + 2\left(\frac{1}{\mu_y}\right)\left(-\frac{\mu_x}{\mu_y^2}\right)\sigma_{xy}$$

This simplifies to:

$$\mathrm{Var}\left(\frac{X}{Y}\right) \approx \frac{\sigma_x^2}{\mu_y^2} + \frac{\mu_x^2\sigma_y^2}{\mu_y^4} - \frac{2\mu_x\sigma_{xy}}{\mu_y^3}$$

This formula can be rewritten in a more intuitive form by factoring out $\left(\mu_x/\mu_y\right)^2$:

$$\mathrm{Var}\left(\frac{X}{Y}\right) \approx \left(\frac{\mu_x}{\mu_y}\right)^2 \left(\frac{\sigma_x^2}{\mu_x^2} + \frac{\sigma_y^2}{\mu_y^2} - \frac{2\sigma_{xy}}{\mu_x\mu_y}\right)$$

This form highlights how the relative variance (or "coefficient of variation") of X and Y contribute to the relative variance of the ratio.

Summary of Formulas

- Mean (Second-Order Approximation):
  $$E\left[\frac{X}{Y}\right] \approx \frac{\mu_x}{\mu_y}\left(1 + \frac{\sigma_y^2}{\mu_y^2} - \frac{\sigma_{xy}}{\mu_x\mu_y}\right)$$
- Variance (First-Order Approximation):
  $$\mathrm{Var}\left(\frac{X}{Y}\right) \approx \left(\frac{\mu_x}{\mu_y}\right)^2 \left(\frac{\mathrm{Var}(X)}{\mu_x^2} + \frac{\mathrm{Var}(Y)}{\mu_y^2} - \frac{2\mathrm{Cov}(X,Y)}{\mu_x\mu_y}\right)$$

Seminal Citations

The theoretical basis for these approximations is the same as for the product, but some works focus specifically on the challenges and properties of ratios.

1. Fieller, E. C. (1932). The Distribution of the Index in a Normal Bivariate Population. *Biometrika*, 24(3/4), 428–440.
   This is a foundational paper that derives the *exact* distribution for the ratio of two normally distributed variables. The complexity of the exact solution (known as Fieller's Theorem) is the primary motivation for using the simpler Taylor series approximation in practice.

2. Hinkley, D. V. (1969). On the Ratio of Two Correlated Normal Random Variables. *Biometrika*, 56(3), 635-639.
   Hinkley's paper provides a detailed analysis of the properties of the ratio of correlated normal variables, examining the accuracy of the normal approximation and providing more insight than the first-order Taylor expansion alone.

3. Kendall, M., & Stuart, A. (1977). *The Advanced Theory of Statistics* (Vol. 1, 4th ed.). Charles Griffin & Company.
   This text is the definitive reference for the underlying theory of the Delta Method (Chapter 10), from which these specific approximations for the ratio are derived.