Areas Between Curves 1

Motivation : Let's extend the application of definite integrals to represent areas between curves.

Definition: The area of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$ where $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is given by:

$$A = \int_a^b [f(x) - g(x)]\, dx$$

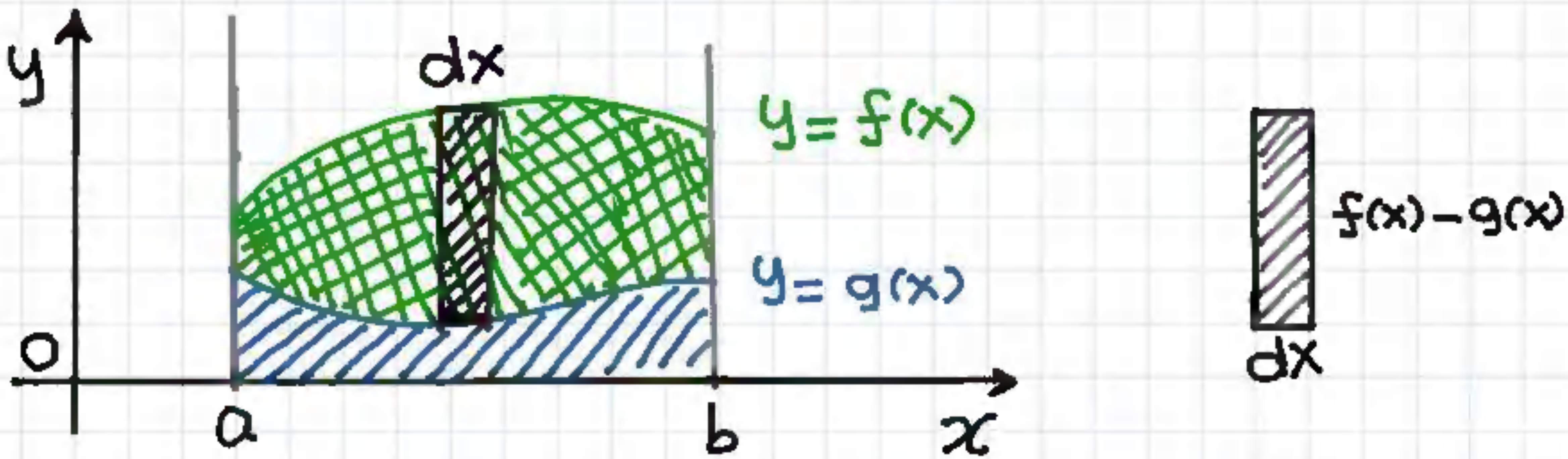
For the special case $g(x) = 0 \Rightarrow A = \int_a^b f(x)\, dx$

Area bounded by $f(x)$ and the x axis

For the case where $f(x)$ and $g(x)$ are both positive

$A =$ [area under $y = f(x)$] $-$ [area under $y = g(x)$]

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$



$dA = [f(x) - g(x)]\,dx$ Area of rectangular slice

$$A = \int_a^b dA = \int_a^b [f(x) - g(x)]\,dx$$

Ex| Find the area bounded by the graphs of

$y = \frac{1}{x}$; $x = 1$, $x = e$ and the x axis $y = 0$

Steps:

1| Sketch the region

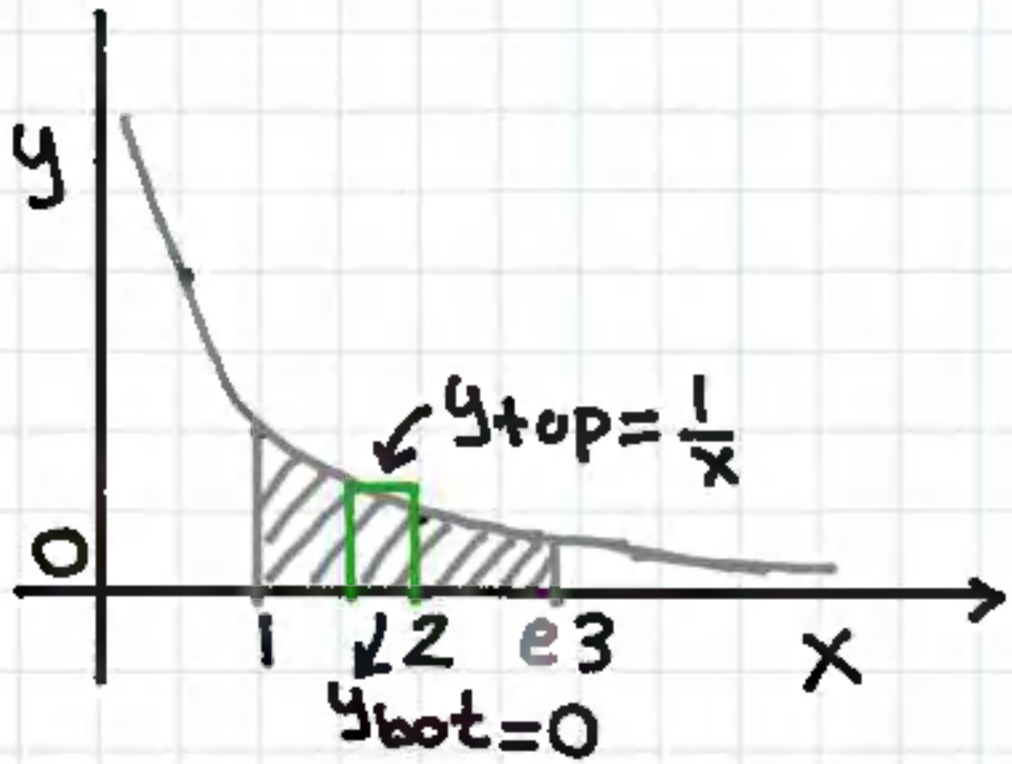
2| Determine limits of integration a and b

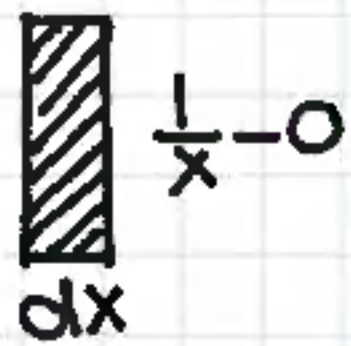
3| set up definite integral $\int_a^b [f(x) - g(x)]\, dx$

4| Integrate

$y = \frac{1}{x}$; $x = 1$, $x = e$ and the x axis $y = 0$

$$\text{Area} = \int_a^b [y_{top} - y_{bot}]\, dx$$





$$dA = \frac{1}{x}\, dx$$

$$A = \int_1^e \left[\frac{1}{x} - 0\right] dx = \int_1^e \frac{1}{x}\, dx = \ln x \Big|_1^e = \overset{1}{\ln e} - \overset{0}{\ln 1} = 1 - 0$$

$$= 1$$

$$\text{Therefore Area} = \int_1^e \frac{1}{x}\, dx = 1$$

Find the area of the region bounded by the parabolas y=4-x^2 and y=x^2+2x solved example

Areas Between Curves 2

Ex| Find the area of the region bounded by the parabolas $y=4-x^2$ and $y=x^2+2x$

solution: Step 1| Find the intersection points of

$y=4-x^2$ and $y=x^2+2x$

$y=y \Rightarrow 4-x^2=x^2+2x \Rightarrow -2x^2-2x+4=0$

$-2[x^2+x-2]=0 \Rightarrow -2[x+2][x-1]=0$

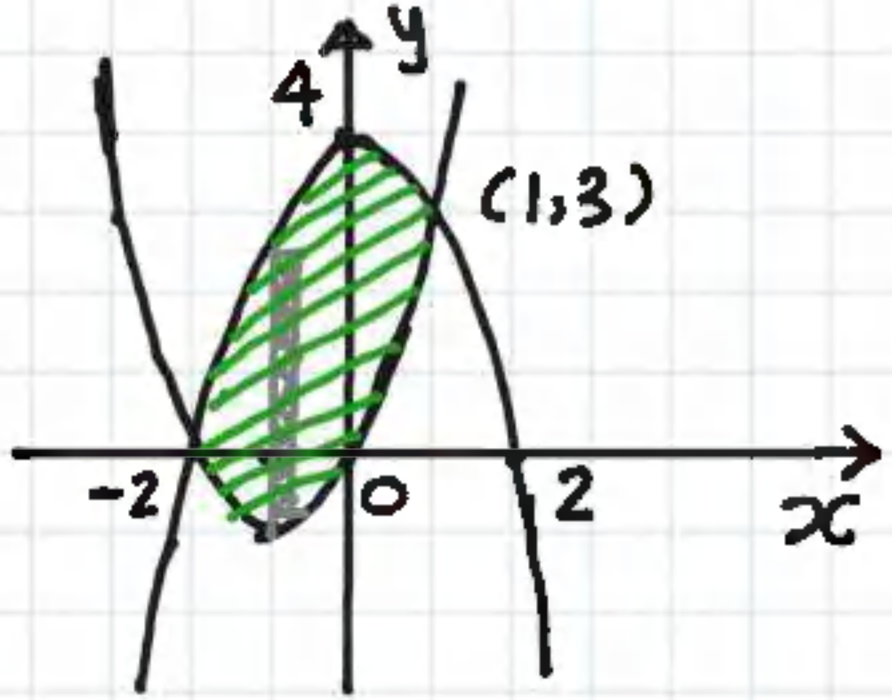
$x+2=0 \Rightarrow x=-2$ and $x-1=0 \Rightarrow x=1$

$x=1$ and $x=-2$ are x coord. of intersection

$x=1 \Rightarrow y=4-x^2 \Rightarrow y=4-1=3$ $(1,3)$

$x=-2 \Rightarrow y=x^2+2x \Rightarrow y=4-4=0$ and $(-2,0)$

step 2) Sketch bounded region $y = 4 - x^2$ and $y = x^2 + 2x$



$y = 4 - x^2$

$x = 0 \quad y = 4$

$y = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$y = x^2 + 2x = x(x+2) = 0$

$x = 0$, $x + 2 = 0 \Rightarrow x = -2$

$y = 0 \qquad\qquad y = 0$

step 3) Set up definite integral to find area

when $-2 < x < 1$ the height of the rectangular element is defined by the difference of top function $y = 4 - x^2$ and the bottom function $y = x^2 + 2x$

$$dA = (4 - x^2 - (x^2 + 2x))\,dx = (4 - 2x^2 - 2x)\,dx$$

$$\text{Area} = \int_{-2}^{1} dA = \int_{-2}^{1} (-2x^2 - 2x + 4)\,dx$$

$$A = \int_{-2}^{1} (-2x^2 - 2x + 4)\,dx$$

$$\overbrace{(4 - x^2)}^{y_{top}} - \overbrace{(x^2 + 2x)}^{y_{bot}}$$

dx

$$A = \left[-\frac{2x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + 4x \right]_{-2}^{1}$$

$$A = \left[-\frac{2}{3} - 1 + 4 \right] - \left[-\frac{2}{3}(-2)^3 - (-2)^2 + 4(-2) \right]$$

$$A = \left[3 - \frac{2}{3} \right] - \left[\frac{16}{3} - 4 - 8 \right] = 3 - \frac{2}{3} - \frac{16}{3} + 12$$

$$A = \frac{15}{1} - \frac{2}{3} - \frac{16}{3} = \frac{45 - 2 - 16}{3} = \frac{27}{3} = 9$$

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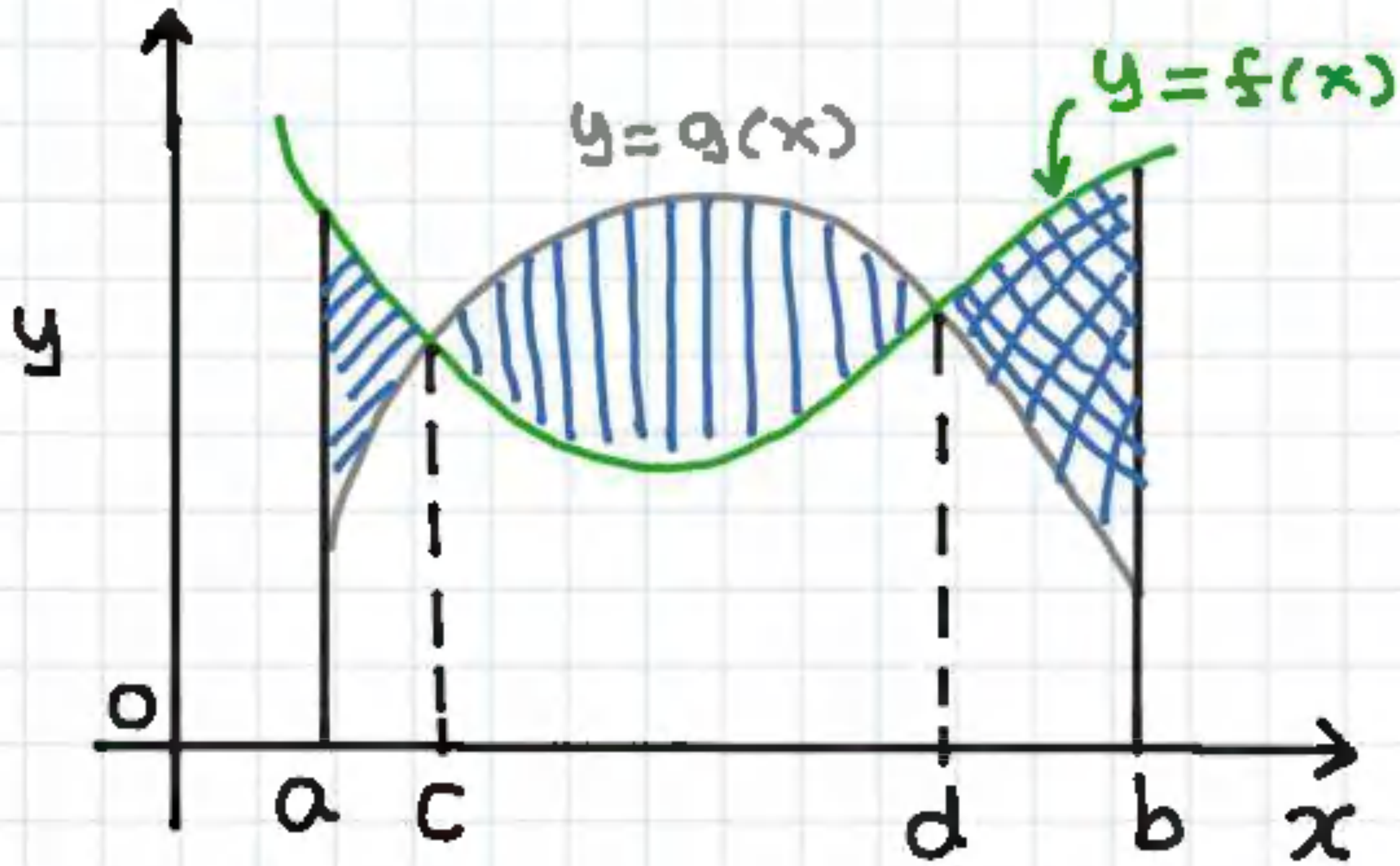
Areas Between Curves 3

Consider the case of finding the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x)$ and $g(x)$ intersect at 2 or more points.

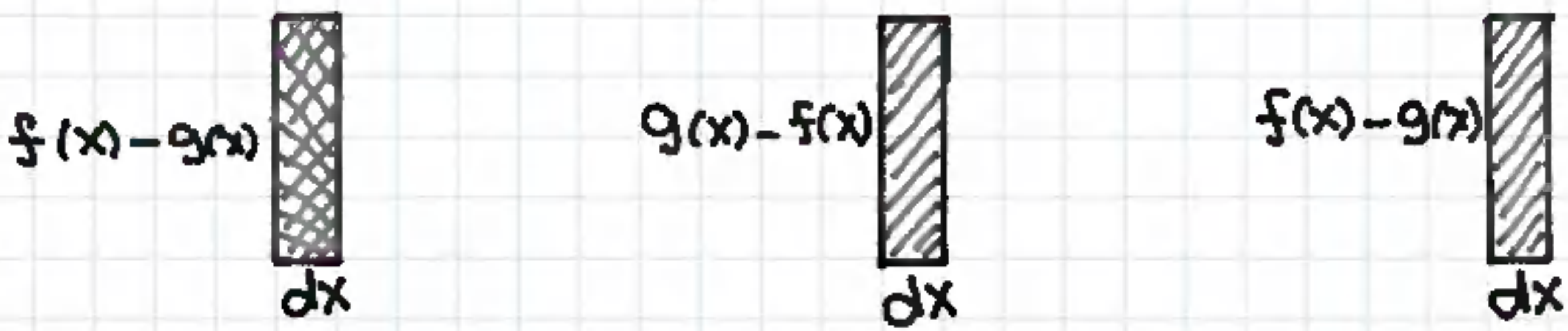


$$|f(x)-g(x)| = \begin{cases} f(x)-g(x) & \text{when } f(x) \geq g(x) \\ g(x)-f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

The area bounded by the curves $y=f(x)$ and $y=g(x)$ and $x=a$, $x=b$ is given by:

$$A = \int_a^b |f(x)-g(x)|\,dx$$

$$A = \int_a^c (f(x)-g(x))\,dx + \int_c^d (g(x)-f(x))\,dx + \int_d^b (f(x)-g(x))\,dx$$



Find the area of the region bounded by the curves y=2x/(1+x^2) , y=x^2,x=0 and x=2

Ex| Find the area of the region bounded by the curves

$y = \frac{2x}{1+x^2}$ and $y = x^2$ and $x=0$, $x=2$

Solution:

step1| Find the intersection points of $y = x^2$ and

$y = \frac{2x}{1+x^2}$

Cross multiply

$y = y \Rightarrow x^2 = \frac{2x}{1+x^2} \Rightarrow 2x = x^2(1+x^2) \Rightarrow 2x = x^2 + x^4$

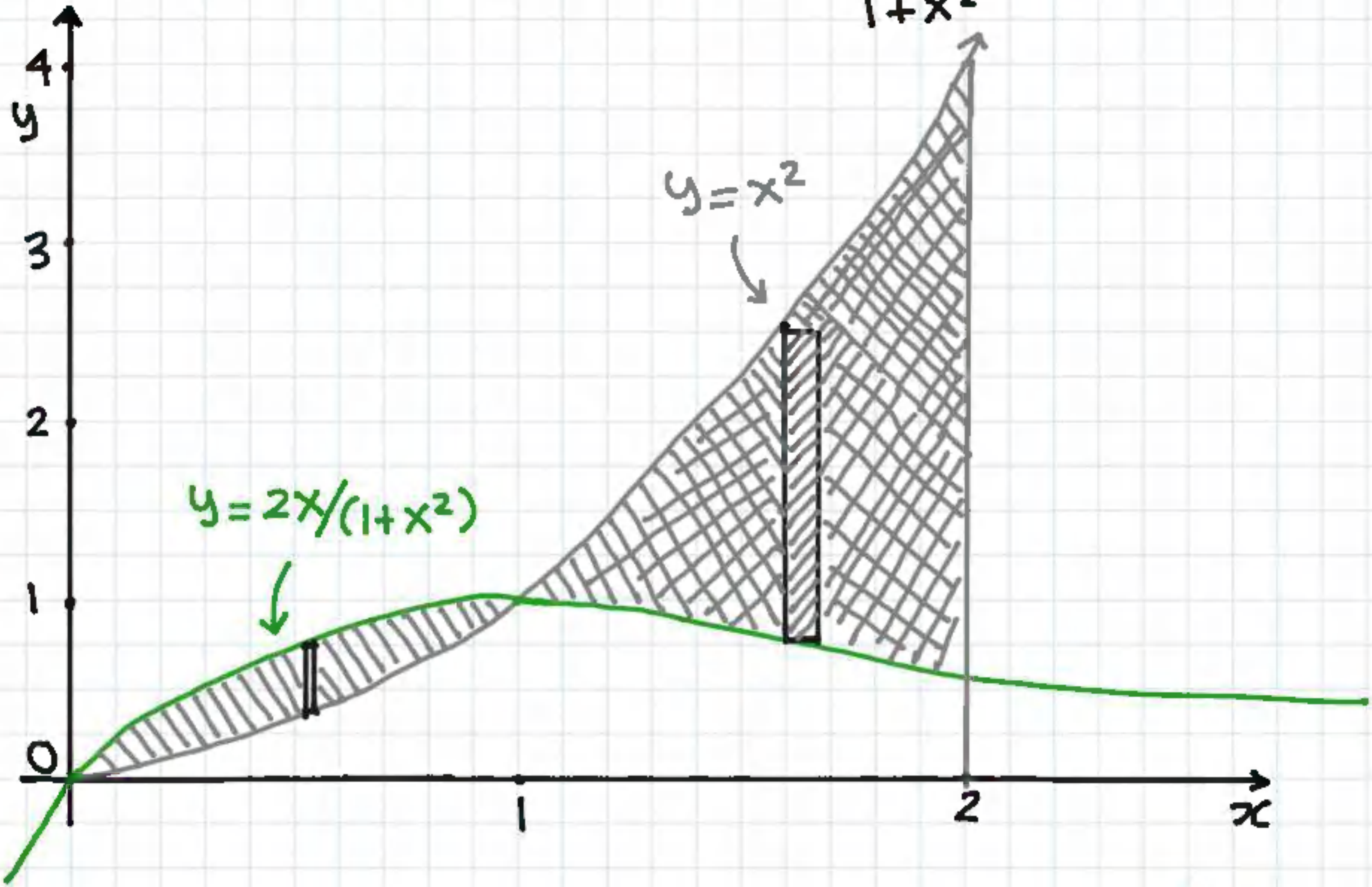
$x^4 + x^2 - 2x = 0 \Rightarrow x(x^3 + x - 2) = 0$

$x = 0$ and $x^3 + x - 2 = 0$ $\qquad$ $x^3 + x - 2 = 0$

Cubic polynomial Guess $x = 1 \Rightarrow 1^3 + 1 - 2 = 0$ ✓✓

$x = 0$ and $x = 1$ are x coord. of intersection.

Step 2| Sketch bounded region $y = \frac{2x}{1+x^2}$ and $y = x^2$



Step 3 | Set up definite integral to find Area

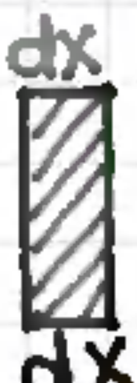
The points of intersection of $y = \dfrac{2x}{1+x^2}$ and $y = x^2$ are $x = 0$ and $x = 1$

Notice $\dfrac{2x}{1+x^2} > x^2$ when $0 < x < 1$ and

$x^2 > \dfrac{2x}{1+x^2}$ when $1 < x < 2$

Therefore the Area of bounded region is :

$$A = \int_0^1 \left(\frac{2x}{1+x^2} - x^2 \right) dx + \int_1^2 \left(x^2 - \frac{2x}{1+x^2} \right) dx$$



$\left(\dfrac{2x}{1+x^2} - x^2 \right)$

$\left(x^2 - \dfrac{2x}{1+x^2} \right)$

$$A = \int_0^1 \left(\frac{2x}{1+x^2} - x^2\right) dx + \int_1^2 \left(x^2 - \frac{2x}{1+x^2}\right) dx$$

$$A = \int_0^1 \frac{2x}{1+x^2} dx - \int_0^1 x^2 dx + \int_1^2 x^2 dx - \int_1^2 \frac{2x}{1+x^2} dx$$

Let $u = 1+x^2$ $\quad \int \frac{du}{u} = Ln|u| = Ln(1+x^2)\Big|_0^1$

$du = 2x\,dx$

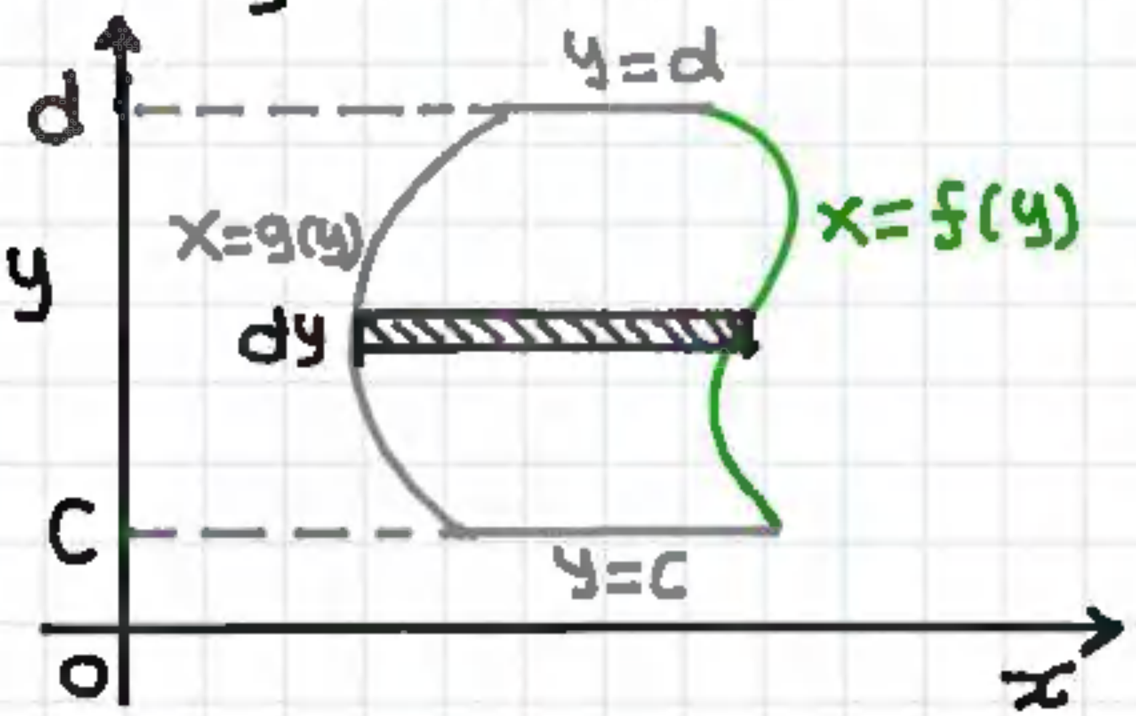
$$A = Ln(1+x^2)\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 - Ln(1+x^2)\Big|_1^2$$

$$A = Ln\,2 - Ln\,1 - \left(\frac{1}{3} - 0\right) + \left(\frac{8}{3} - \frac{1}{3}\right) - \left(Ln(5) - Ln\,2\right)$$

$$A = Ln2 - 0 - \frac{1}{3} + \frac{8}{3} - \frac{1}{3} - Ln5 + Ln2 = \underline{2 + 2Ln2 - Ln5}$$

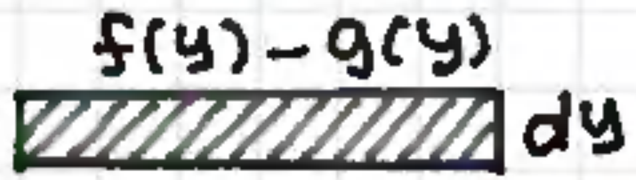
Areas Between Curves 4

Motivation: Integrating with respect to y, It is sometimes easier to calculate the area between two curves by regarding x as a function of y and using horizontal rectangular elements.



$$A = \int_c^d (f(y) - g(y))\,dy$$

$$A = \int_c^d (x_{right} - x_{left})\,dy$$



Ex| Find the area enclosed by the parabolas $x=y^2$ and $x=2y^2-1$.

Solution: step 1| Find intersection points of $x=y^2$ and $x=2y^2-1$

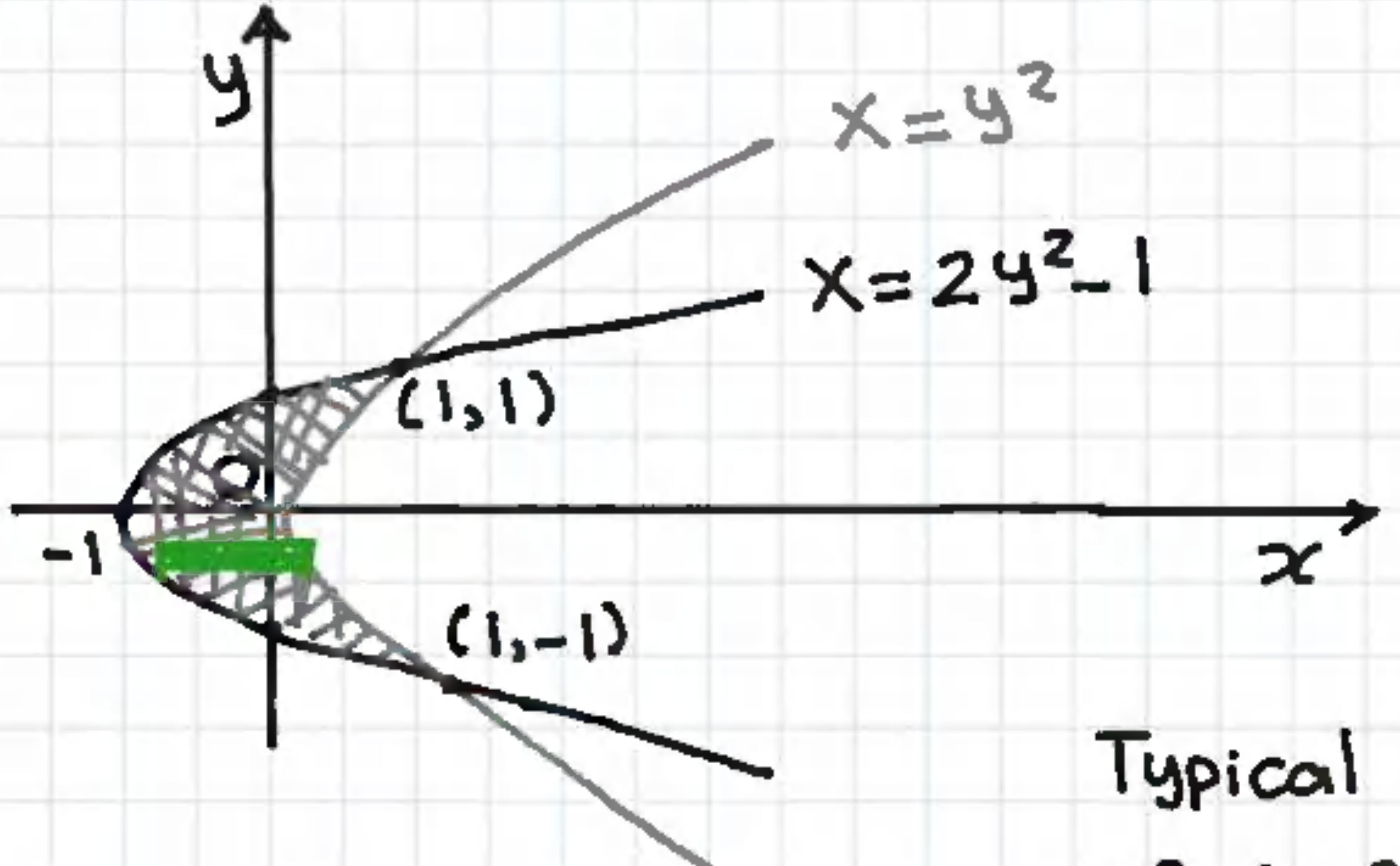
$y^2=2y^2-1 \Rightarrow y^2-2y^2=-1 \Rightarrow -y^2=-1 \Rightarrow y^2=1 \Rightarrow y=\pm 1$

$y=1 \Rightarrow x=y^2 \Rightarrow x=1 \Rightarrow x=1, y=1$

$y=-1 \Rightarrow x=2y^2-1 \Rightarrow x=1, y=-1$

step 2| Sketch region bounded by $x=y^2$ and $x=2y^2-1$

Step 2] Sketch region bounded by $x=y^2$ and $x=2y^2-1$



Typical horizontal slice

$$dA=(y^2-(2y^2-1))\,dy$$

step 3| Set up definite integral to Calculate Area

$$A = \int_{-1}^{1} dA = \int_{-1}^{1} (y^2 - (2y^2 - 1))\,dy = \int_{-1}^{1} (-y^2 + 1)\,dy$$

$$A = 2\int_{0}^{1} (1 - y^2)\,dy$$

Region Symmetric with respect to x axis

$$A = 2\left[y - \frac{y^3}{3}\right]_0^1 = 2\left[1 - \frac{1}{3} - (0-0)\right] = 2[2/3] = 4/3$$

Area $= 4/3$

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Find the area of the region bounded by the curves x=sin(πy/2),x=y^2 and y=2 solved example

Areas between Curves 5

Ex] Find the area of the region bounded by the curves

$x = \sin(\frac{\pi}{2} y)$ and $x = y^2$ and $y = 2$

Solution: step 1] Find the points of intersection of

$x = \sin(\frac{\pi y}{2})$ and $x = y^2$

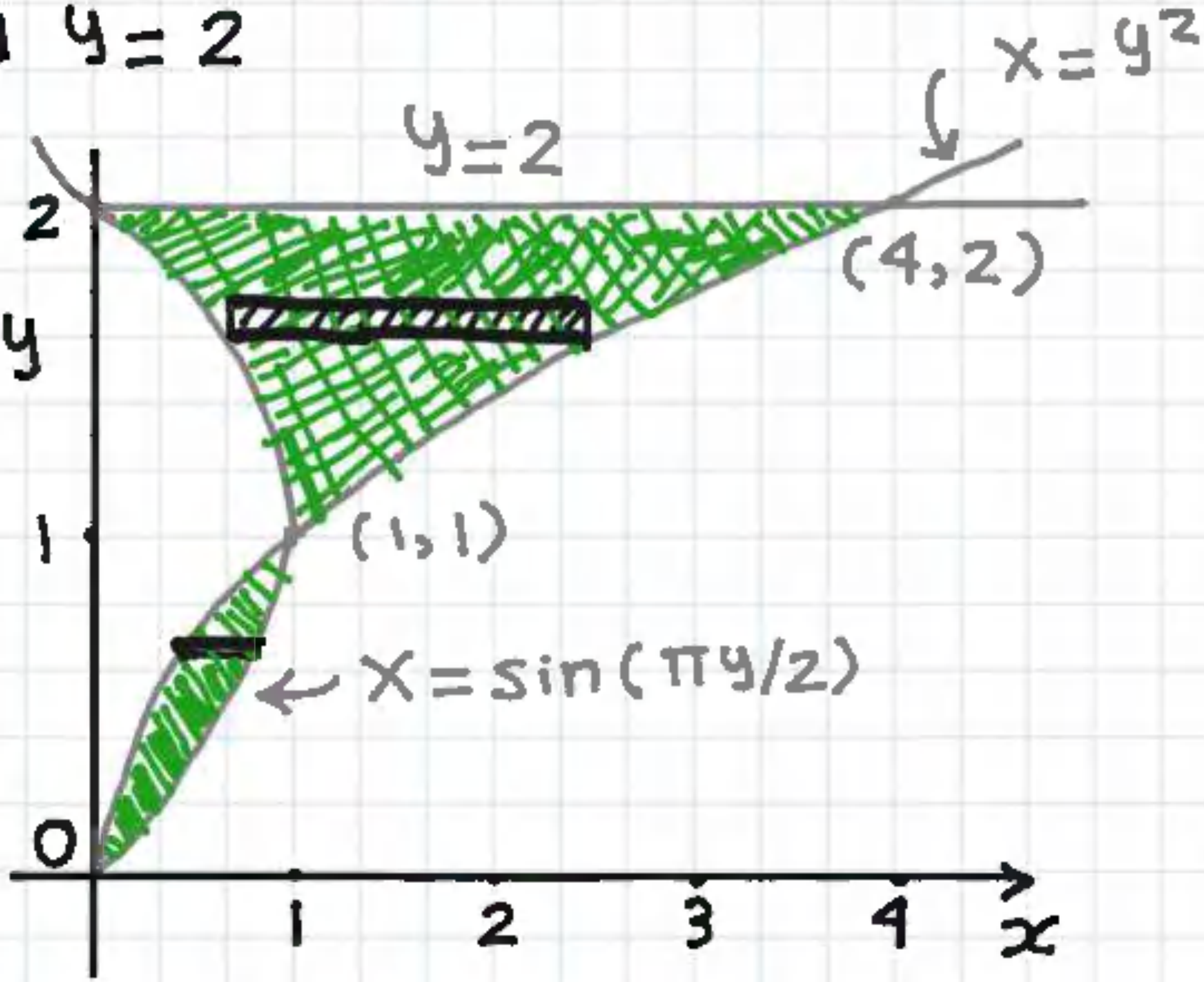
$\sin(\pi y/2) = y^2 \Rightarrow$ Cannot solve algebraically
let's Guess!

$y = 0 \Rightarrow \sin(0) = 0 \Rightarrow y = 0 \Rightarrow x = y^2 \Rightarrow x = 0$

$y = 1 \Rightarrow \sin(\pi/2) = 1 \Rightarrow y = 1 \Rightarrow x = \sin(\frac{\pi y}{2}) \Rightarrow x = 1$

$\therefore$ $x = 0, y = 0$ and $x = 1, y = 1$ Intersection points

Step 2] Sketch region bounded by $x = \sin(\frac{\pi y}{2})$, $x = y^2$ and $y = 2$



$y^2 - \sin(\pi y/2)$

dy

$x_{right} - x_{left}$

$dA = (y^2 - \sin(\frac{\pi y}{2}))dy$

when $1 \leq y \leq 2$

and similiarly

$dA = (\sin(\frac{\pi y}{2}) - y^2)dy$

when $0 \leq y \leq 1$

Step 3| Set up definite integral to find Area

$$A = \int_0^1 \left(\sin\left(\frac{\pi y}{2}\right) - y^2\right) dy + \int_1^2 \left(y^2 - \sin\left(\frac{\pi y}{2}\right)\right) dy$$

$$A = \int_0^1 \sin(\pi y/2)\, dy - \int_0^1 y^2\, dy + \int_1^2 y^2\, dy - \int_1^2 \sin\left(\frac{\pi y}{2}\right) dy$$

Apply U-Subst. $U = \pi y/2 \quad du = \frac{\pi}{2} dy \Rightarrow dy = \frac{2}{\pi} du$

$$A = \left[-\cos(\pi y/2)\right]\cdot \frac{2}{\pi}\Big|_0^1 - \frac{y^3}{3}\Big|_0^1 + \frac{y^3}{3}\Big|_1^2 + \left[\cos\left(\frac{\pi y}{2}\right)\cdot\frac{2}{\pi}\right]_1^2$$

$$A = -\frac{2}{\pi}\Big[\underbrace{\cos(\pi/2)}_{=0} - \underbrace{\cos 0}_{=1}\Big] - \left(\frac{1}{3} - 0\right) + \frac{8}{3} - \frac{1}{3}$$
$$+ \, 2/\pi\Big[\underbrace{\cos\pi}_{=-1} - \underbrace{\cos(\pi/2)}_{=0}\Big]$$

$$A = -\frac{2}{\pi}[0-1] - \frac{1}{3} + \frac{8}{3} - \frac{1}{3} + \frac{2}{\pi}[-1-0]$$

$$A = \frac{2}{\pi} + 2 - \frac{2}{\pi} = 2$$

Concept Review

when $0 \leq y \leq 1 \Rightarrow \sin(\pi y/2) > y^2$

when $1 \leq y \leq 2 \Rightarrow y^2 > \sin(\pi y/2)$

$$\therefore\ A = \int_0^1 (\sin(\pi y/2) - y^2)\,dy + \int_1^2 (y^2 - \sin(\pi y/2))\,dy$$

$$A = \int_0^1 (x_{right} - x_{left})\,dy + \int_1^2 (x_{right} - x_{left})\,dy$$

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Areas Between Curves 6

Ex| Find the area of the region bounded by the curves $y = x^3 - x$, $y = \sin(\pi x)$, $x=0$ and $x=2$

Solution : Step 1| Find the intersection points of $y = x^3 - x$ and $y = \sin(\pi x)$

$y = y \Rightarrow x^3 - x = \sin(\pi x)$ Can't solve with Algebra!

Let's Guess.

$y = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$ Factorize

$y = \sin(\pi x)$

$x = -1 \Rightarrow y = \sin(-\pi) = 0$

$x = 0 \Rightarrow y = \sin(0) = 0$

$x = 1 \Rightarrow y = \sin(\pi) = 0$

Notice $y = x^3 - x$ and $y = \sin(\pi x)$ both share the same x intercepts $x=-1$, $x=0$ and $x=1$

$y = x^3 - x = x(x-1)(x+1) = 0 \Rightarrow x=-1, x=0$ and $x=1$

$y = \sin(\pi x) = 0 \Rightarrow x=-1, x=0$ and $x=1$

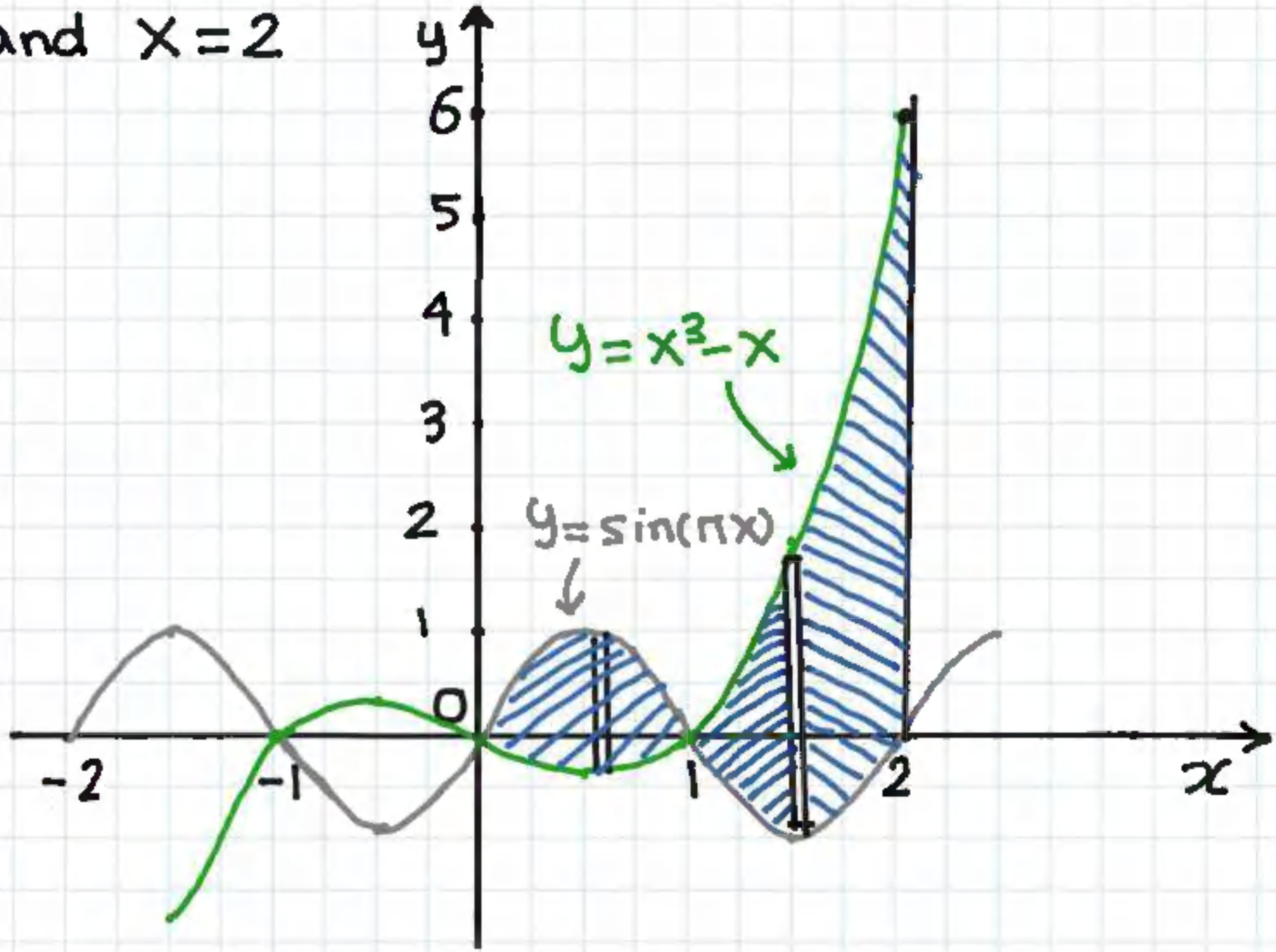
Therefore the points of intersection of $y = \sin(\pi x)$ and $y = x^3 - x$ are $x=-1, x=0$ and $x=1$

$x=-1$ $y=0$; $x=0, y=0$ and $x=1, y=0$

Points of intersection of $y = x^3 - x$ and $y = \sin(\pi x)$

Step 2 | Sketch bounded region $y = x^3 - x$, $y = \sin(\pi x)$

$x = 0$ and $x = 2$



Step 3 | Set up definite integral to find the Area.

The points of intersection of $y = x^3 - x$ and $y = \sin(\pi x)$ are $x = -1$, $x = 0$ and $x = 1$

Notice $\sin(\pi x) > x^3 - x$ when $0 < x < 1$ and

$x^3 - x > \sin(\pi x)$ when $1 < x < 2$

Therefore the area of bounded region is :

$$A = \int_0^1 [\sin(\pi x) - (x^3 - x)]\,dx + \int_1^2 [(x^3 - x) - \sin(\pi x)]\,dx$$

$[\sin(\pi x) - (x^3 - x)]$

$[(x^3 - x) - \sin(\pi x)]$

$$A = \int_0^1 [\sin(\pi x) - (x^3 - x)]\,dx + \int_1^2 [(x^3 - x) - \sin(\pi x)]\,dx$$

Let $u = \pi x$ $\quad du = \pi\,dx$ $\quad$ Apply U-sub to $\int_0^1 [\sin(\pi x)]\,dx$

$$A = \frac{-\cos(\pi x)}{\pi} - \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_0^1 + \frac{x^4}{4} - \frac{x^2}{2} - -\frac{1}{\pi}\cos(\pi x)\Bigg|_1^2$$

$$A = \frac{-\cos(\pi)}{\pi} - \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{-\overset{1}{\cos(0)}}{\pi} - (0 - 0)\right)$$

$$+ \frac{16}{4} - \frac{4}{2} + \frac{1}{\pi}\underset{1}{\cos(2\pi)} - \left(\frac{1}{4} - \frac{1}{2} + \frac{1}{\pi}\overset{-1}{\cos(\pi)}\right)$$

$$A = \frac{1}{\pi} + \frac{1}{4} + \frac{1}{\pi} + 4 - 2 + \frac{1}{\pi} + \frac{1}{4} + \frac{1}{\pi}$$

$$A = \frac{4}{\pi} + 2 + \frac{1}{2} \cong 3.77$$

Find the area of the region bounded between the curves y=cosx,y=1/2,x=0,x=π solved example

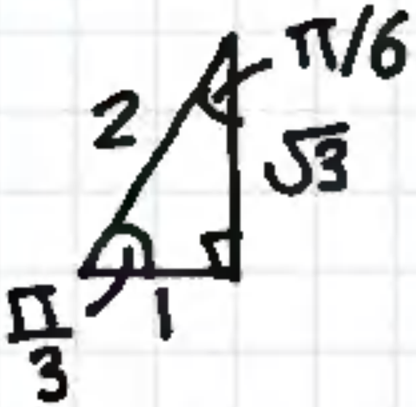
Areas between Curves 7

Ex) Find the area of the region bounded between the curves $y=\cos x$ and $y=1/2$ and between $x=0$ and $x=\pi$.

Solution: step 1) Find the intersection points of $y=\cos x$ and $y=1/2$.

$y=y \Rightarrow \cos x = 1/2 \qquad 0 < x < \pi$

$\cos x = \frac{1}{2} \Rightarrow x = \pi/3$

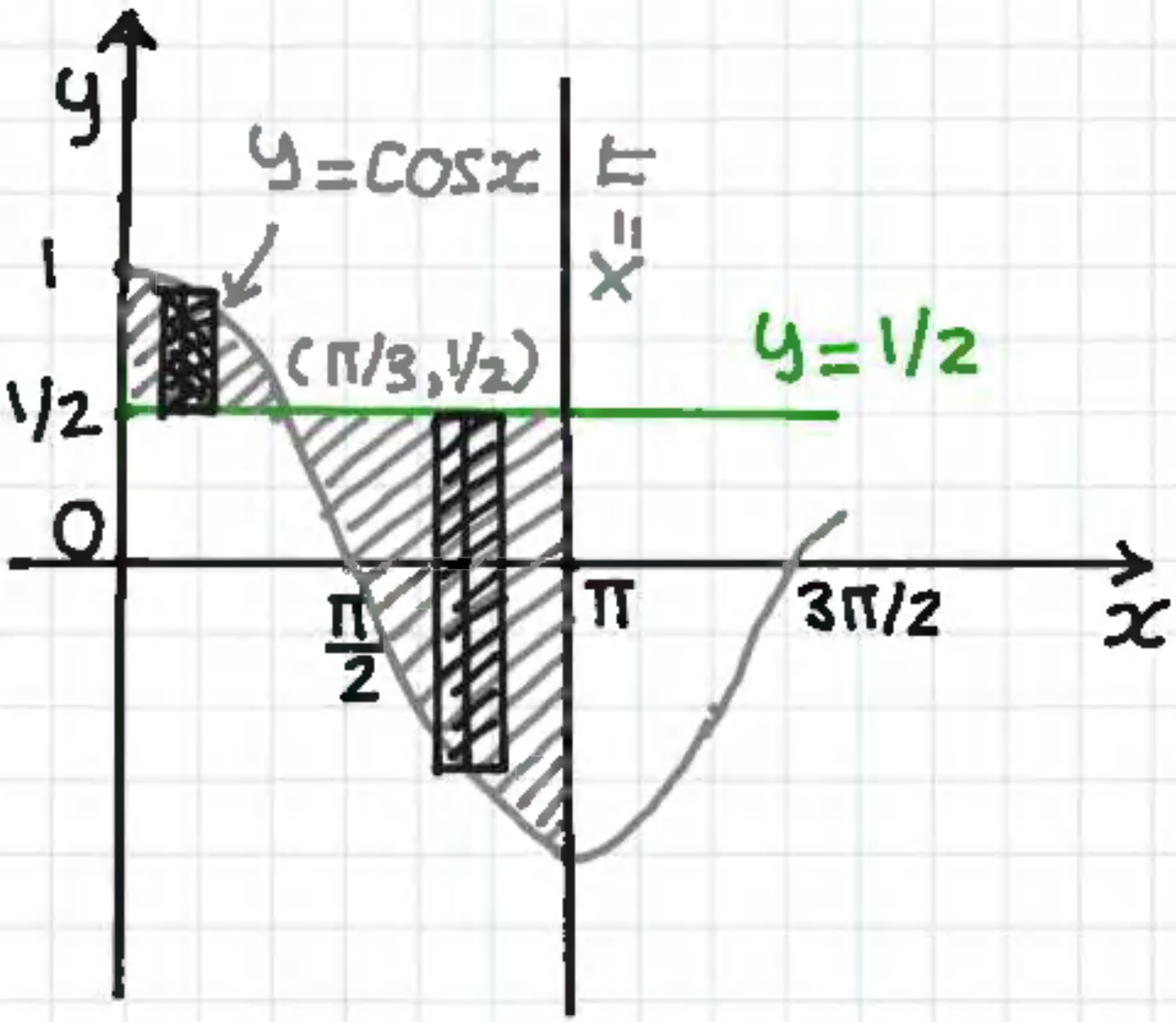


$\cos(\pi/3) = 1/2$

Recall: 30-60-90 triangle

Step 2] Sketch bounded region $y = \cos x$, $y = \frac{1}{2}$

$x = 0$ and $x = \pi$

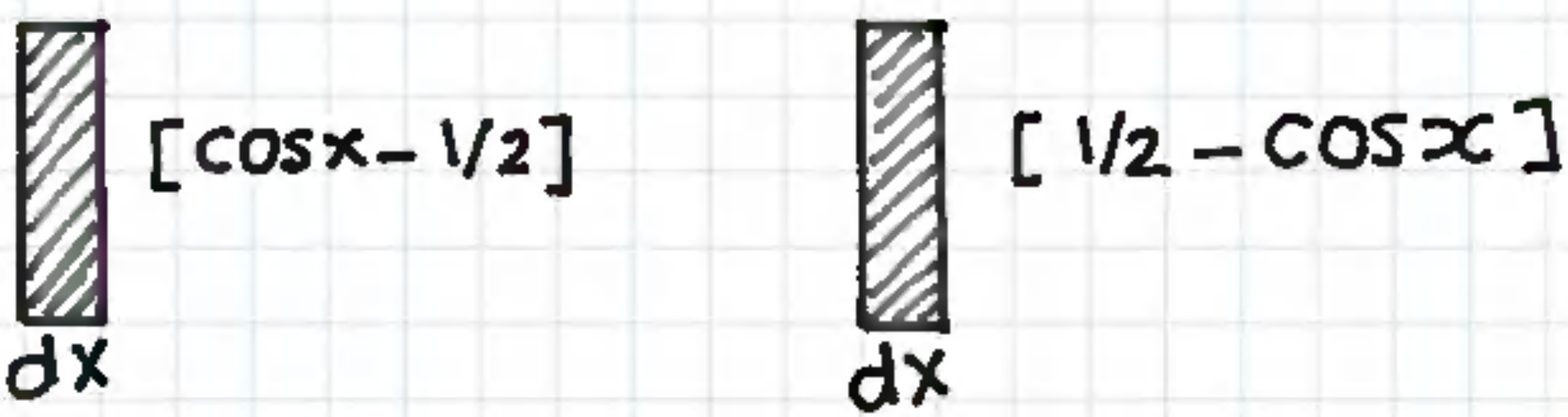


Step 3| Set up definite integral to find Area and Solve

The points of intersection of $y = \cos x$ and $y = 1/2$ is $x = \pi/3$ $y = 1/2$ on the interval $0 < x < \pi$

Notice $\cos x > \frac{1}{2}$ when $0 < x < \pi/3$ and $\frac{1}{2} > \cos x$ when $\frac{\pi}{3} < x < \pi$; therefore Area is

$$A = \int_0^{\pi/3} \left[\cos x - \frac{1}{2}\right] dx + \int_{\pi/3}^{\pi} \left[\frac{1}{2} - \cos x\right] dx$$



$$A = \int_0^{\pi/3} \left[\cos x - \frac{1}{2}\right] dx + \int_{\pi/3}^{\pi} \left[\frac{1}{2} - \cos x\right] dx$$

$$A = \left[\sin x - \frac{1}{2}x\right]_0^{\pi/3} + \left[\frac{1}{2}x - \sin x\right]_{\pi/3}^{\pi}$$

$$A = \underbrace{\sin(\pi/3)}_{\sqrt{3}/2} - \frac{\pi}{6} - \left(\underbrace{\sin 0}_{0} - 0\right) + \left[\frac{\pi}{2} - \underbrace{\sin \pi}_{0}\right] - \left[\frac{\pi}{6} - \underbrace{\sin\frac{\pi}{3}}_{\sqrt{3}/2}\right]$$

$$A = \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\pi}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} + \frac{\pi}{2} \cong 2.26$$

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Find the values of c such that the area of the region bounded by the parabolas x=c^2-y^2 and x=y^2-c^2 is 72 solved example

Areas Between Curves 8

Ex] Find the values of C such that the area of the region bounded by the parabolas $x=C^2-y^2$ and $x=y^2-C^2$ is 72.

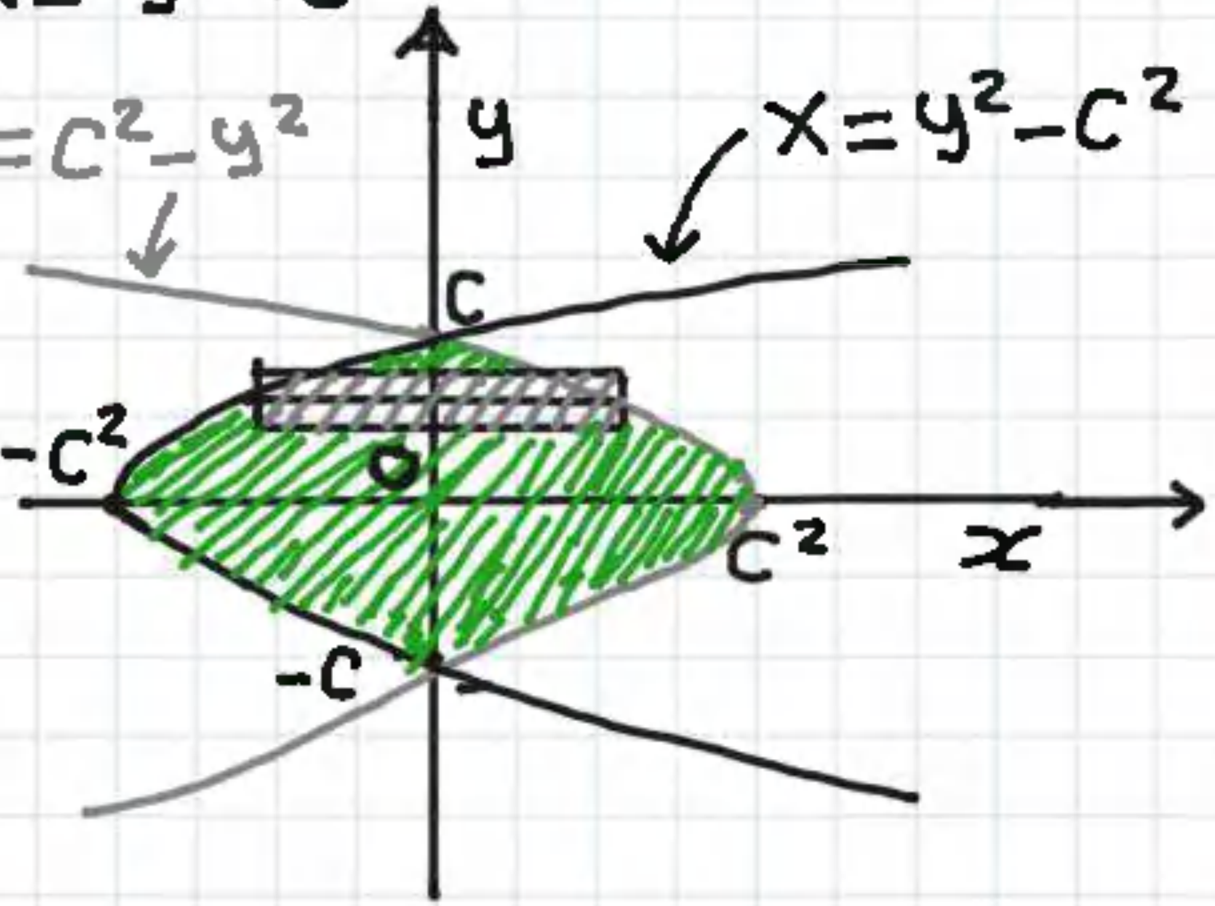
Solution: step 1] Find intersection points of $x=C^2-y^2$ and $x=y^2-C^2$

$x=x \Rightarrow C^2-y^2=y^2-C^2 \Rightarrow 2C^2=2y^2 \Rightarrow y^2=C^2$

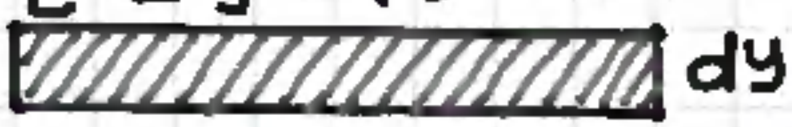
$\Rightarrow y=\pm C \qquad x=y^2-C^2 \Rightarrow x=C^2-C^2=0$

$x=0 \quad y=C$ and $x=0 \quad y=-C$ are the intersection points.

Step 2| Sketch region bounded by $x = c^2 - y^2$ and $x = y^2 - c^2$



Typical horizontal slice



$$dA = (c^2 - y^2 - (y^2 - c^2))\,dy$$

Step 3] Set up definite integral and solve for C

$$A = \int_{-C}^{C} dA = \int_{-C}^{C} [C^2 - y^2 - (y^2 - C^2)]\,dy$$

$$A = \int_{-C}^{C} [2C^2 - 2y^2]\,dy = 72$$

Integrate and solve for C such that bounded area = 72

$$A = 2\int_{0}^{C} [2C^2 - 2y^2]\,dy = 72$$

Region Symmetric with respect to x axis.

$$A = 2\left[2C^2 y - \frac{2y^3}{3}\right]_0^C = 72$$

$$A = 2\left[2C^3 - \frac{2C^3}{3} - (0 - 0)\right] = 72$$

$$A = \frac{4C^3}{1} - \frac{4C^3}{3} = 72$$

$$A = \frac{12C^3 - 4C^3}{3} = 72 \Rightarrow \frac{8C^3}{3} = 72 \Rightarrow C^3 = 72 \times \frac{3}{8}$$

$$C^3 = 27 \Rightarrow C = \sqrt[3]{27} \Rightarrow C = 3$$

Therefore for the total area of the bounded region to be 72, the value of C must be 3.

Find the area of the region bounded by the curves y=arctanx , y=π/4,x=0,x=2 solved example

Areas between curves 9

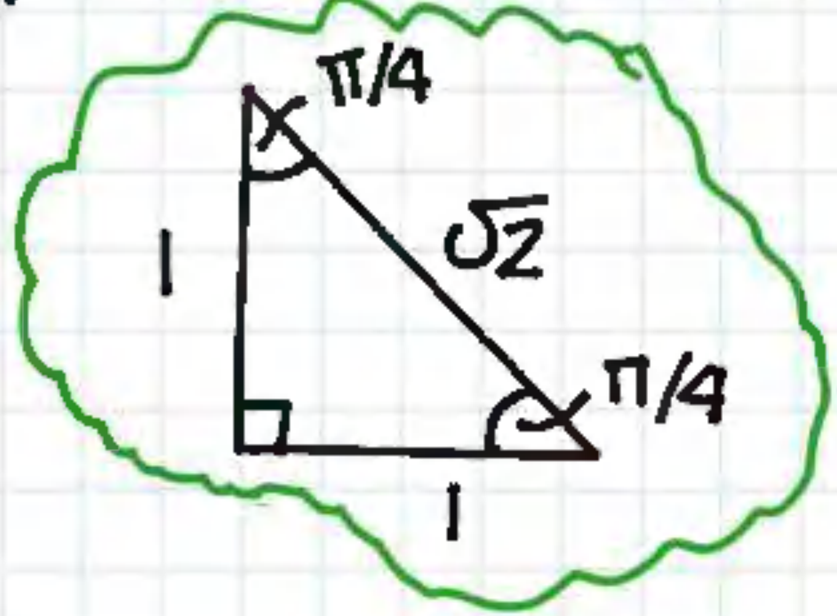
Ex] Find the area of the region bounded by the curves $y = \tan^{-1}x$, $y = \pi/4$ and $x=0$, $x=2$

Solution: Step 1] Find the intersection points of $y = \tan^{-1}x$ and $y = \pi/4$

$y = y \Rightarrow \tan^{-1}x = \pi/4 \Rightarrow \tan(\tan^{-1}x) = \tan(\pi/4)$

$x = \tan(\pi/4) \Rightarrow x = 1 \quad y = \pi/4$

Special Triangle

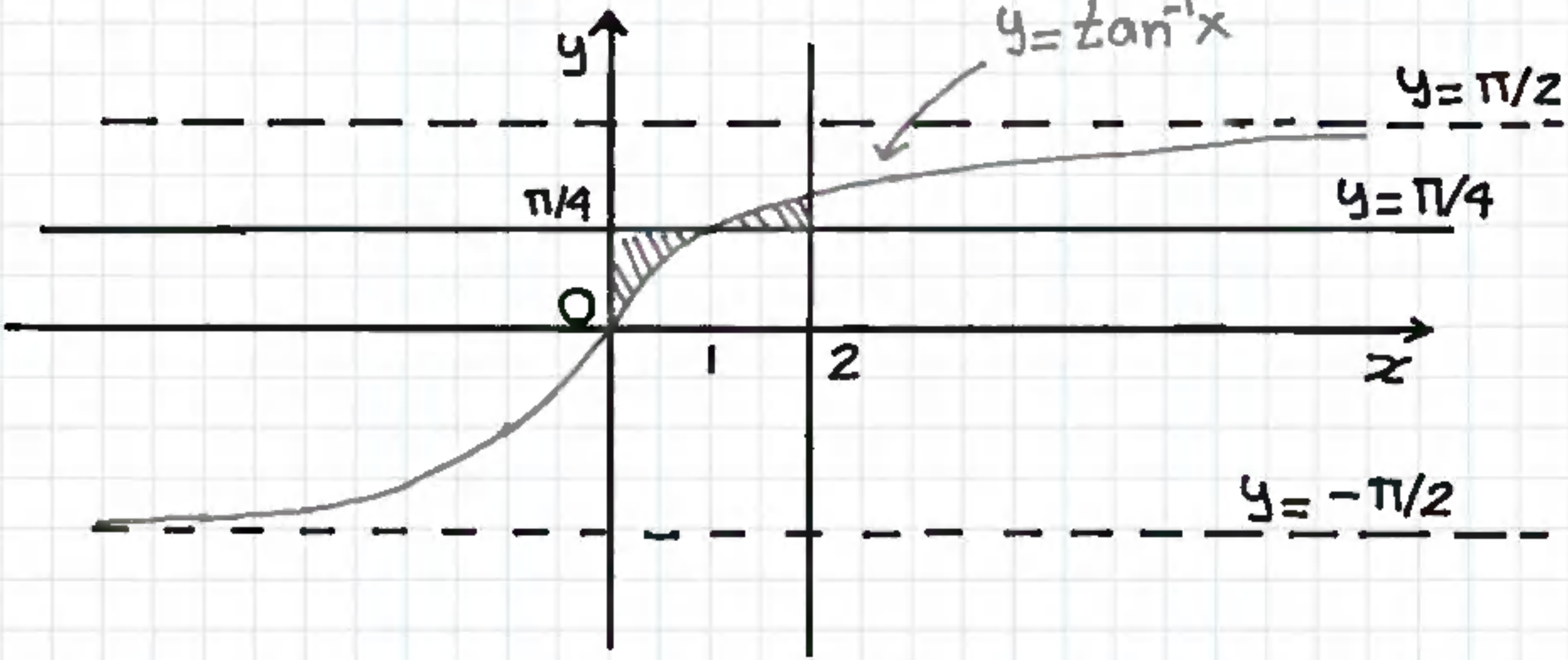


Points of intersection of $y = \tan^{-1}x$ and $y = \pi/4$, $x=0$, $x=2$

$x=1$, $y = \pi/4$; $x = 2 \Rightarrow y = \tan^{-1}x$

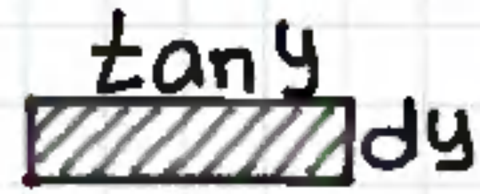
$x = 2 \Rightarrow y = \tan^{-1}2$

step 2] Sketch region bounded by $y = \tan^{-1}x$, $y = \pi/4$ and $x=0$ and $x=2$



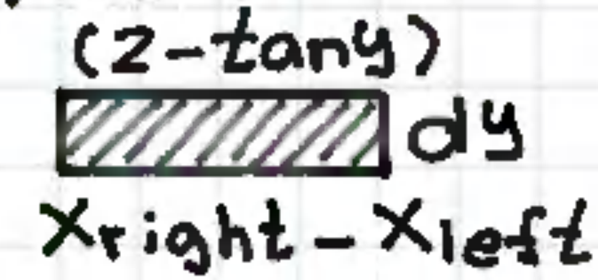
Since it is easier to integrote $x = \tan y$ with respect to y rather than integrating $y = \tan^{-1}x$ w.r.t x

we will apply horizontal rectangular elements and integrate with respect to y



For $0 < y < \frac{\pi}{4}$ $dA = \tan y \, dy$

For $\frac{\pi}{4} < y < \tan^{-1} 2$ $dA = (2 - \tan y)\, dy$



$$A = \int_0^{\pi/4} \tan y \, dy + \int_{\pi/4}^{\tan^{-1} 2} [2 - \tan y]\, dy$$

Set up two separate definite integrals with respect to y and add up to find area bounded.

$$A = \int_0^{\pi/4} \tan y \, dy + \int_{\pi/4}^{\tan^{-1}2} [2 - \tan y] \, dy$$

Let's apply U-Subst. to find $\int \tan y \, dy$

$$\int \tan y \, dy = \int \frac{\sin y}{\cos y} \, dy \qquad \text{Let } u = \cos y$$
$$du = -\sin y \, dy$$

$$\int \tan y \, dy = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos y| + C$$

$$A = -\ln|\cos y| \Big|_0^{\pi/4} + [2y - -\ln|\cos y|]_{\pi/4}^{\tan^{-1}2}$$

$$A = -\ln(\overset{1/\sqrt{2}}{\cos(\pi/4)}) + \ln(\overset{1}{\cos 0}) + 2\tan^{-1}2 + \ln|\cos(\tan^{-1}2)|$$
$$- [\frac{2\pi}{4} + \ln(\cos(\pi/4))]$$

$$A = -\ln(1/\sqrt{2}) + \underbrace{\ln 1}_{0} + 2\tan^{-1}2 + \ln(\cos(\tan^{-1}2)) - \pi/2$$
$$- \ln(1/\sqrt{2})$$

$$A = -2\ln(1/\sqrt{2}) + 2\tan^{-1}2 - \frac{\pi}{2} + \ln(\cos(\tan^{-1}2))$$

Calculator ready answer

$$A \cong 0.6931 + 2.2143 - 1.5708 + -0.8047$$

$$A \cong 0.532$$

Find the area of the region bounded by the curves y=e^(1-x),y=e^x,x=0,x=1 solved example

Areas Between Curves 10

Ex| Find the area of the region bounded by the curves $y=e^{1-x}$, $y=e^x$ and between $x=0$ and $x=1$

Solution: step 1| Find the intersection points of $y=e^{1-x}$ and $y=e^x$

$y=y \Rightarrow e^{1-x}=e^x \Rightarrow \ln e^{1-x}=\ln e^x \Rightarrow 1-x=x$

$1=2x \Rightarrow x=1/2$

$x=\frac{1}{2} \Rightarrow y=e^x \Rightarrow y=e^{1/2}$

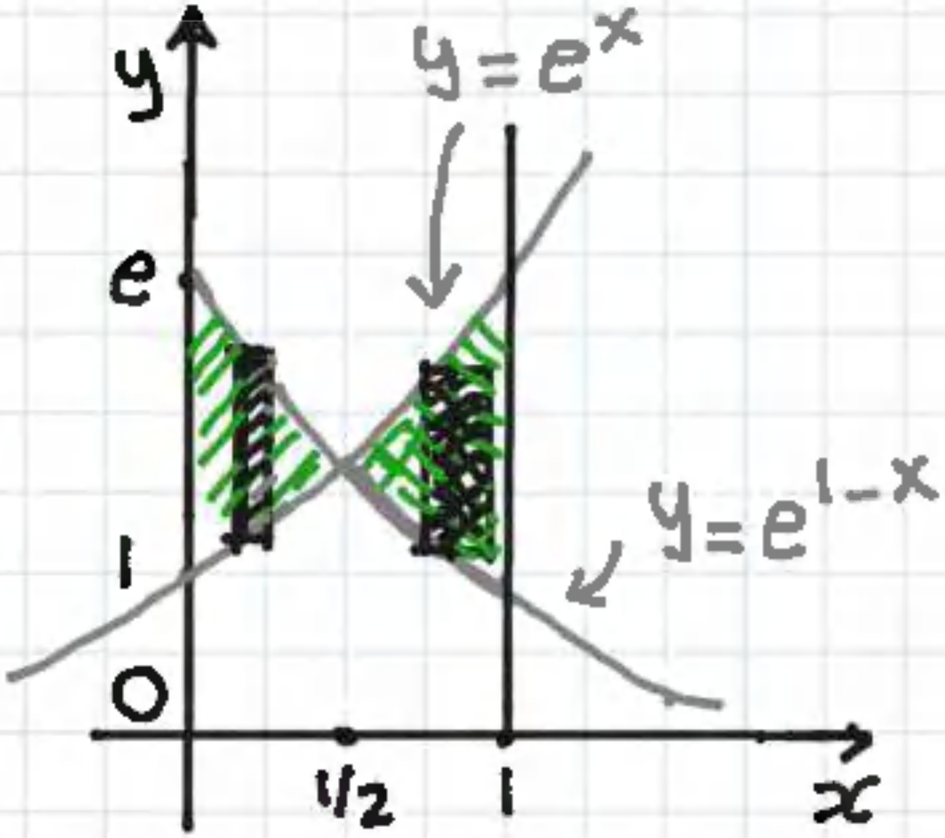
$x=1/2$, $y=e^{1/2}$

Recall: $\ln e^x = x$

$e^{\ln x} = x$

$\ln 1 = 0$ $\quad \ln e = 1$

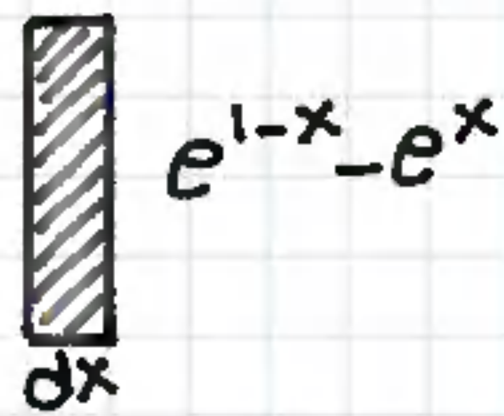
Step 2] Sketch bounded region $y=e^{1-x}$, $y=e^x$, $x=0$ and $x=1$



Notice when $0<x<1/2$

$e^{1-x} > e^x$, therefore

$dA=(e^{1-x}-e^x)\,dx$



And when $\frac{1}{2}<x<1$

$e^x > e^{1-x}$, therefore

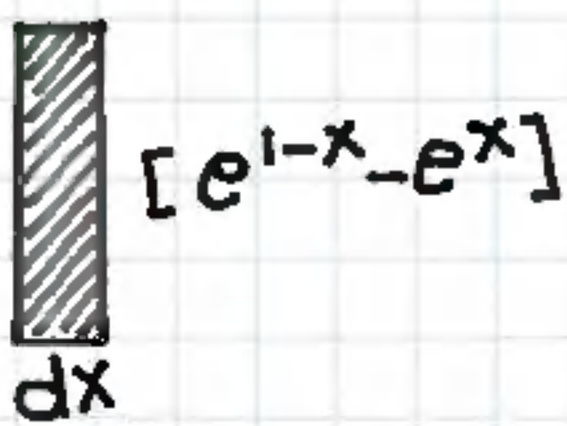
$dA=(e^x-e^{1-x})\,dx$

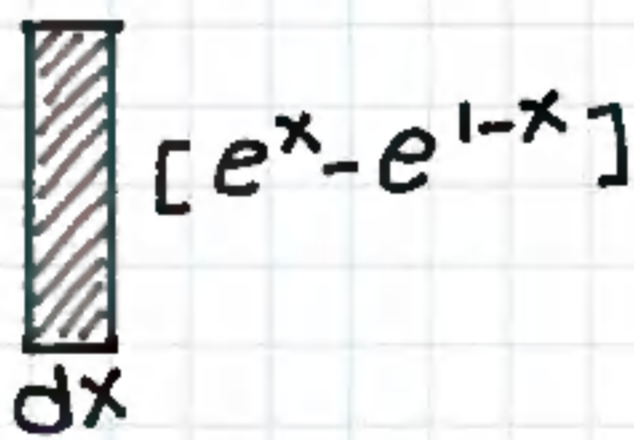
Step 3| Set up definite integral to find area and solve.

The points of intersection of $y=e^{1-x}$ and $y=e^{x}$ is $x=\frac{1}{2}$ $y=e^{1/2}$ on the interval $0<x<1$

Notice $e^{1-x}>e^{x}$ on $0<x<1/2$ and $e^{x}>e^{1-x}$ on $\frac{1}{2}<x<1$; therefore Area is:

$$A=\int_{0}^{1/2}[e^{1-x}-e^{x}]\,dx+\int_{1/2}^{1}[e^{x}-e^{1-x}]\,dx$$





$$A = \int_0^{1/2} [e^{1-x} - e^x]\,dx + \int_{1/2}^{1} [e^x - e^{1-x}]\,dx$$

Apply U-Subst. to find $\int e^{1-x}\,dx$

$u = 1-x$
$du = -dx$

$$\int e^{1-x}\,dx = -\int e^u\,du = -e^u + C = -e^{1-x} + C$$

$$\int e^{1-x}\,dx = -e^{1-x} + C$$

$$A = \left[-e^{1-x} - e^x\right]_0^{1/2} + \left[e^x - -e^{1-x}\right]_{1/2}^{1}$$

$$A = \left[-e^{1/2} - e^{1/2} - (-e^1 - e^0)\right] + \left[e^1 + e^0 - (e^{1/2} + e^{1/2})\right]$$

$$A = -2e^{1/2} + e + 1 + e + 1 - 2e^{1/2}$$

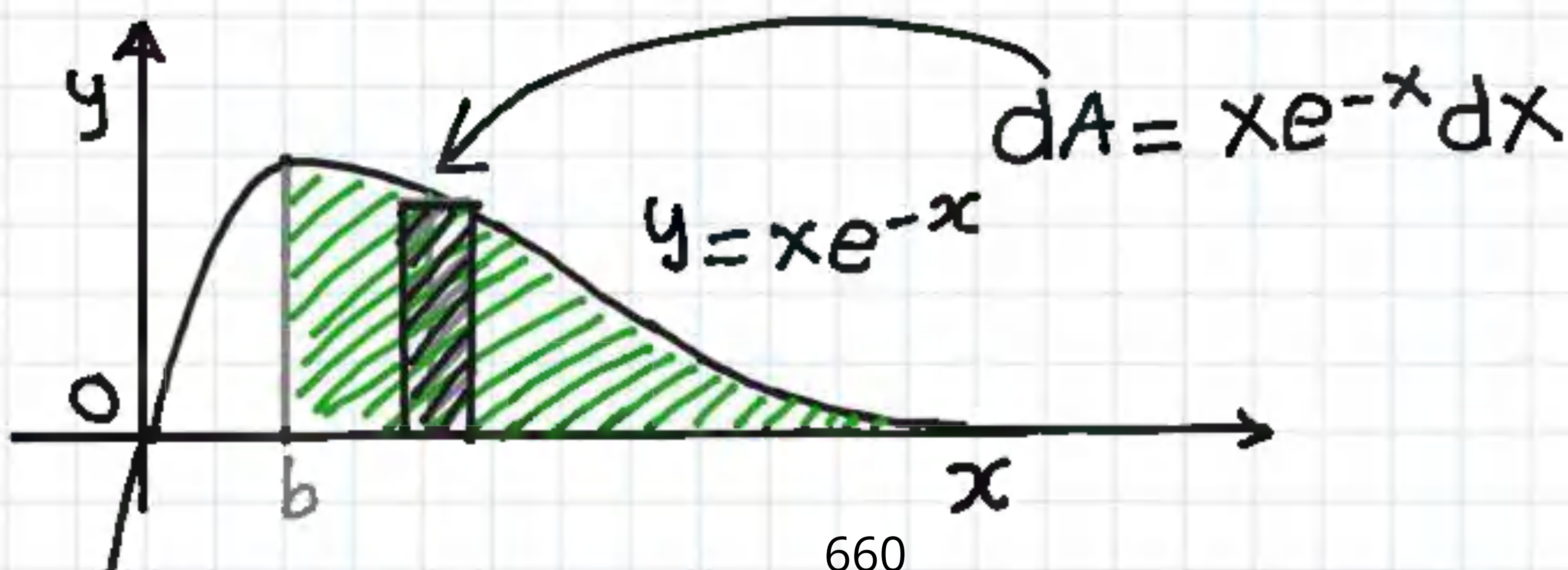
$$A = 2e + 2 - 4e^{1/2} \cong 0.842$$

Let R be the region bounded by y=xe^(-x) and y=0 on [b,infinity), find b such that area of the region is 2/e , solved example

Areas between curves II

Ex] Let R be the region bounded by the graph of $y = xe^{-x}$ and the x axis on the interval $[b, \infty)$ where b is a constant. Find the value of b such that the area of the region is $2/e$.

Solution: Step1] Sketch bounded region $y = xe^{-x}$, $y = 0$ and $x = b$



step2 | Set up definite integral and solve for b

$$A = \int_b^{\infty} xe^{-x}\,dx = \frac{2}{e}$$

Recall: This is an improper integral upper limit of integration approaches ∞

Strategy: Apply integration by parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

let $u = x$ $du = dx$ $dv = e^{-x}dx$ $v = -e^{-x}$

$$A = -xe^{-x}\Big|_b^{\infty} - \int_b^{\infty} -e^{-x}\,dx = \frac{2}{e}$$

$$A = -xe^{-x}\Big|_b^{\infty} + \int_b^{\infty} e^{-x}\,dx = \frac{2}{e}$$

$$A = \left(-xe^{-x} + -e^{-x}\right)\Big|_b^{\infty} = \frac{2}{e}$$

$$A = \lim_{p\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_b^{p} = \frac{2}{e}$$

$$A = \lim_{p\to\infty} -pe^{-p} - e^{-p} - \left(-be^{-b} - e^{-b}\right) = \frac{2}{e}$$

$$A = \lim_{p\to\infty} \frac{-p}{e^p} - \frac{1}{e^p} + be^{-b} + e^{-b} = \frac{2}{e}$$

Since $e^p >> p$ as $p \to \infty \Rightarrow \lim_{p\to\infty} \frac{p}{e^p} = 0$

Similiarly $\lim_{p\to\infty} \frac{1}{e^p} = 0$

$$A = \lim_{p \to \infty} \underbrace{\frac{-p}{e^p}}_{0} - \underbrace{\frac{1}{e^p}}_{0} + be^{-b} + e^{-b} = \frac{2}{e}$$

$$A = be^{-b} + e^{-b} = \frac{2}{b} \Rightarrow A = \frac{1}{e^b}(b+1) = \frac{2}{e}$$

We can't use algebra to solve above equation, so let's guess by inspection $\Rightarrow$ Lets Try $b = 1$

plug $b=1$ into $A = \frac{1}{e^b}(b+1) = \frac{2}{e}$

$$A = \frac{1}{e^1}(1+1) = \frac{2}{e}$$

Therefore $b=1$ such that bounded area is $\frac{2}{e}$

Areas between Curves 12

Ex| Find the area of the region bounded by the curves $y=\sqrt{4-x^2}$ and $y=x^2-4$

Solution: Step 1| Find the intersection points of $y=\sqrt{4-x^2}$ and $y=x^2-4$

$$y=y \Rightarrow \sqrt{4-x^2}=x^2-4 \Rightarrow \left(\sqrt{4-x^2}\right)^2=(x^2-4)^2$$

$$4-x^2=(x^2-4)^2 \Rightarrow (x^2-4)^2-(4-x^2)=0$$

$$\Rightarrow (x^2-4)^2+x^2-4=0 \Rightarrow (x^2-4)[(x^2-4)+1]=0$$

$$\Rightarrow (x^2-4)(x^2-3)=0 \Rightarrow x^2-4=0 \Rightarrow x^2=4$$

$$\sqrt{x^2}=\sqrt{4} \Rightarrow |x|=2 \Rightarrow x=\pm 2$$

$$x^2-3=0 \Rightarrow x=\pm\sqrt{3}$$

Let's also look at x intercepts:

$y=\sqrt{4-x^2}=0 \Rightarrow 4-x^2=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

$y=x^2-4=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

Notice we must reject $x=\pm\sqrt{3}$

Since $x=\sqrt{3} \Rightarrow y=\sqrt{4-x^2}=\sqrt{4-3}=\sqrt{1}=1$

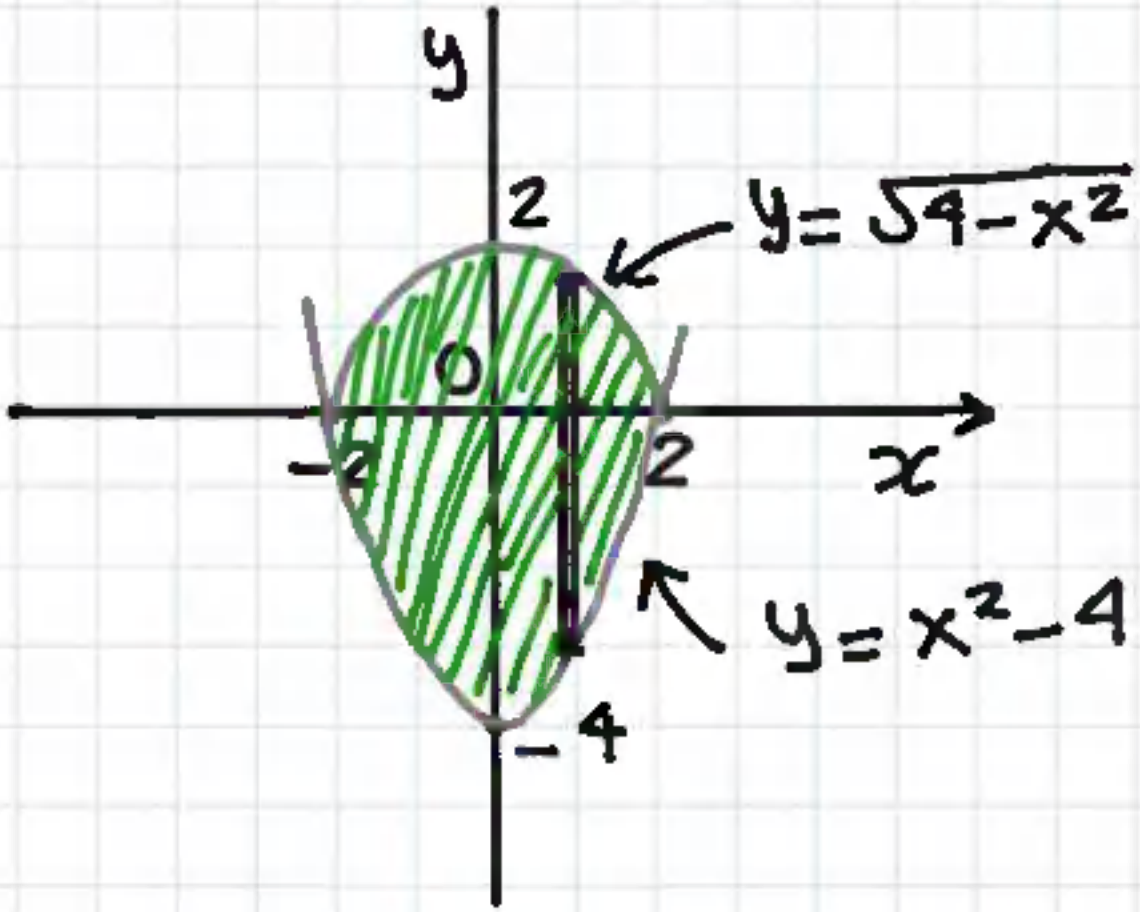
$x=\sqrt{3} \Rightarrow y=x^2-4=3-4=-1$

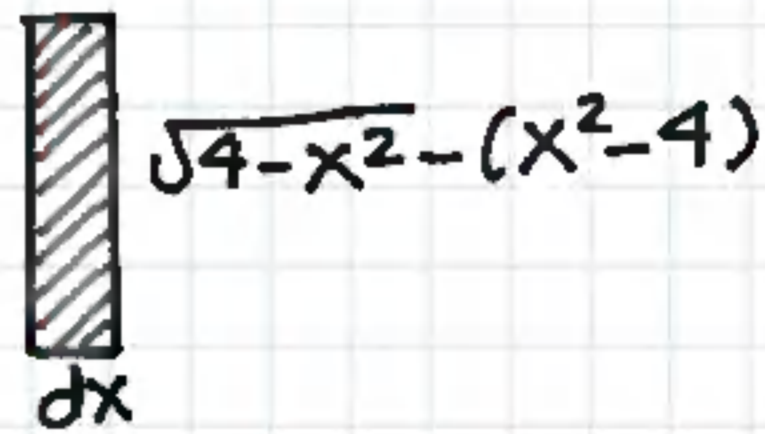
Not Equal

Similiarly $x=-\sqrt{3}$ is rejected

Therefore the points of intersection of $y=\sqrt{4-x^2}$ and $y=x^2-4$ are $(-2,0)$ and $(2,0)$

Step 2] Sketch bounded region $y=\sqrt{4-x^2}$, $y=x^2-4$





$$dA=(y_{top}-y_{bot})\,dx$$

$$dA=\left(\sqrt{4-x^2}-(x^2-4)\right)dx$$

Step 3| Set up definite integral to find area and solve

$$A = \int_{-2}^{2} \left[\sqrt{4-x^2} - (x^2-4) \right] dx$$

$$A = \int_{-2}^{2} \sqrt{4-x^2}\, dx - \int_{-2}^{2} (x^2-4)\, dx$$

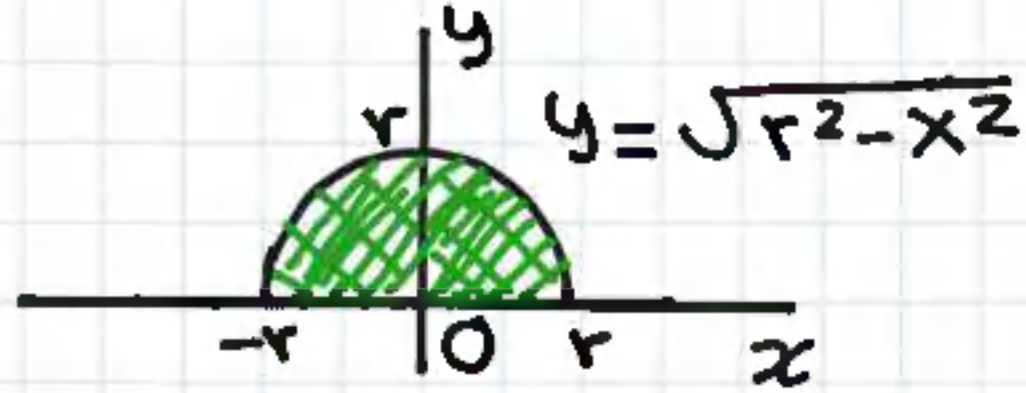
$$A = \frac{\pi(2)^2}{2} - \left[\frac{x^3}{3} - 4x \right]_{-2}^{2}$$

$$A = 2\pi - \left[\frac{8}{3} - 8 - \left(-\frac{8}{3} + 8 \right) \right] = 2\pi - \left[\frac{16}{3} - 16 \right]$$

$$A = 2\pi - \left[\frac{16-48}{3} \right] = 2\pi + \frac{32}{3} \cong 16.95$$

Integration Review

$$\int_{-r}^{r} \sqrt{r^2-x^2}\,dx = \frac{\pi r^2}{2}$$



$y=\sqrt{r^2-x^2}$ is top half of circle

$$y^2=\left(\sqrt{r^2-x^2}\right)^2 \Rightarrow y^2=r^2-x^2 \Rightarrow x^2+y^2=r^2$$

$x^2+y^2=r^2$ is the equation of a circle of radius r

Therefore $y=\sqrt{r^2-x^2}$ is the equation of top half of circle and hence $\displaystyle\int_{-r}^{r}\sqrt{r^2-x^2}\,dx=\frac{\pi r^2}{2}$

Find the area of the region bounded by the curves y= Lnx/x , y=1/x, x=1, x=4 solved example

Areas Between Curves 13

Ex] Find the area of the region bounded by the curves

$y=\frac{\ln x}{x}$, $y=\frac{1}{x}$, $x=1$, $x=4$

Solution: step1] Find the intersection points of

$y=\frac{\ln x}{x}$ and $y=\frac{1}{x}$

$y=y \Rightarrow \frac{\ln x}{x}=\frac{1}{x} \Rightarrow x\ln x = x \Rightarrow x\ln x - x = 0$

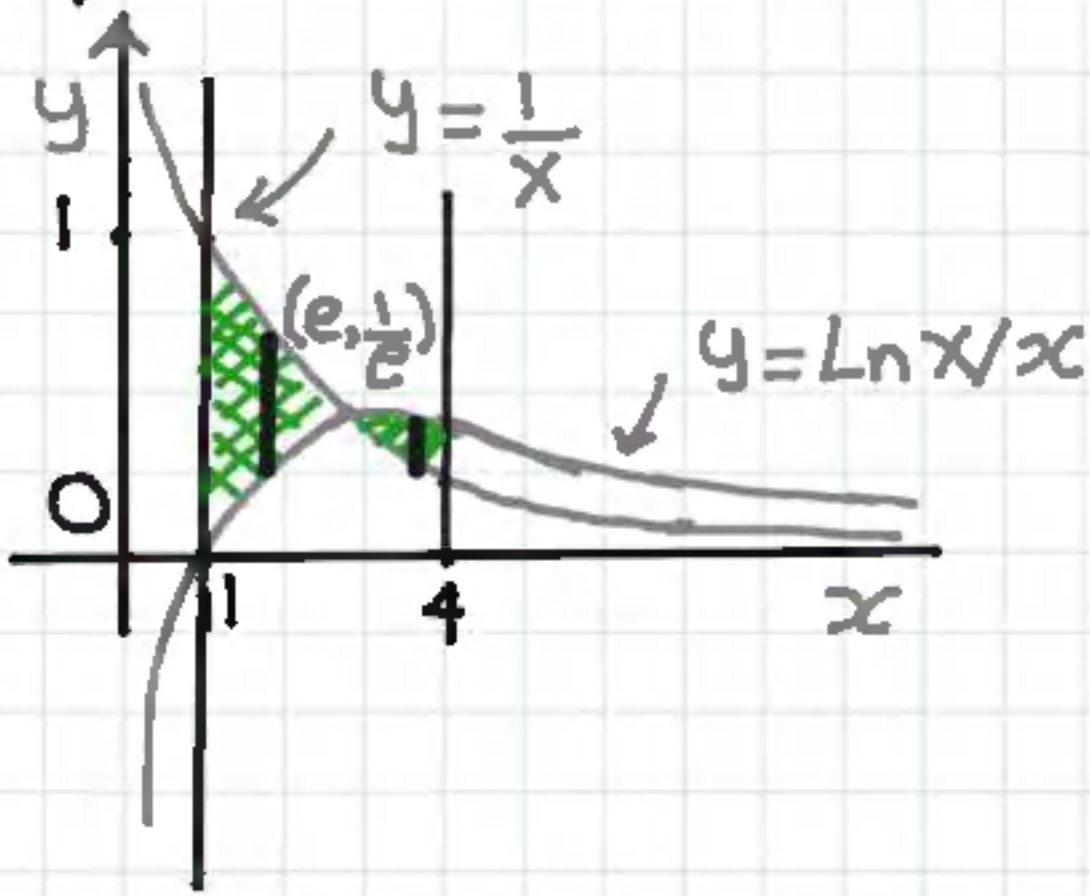
$x\ln x - x = 0 \Rightarrow x(\ln x - 1)=0 \Rightarrow x=0 \ , \ \ln x - 1 = 0$

$\ln x - 1 = 0 \Rightarrow \ln x = 1 \Rightarrow e^{\ln x}=e^1 \Rightarrow x=e \Rightarrow y=\frac{1}{x} \Rightarrow y=\frac{1}{e}$

Reject $x=0$ since outside domain of $y=\frac{\ln x}{x}$

Step 2] sketch bounded region $y = \frac{\ln x}{x}$, $y = \frac{1}{x}$

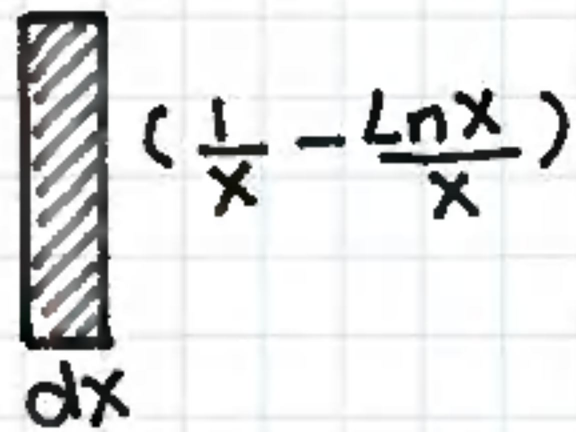
$x = 1$, $x = 4$



Notice when $1 < x < e$

$\frac{1}{x} > \frac{\ln x}{x}$, therefore

$$dA = \left(\frac{1}{x} - \frac{\ln x}{x}\right) dx$$



And when $e < x < 4$

$\frac{\ln x}{x} > \frac{1}{x}$, therefore

$$dA = \left(\frac{\ln x}{x} - \frac{1}{x}\right) dx$$

step 3] Set up definite integral to find area and solve.

The points of intersection of $y = \frac{\ln x}{x}$ and $y = \frac{1}{x}$ is $x = e$, $y = \frac{1}{e}$ on the interval $1 < x < 4$

Notice $\frac{1}{x} > \frac{\ln x}{x}$ on $1 < x < e$ and $\frac{\ln x}{x} > \frac{1}{x}$

On $e < x < 4$; therefore Area is:

$$A = \int_1^e \left(\frac{1}{x} - \frac{\ln x}{x} \right) dx + \int_e^4 \left(\frac{\ln x}{x} - \frac{1}{x} \right) dx$$

Let's first find $\int \frac{\ln x}{x} dx$ Apply U-Subst.

$u = \ln x \quad du = \frac{1}{x} dx \quad \Rightarrow \quad \int u\, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$

$$A = \int_1^e \left(\frac{1}{x} - \frac{\ln x}{x}\right) dx + \int_e^4 \left(\frac{\ln x}{x} - \frac{1}{x}\right) dx$$

$$A = \left[\ln x - \frac{(\ln x)^2}{2}\right]_1^e + \left[\frac{(\ln x)^2}{2} - \ln x\right]_e^4$$

$$A = \left[\ln e - \frac{(\ln e)^2}{2} - \left(\ln 1 - \frac{(\ln 1)^2}{2}\right)\right]$$

$$+ \left[\frac{(\ln 4)^2}{2} - \ln 4 - \left(\frac{(\ln e)^2}{2} - \ln e\right)\right]$$

$$A = \left[1 - \frac{1}{2} - \left(0 - \frac{0}{2}\right)\right] + \left[\frac{(\ln 4)^2}{2} - \ln 4 - \left(\frac{1}{2} - 1\right)\right]$$

$$A = \frac{1}{2} + \frac{(\ln 4)^2}{2} - \ln 4 + \frac{1}{2} = 1 + \frac{(\ln 4)^2}{2} - \ln 4 \cong 0.57$$

Review of Log properties

$\int \frac{1}{x} dx = Ln|x| + C$

Domain of $y = Ln x$ is $x > 0$

Since domain of $y = \frac{Ln x}{x}$ is $x > 0$ we dropped the absolute value $|x| = x$ when $x > 0$

$\int_1^e \frac{1}{x} dx = Ln|x| \Big|_1^e = Ln|e| - Ln|1| = \underbrace{Ln e}_{1} - \underbrace{Ln 1}_{0} = 1$

Recall: $Ln 1 = 0$, $Ln e = 1$

$e^{Ln x} = x$, $Ln e^x = x$

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