Taylor Series Approximation for the Product of Two Estimates

Gemini summary
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In statistics and error propagation, it is often necessary to determine the mean and variance of a quantity that is the product of two other measured quantities (or estimates), each with its own uncertainty. If we have two random variables, and , we can approximate the statistical properties of their product, , using a Taylor series expansion. This technique is a specific application of a broader statistical tool known as the **Delta Method**.

The General Approach

The method involves expanding the function—in this case, the product —as a Taylor series around the means of the estimates. The general second-order Taylor expansion is:

$$f(X,Y) \approx f(\mu_x,\mu_y) + (X-\mu_x) f_x + (Y-\mu_y) f_y + \\ + \frac{1}{2}\left[(X-\mu_x)^2 f_{xx} + 2(X-\mu_x)(Y-\mu_y) f_{xy} + (Y-\mu_y)^2 f_{yy} \right]$$

where f_x, f_y, f_{yx}, etc., represent the partial derivatives of the function evaluated at the point (μ_x,μ_y). For our function, $f(X,Y) = XY$, the partial derivatives are:

- $f_x = \frac{\partial}{\partial X}(XY) = Y$, which evaluates to μ_y at (μ_x,μ_y)
- $f_y = \frac{\partial}{\partial Y}(XY) = X$, which evaluates to μ_x at (μ_x,μ_y)
- $f_{xx} = \frac{\partial^2}{\partial X^2}(XY) = 0$
- $f_{yy} = \frac{\partial^2}{\partial Y^2}(XY) = 0$
- $f_{xy} = \frac{\partial^2}{\partial X \partial Y}(XY) = 1$

Approximation for the Mean (Expected Value)

To find the approximate expected value of the product E[XY], we take the expectation of the

Taylor series expansion. Using the second-order expansion gives a more accurate result than simply using the first order.

Substituting the derivatives into the expansion:

$$XY \approx \mu_x\mu_y + (X-\mu_x)\mu_y + (Y-\mu_y)\mu_x + (X-\mu_x)(Y-\mu_y)$$

Now, taking the expectation of this expression:

$$E[XY] \approx E[\mu_x\mu_y] + E[(X-\mu_x)\mu_y] + E[(Y-\mu_y)\mu_x] + E[(X-\mu_x)(Y-\mu_y)]$$

Since $E[X-\mu_x]=0$ and $E[Y-\mu_y]=0$, the middle two terms drop out. The final term is, by definition, the covariance of X and Y, denoted σ_{xy}. This leaves us with the second-order approximation:

$$E[XY] \approx \mu_x\mu_y + \sigma_{xy}$$

Approximation for the Variance

The variance of the product $\mathrm{Var}(XY)$ is typically approximated using only the first-order terms of the Taylor expansion. The general formula for the variance of a function $f(X,Y)$ is:

$$\mathrm{Var}(f(X,Y)) \approx (f_x)^2\,\mathrm{Var}(X) + (f_y)^2\,\mathrm{Var}(Y) + 2f_xf_y\mathrm{Cov}(X,Y)$$

Substituting the first partial derivatives and using the standard notation for variance and covariance:

$$\mathrm{Var}(XY) \approx \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2 + 2\mu_x\mu_y\sigma_{xy}$$

This formula is widely used for error propagation when dealing with the product of two measurements.

Summary of Formulas

- Mean (Second-Order Approximation):
  $E[XY] \approx \mu_x\mu_y + \mathrm{Cov}(X,Y)$
- Variance (First-Order Approximation):

  $\mathrm{Var}(XY) \approx \mu_y^2\mathrm{Var}(X) + \mu_x^2\mathrm{Var}(Y) + 2\mu_x\mu_y\mathrm{Cov}(X,Y)$

Seminal Citations

The development of these approximations is part of the broader history of statistical theory and error propagation, with roots going back to Gauss. However, for the specific problem of the moments of a product of random variables, the following works are considered foundational.

1. **Goodman, L. A. (1960). On the Exact Variance of Products. *Journal of the American Statistical Association*, 55(292), 708–713.**

    This is arguably the most cited paper on this topic. Goodman provides the *exact* formula for the variance of the product of two random variables, not just the approximation. He then shows how the first-order Taylor series approximation presented above can be derived from the exact formula, providing a rigorous justification for its use.

2. **Kendall, M., & Stuart, A. (1977). *The Advanced Theory of Statistics* (Vol. 1, 4th ed.). Charles Griffin & Company.**

    This classic and comprehensive multi-volume text on statistical theory provides a detailed treatment of the Delta Method (Chapter 10). It explains the use of Taylor series to find the moments of functions of random variables in a general form, establishing the theoretical underpinnings for specific applications like the product of estimates.

3. **Klein, L. R. (1953). *A Textbook of Econometrics*. Row, Peterson and Company.**

    Klein's influential textbook was one of the earliest to explicitly introduce the Taylor series method for approximating the variances of estimators to the field of econometrics. This work was crucial for popularizing the technique among social scientists for analyzing the uncertainty of statistical models.