Average value | PDF 107

Average value of a function

Motivation: If we have 5 temperature readings on Jan.1/2020 in Vancouver recorded at various times of the day as follows:

1°C, 4°C, 7.5°C, 6.7°C, 4.5°C

we can calculate the average daily temperature as:

$$T_{avg.} = \frac{1+4+7.5+6.7+4.5}{5} = 4.74\,°C$$

Let $f(t)$ be the recorded temperature during a 24 hour cycle then $T_{avg} = \frac{1}{24}\sum_{i=1}^{24} f(t_i)$

$T_{avg} = \frac{1}{24} \sum_{i=1}^{24} f(t_i)$; instead of recording 24 hourly temperatures lets make n temperature recordings during the day to improve our estimate of the average daily temperature. (ie every minute, second, ...)

$T_{avg} = \frac{1}{n} \sum_{i=1}^{n} f(t_i)$ Looks like Riemann Sum

$\Delta t = \frac{b-a}{n}$; $n = \frac{b-a}{\Delta t}$; $\frac{1}{n} = \frac{\Delta t}{b-a}$

$T_{avg} = \frac{\Delta t}{b-a} \sum_{i=1}^{n} f(t_i) = \frac{1}{b-a} \sum_{i=1}^{n} f(t_i) \Delta t$

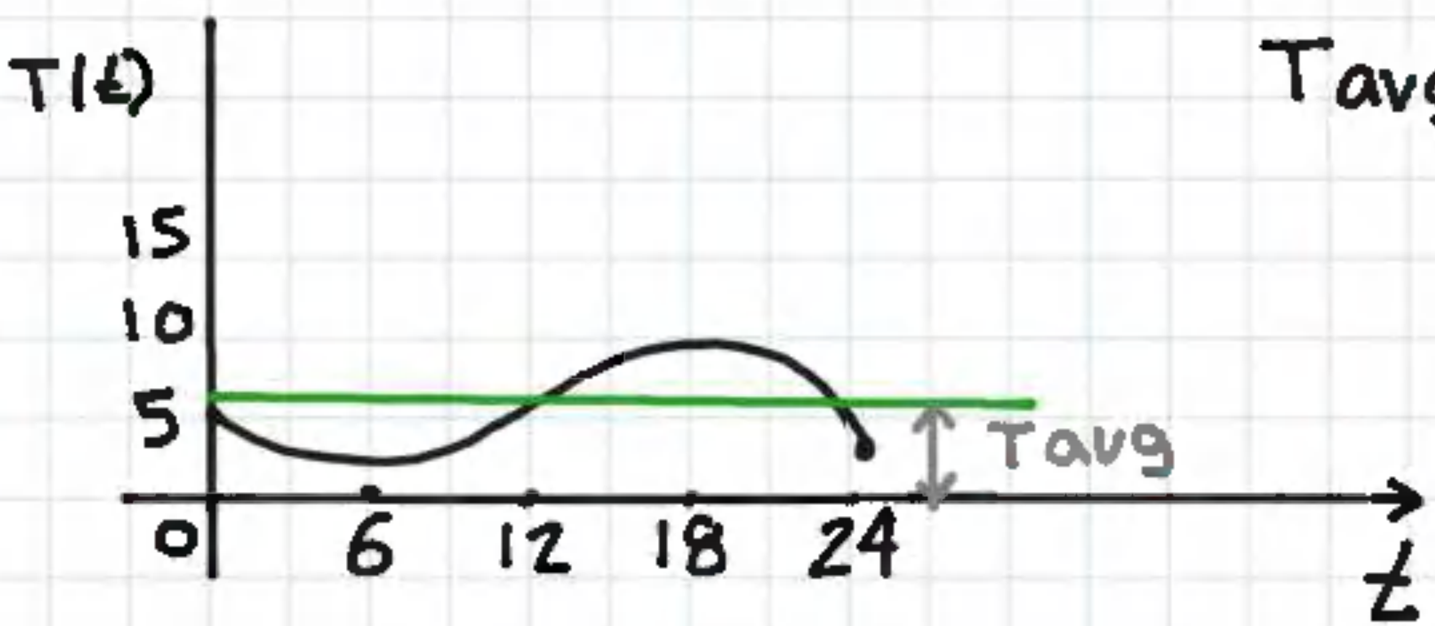
Let $n \to \infty$ $T_{avg} = \lim_{n \to \infty} \frac{1}{b-a} \sum_{i=1}^{n} f(t_i) \Delta t = \frac{1}{b-a} \int_a^b f(t)\, dt$

Therefore for a 24 hour cycle $T_{avg} = \frac{1}{24}\int_0^{24} f(t)\,dt$

In general $T_{avg} = \frac{1}{b-a}\int_a^b f(t)\,dt$

We can now define the average value of a function as:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



$$T_{avg} = \frac{1}{24}\int_0^{24} f(t)\,dt$$

Average value of a function geometric idea explained by a diagram

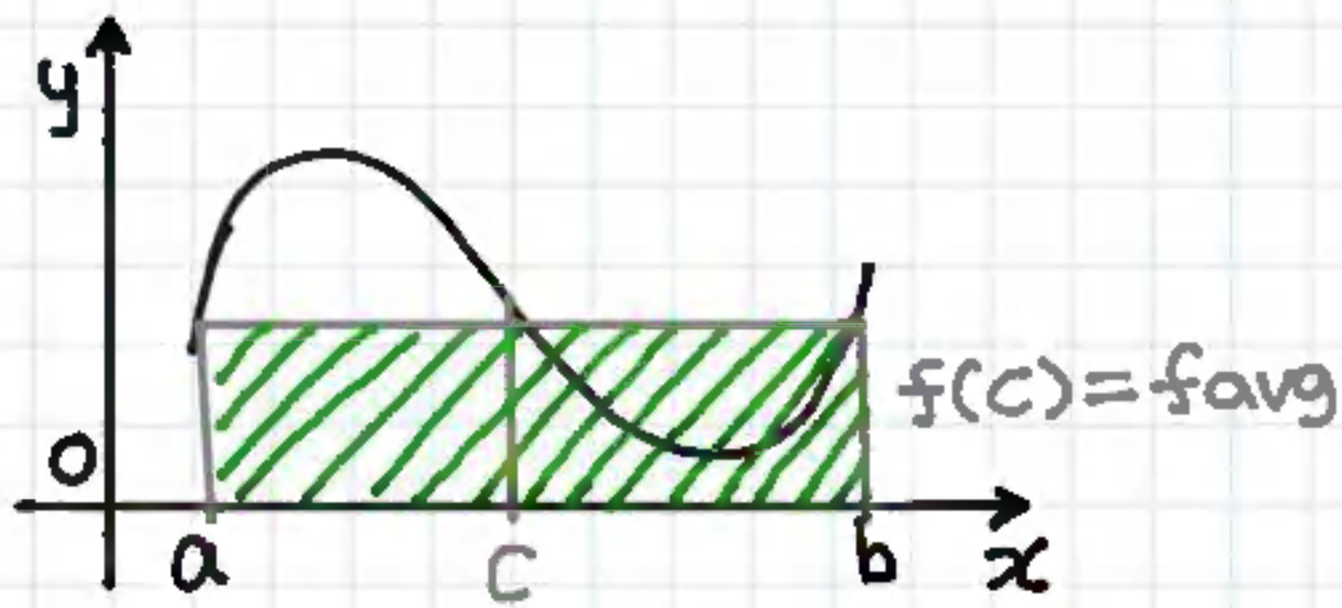
Average Value Theorem : If $f(x)$ is continuous on $[a,b]$, then there exists $x=c$ in $[a,b]$ such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or equivalently $f_{avg}(b-a) = \int_a^b f(x)\,dx$

Geometric Picture : Area of rectangle $f_{avg}(b-a)$ is exactly equal to area under graph of $f(x)$ from $[a,b]$

$$f(c)(b-a) = \int_a^b f(x)\,dx$$



Find the average value of f(x)=√x on [1,4] and find c such that favg=f(c) solved

Ex.| Given $f(x)=\sqrt{x}$ on $[1,4]$ find the following.

A) Find $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

B) Find $x=c$ such that $f_{avg}=f(c)$

C) Sketch $f(x)=\sqrt{x}$ and a rectangle whose area is exactly equal to the area under graph of $f(x)=\sqrt{x}$

A| Solution

$$f_{avg} = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\int_1^4 x^{1/2}\,dx = \frac{1}{3}\,x^{3/2}\cdot\frac{2}{3}\Big|_1^4$$

$$f_{avg} = \frac{2}{9}\,x^{3/2}\Big|_1^4 = \frac{2}{9}\,4^{3/2} - \frac{2}{9}\cdot 1^{3/2} = \frac{2}{9}\cdot 8 - \frac{2}{9} = \frac{14}{9}$$

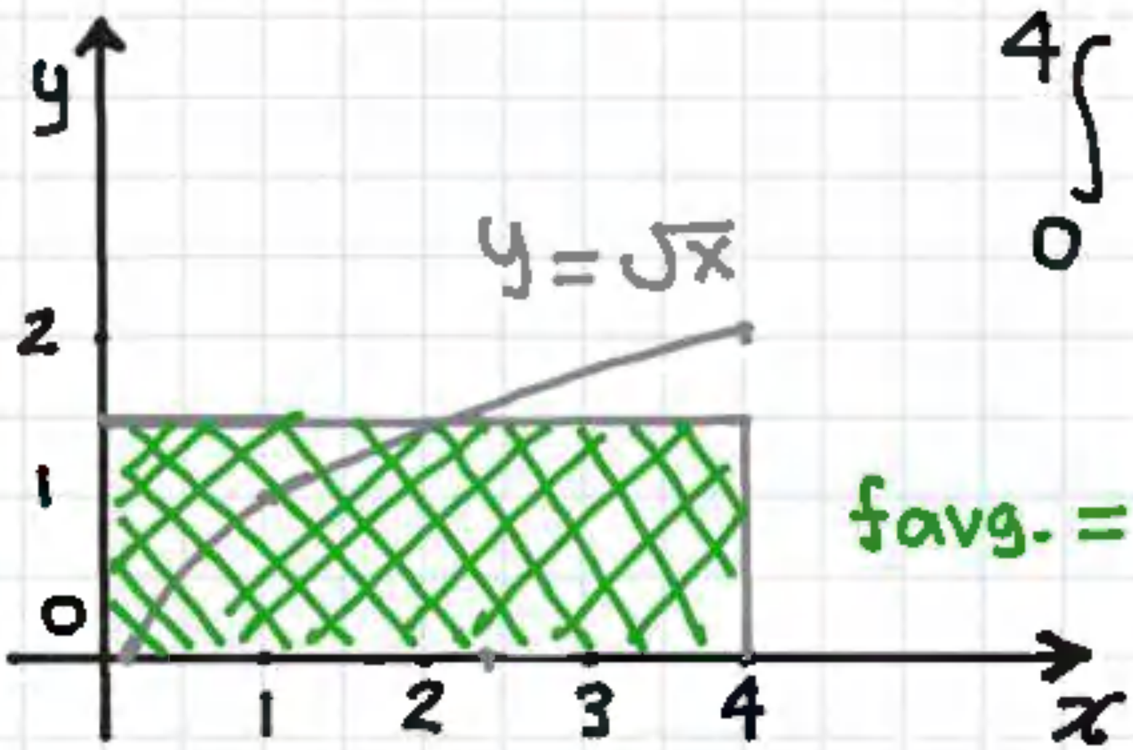
$$\therefore f_{avg} = \frac{14}{9} \cong 1.56$$

B] Find $x = c$ such that $f_{avg} = f(c)$

Solution: $f_{avg} = \frac{14}{9} \Rightarrow$ Set $f(c) = \frac{14}{9}$ and solve for c

$f(c) = \sqrt{c} = \frac{14}{9} \Rightarrow c = \left(\frac{14}{9}\right)^2 \cong 2.42 \Rightarrow x = c = 2.42$

C] Geometric Picture of Rectangle



$$\int_0^4 \sqrt{x}\,dx = f_{avg}(4-0) = \frac{14}{9} \times 4$$
$$\cong 6.\overline{222}$$

KEY CONCEPT

Area of rectangle is equal to area under $y = \sqrt{x}$

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Average Value 2

Ex| Find the values of x on the interval $[0,2]$ such that the function $f(x) = x(2-x)$ has the same y value as the average value of the function.

Solution: We apply $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$f_{avg} : \frac{1}{2-0}\int_0^2 x(2-x)\,dx$$

$$f_{avg} = \frac{1}{2}\int_0^2 (2x-x^2)\,dx = \frac{1}{2}\left[x^2-\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left[2^2-\frac{2^3}{3}\right]-\frac{1}{2}[0-0] = \frac{1}{2}\left[\frac{4}{1}-\frac{8}{3}\right] = \frac{1}{2}\left[\frac{12-8}{3}\right] = \frac{2}{3}$$

$$f_{avg} = 2/3$$

Now we have to find the values of $x = c$ such that

$f_{avg} = f(c)$

We found $f_{avg} = 2/3$ and $f(c) = c(2-c)$

Equating $f_{avg} = f(c)$ we obtain $\frac{2}{3} = c(2-c)$

$\frac{2}{3} = 2c - c^2 \Rightarrow c^2 - 2c + 2/3 = 0$

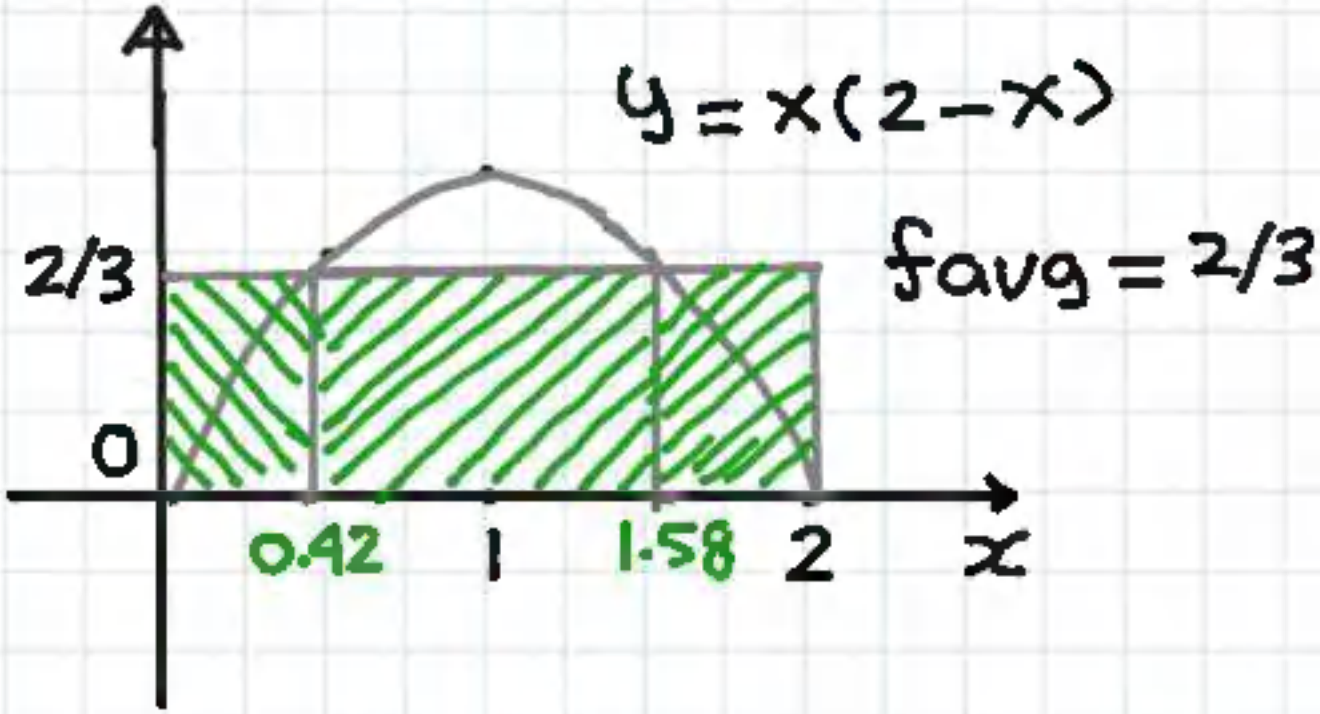
Apply quadratic formula to find c

$$c = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2/3)}}{2} = \frac{2 \pm \sqrt{4 - 8/3}}{2}$$

Solving for c we obtain $c \simeq 0.42$, $c \simeq 1.58$

For these two x values $f(c) = f_{avg}$

ie) $f(0.42) = 2/3$ and $f(1.58) = 2/3$



Key concept: Area of Rectangle is equal to the area bounded by $y = x(2-x)$ and x axis.

$$f_{avg}(b-a) = \int_a^b f(x)\,dx$$

$$\frac{2}{3}(2-0) = \int_0^2 x(2-x)\,dx$$

Average value of function word problem finding the average temperature

Ex| On March 7/2020 in Vancouver, Canada the temperature (in °C) t hours after midnight can be modeled by the function:

$T(t) = 4.5 - 2.5\cos(\pi t/12)$

Find the average temperature during the time interval 12:00 am to 12:00 pm

Solution: 12:00 am corresponds to $t=0$

12:00 pm corresponds to $t=12$

$$T_{avg} = \frac{1}{12-0}\int_0^{12} \left(4.5 - 2.5\cos(\pi t/12)\right)dt$$

$$T_{avg} = \frac{1}{12}\int_0^{12} (4.5 - 2.5\cos(\pi t/12))\,dt$$

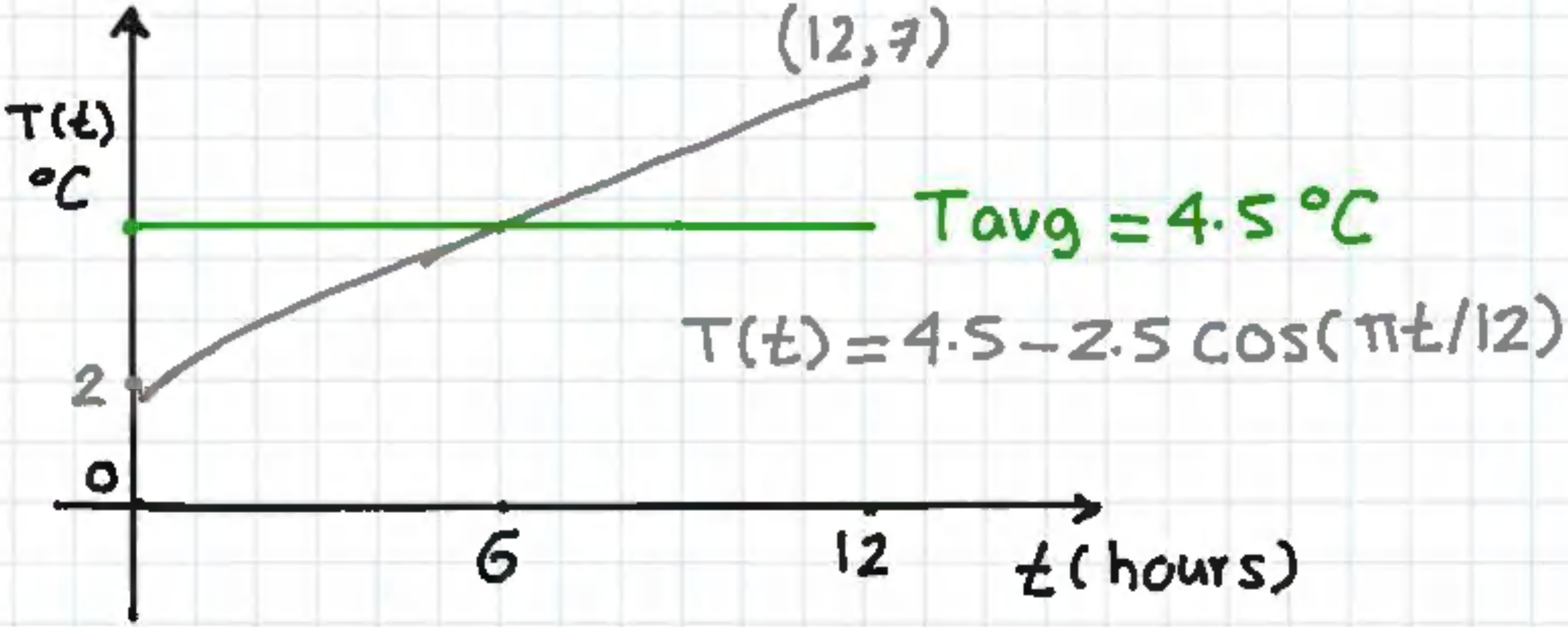
$$T_{avg} = \frac{1}{12}\left[4.5t - 2.5\,\frac{\sin(\pi t/12)}{\pi/12}\right]\Big|_0^{12}$$

$$T_{avg} = \frac{1}{12}\left[\frac{9}{2}t - \frac{5}{2}\cdot\frac{12}{\pi}\sin(\pi t/12)\right]\Big|_0^{12}$$

$$T_{avg} = \frac{1}{12}\left[\frac{9}{2}(12) - \frac{30}{\pi}\sin\left(\frac{\cancel{12}\pi}{\cancel{12}}\right)\right] - \frac{1}{12}\left[0 - \frac{30}{\pi}\sin 0\right]$$

$$T_{avg} = \frac{1}{12}[54 - 0] - \frac{1}{12}[0 - 0] = \frac{54}{12} = 4.5\,°C$$

Therefore the average temperature from 12:00 am to 12:00 pm is 4.5 °C.



Key Concept: We have shown that the average temperature in Vancouver, Canada on March 7/2020 is 4.5 °C according to the temperature model.

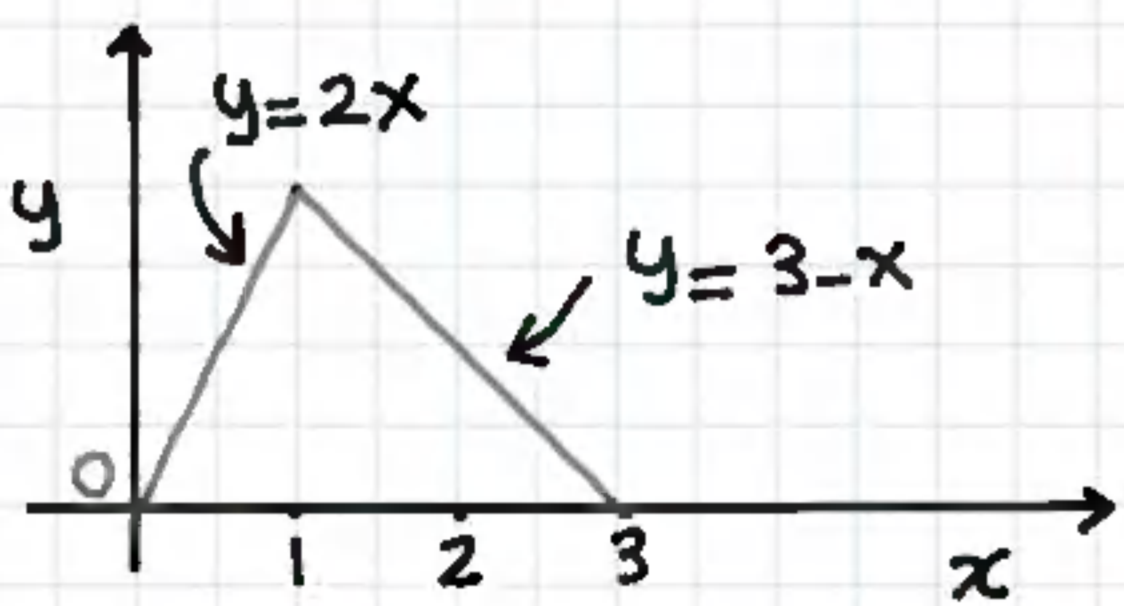
Average value of a piecewise continuous function solved example

Average Value 3

Ex| Let $f(x)$ be defined as a piecewise continuous function $f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 3-x & 1 \leq x \leq 3 \end{cases}$

Find the average value of the function $f(x)$ on the interval $[0,3]$

Solution: Lets start by sketching $f(x)$



$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{avg} = \frac{1}{3-0}\left[\int_0^1 2x\,dx + \int_1^3 (3-x)\,dx\right]$$

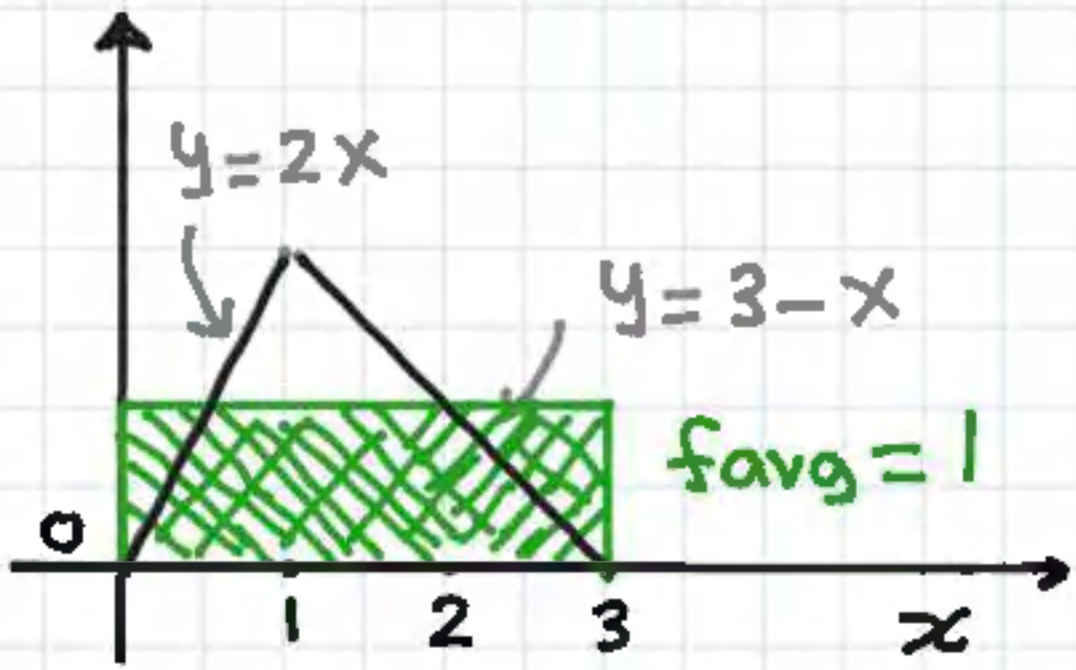
$$f_{avg} = \frac{1}{3}\left[\frac{\cancel{2}x^2}{\cancel{2}}\Big|_0^1\right] + \frac{1}{3}\left[3x - \frac{x^2}{2}\right]\Big|_1^3$$

$$f_{avg} = \frac{1}{3}[1-0] + \frac{1}{3}\left[9 - \frac{9}{2} - \left(3 - \frac{1}{2}\right)\right]$$

$$f_{avg} = \frac{1}{3} + \frac{1}{3}\left[6 - \frac{9}{2} + \frac{1}{2}\right] = \frac{1}{3} + \frac{6}{3} - \frac{9}{6} + \frac{1}{6} = \frac{2+12-9+1}{6}$$

$$f_{avg} = 1$$

Lets illustrate with a diagram



$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 3-x & 1 \le x \le 3 \end{cases}$$

Key Concept: Area of Shaded Rectangle is equal to the area under f(x) and above x axis.

Ex Find the value of b such that the average value of $f(x) = 3x^2 + 2x + 1$ on the interval $[0, b]$ is equal to $f_{avg} = 7$

Solution: $f_{avg} = \frac{1}{b-0}\int_0^b (3x^2 + 2x + 1)\,dx = 7$

$$f_{avg} = \frac{1}{b}\left[\frac{\cancel{3}x^3}{\cancel{3}} + \frac{\cancel{2}x^2}{\cancel{2}} + x\right]\Big|_0^b = 7$$

$$f_{avg} = \frac{1}{b}\left[b^3 + b^2 + b\right] - \frac{1}{b}\left[0 + 0 + 0\right] = 7$$

$$f_{avg} = b^2 + b + 1 = 7 \Rightarrow b^2 + b - 6 = 0 \Rightarrow (b+3)(b-2) = 0$$

$b + 3 = 0 \Rightarrow b = -3$ Reject since $b = -3 < 0$

$b - 2 = 0 \Rightarrow b = 2$ Accept $[0, b] = [0, 2]$

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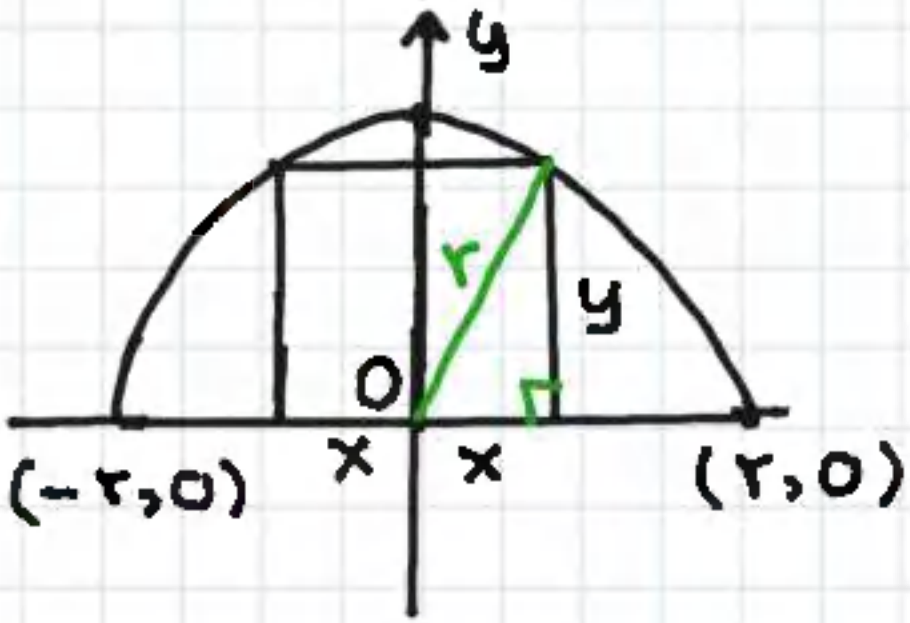
Find the average area of an inscribed rectangle within a semicircle of radius r with one base on the x axis solved example.

Average Value of Function 4

Ex| Find the average area of an inscribed rectangle within a semicircle of radius r with one base on the x axis. See Diagram below.



Solution:

$Area = 2x \cdot y$ Area of rectangle

$x^2 + y^2 = r^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$

$y = \sqrt{r^2 - x^2}$ semi-circle

$A = 2x\sqrt{r^2 - x^2}$ $0 \leq x \leq r$

Now find average value of the Area function

$$A_{avg} = \frac{\int_0^r A(x)\,dx}{r-0} = \frac{\int_0^r 2x\sqrt{r^2-x^2}\,dx}{r}$$

$$A_{avg} = \frac{1}{r}\int_0^r 2x\sqrt{r^2-x^2}\,dx$$

Apply U-Substitution

$U = r^2 - x^2 \qquad du = 0 - 2x\,dx$

Change limits of integration: X limits to U limits

$x = 0 \qquad U = r^2 - x^2 \qquad U = r^2$

$x = r \qquad U = r^2 - x^2 \qquad U = 0$

$$A_{avg} = \frac{1}{r}\int_0^r 2x\sqrt{r^2-x^2}\,dx$$

Recall: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$$A_{avg} = \frac{-1}{r}\int_{r^2}^{0} \sqrt{u}\,du = \frac{1}{r}\int_0^{r^2} u^{1/2}\,du = \frac{1}{r}\, u^{3/2}\cdot\frac{2}{3}\Big|_0^{r^2}$$

$$A_{avg} = \frac{2}{3r}\, u^{3/2}\Big|_0^{r^2} = \frac{2}{3r}\left[(r^2)^{3/2} - 0\right]$$

$$A_{avg} = \frac{2}{3r}(r^{\cancel{2}})^{3/\cancel{2}} = \frac{2}{3r}\cdot r^3 = \frac{2}{3}r^2$$

Therefore the average Area of an inscribed rectangle within a semicircle of radius r is

$$A_{avg} = \frac{2}{3}r^2$$

Compare to Area of semi-circle

$$A = \frac{\pi r^2}{2}$$