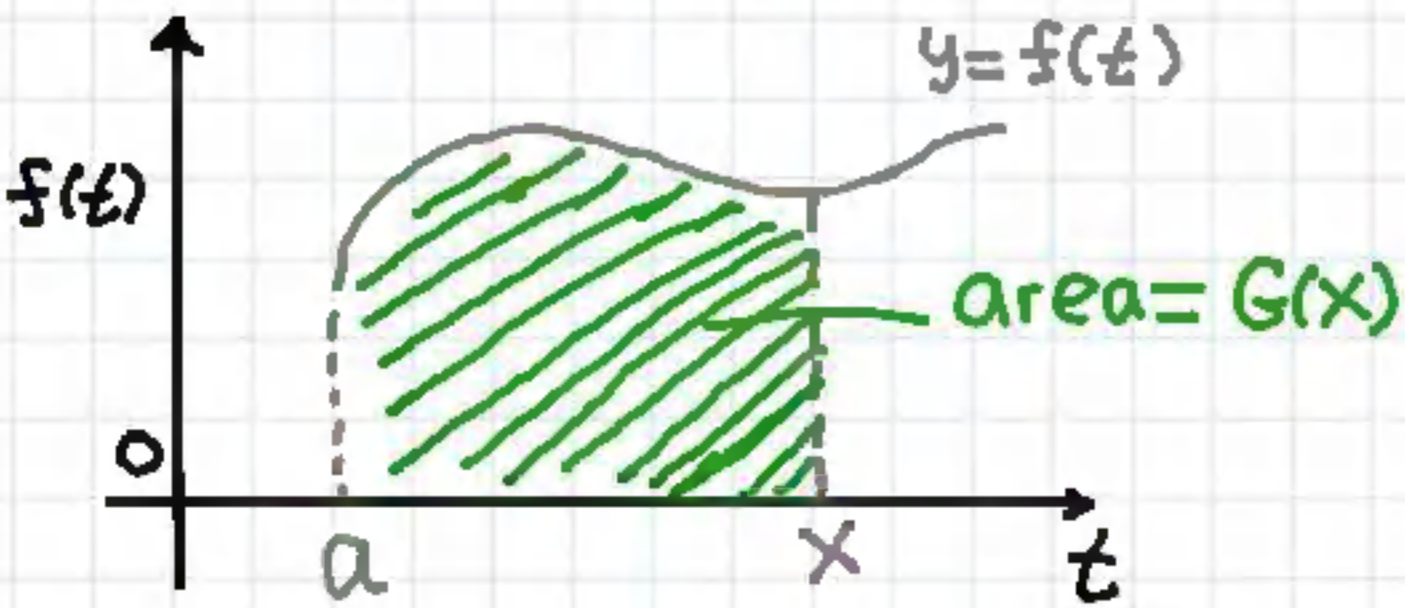
Fundamental Theorem of Calculus 1 (Theory + Examples)

Theorem: Assume $f(t)$ is continuous on $[a, b]$

Part I: If the function $G(x)$ is defined by

$G(x) = \int_a^x f(t)\,dt$ for all x in $[a,b]$, then $G(x)$ is continuous on $[a,b]$ and differentiable on (a,b).

$$G'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



Note: If $f(t) > 0$ then

$G(x) = \int_a^x f(t)\,dt = \text{Area}$

$G(a) = \int_a^a f(t)\,dt = 0$

$G(b) = \int_a^b f(t)\,dt = \text{Area}$

EX| Given $G(x)=\int_2^x te^{-t}\,dt$; Find $G'(x)$?

Solution: Apply F.T.C Part I

$$G'(x)=\frac{d}{dx}\int_2^x te^{-t}\,dt = xe^{-x}$$

Part II Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$$

Example: Evaluate $\int_0^1 [ax^2+bx+C]\,dx$

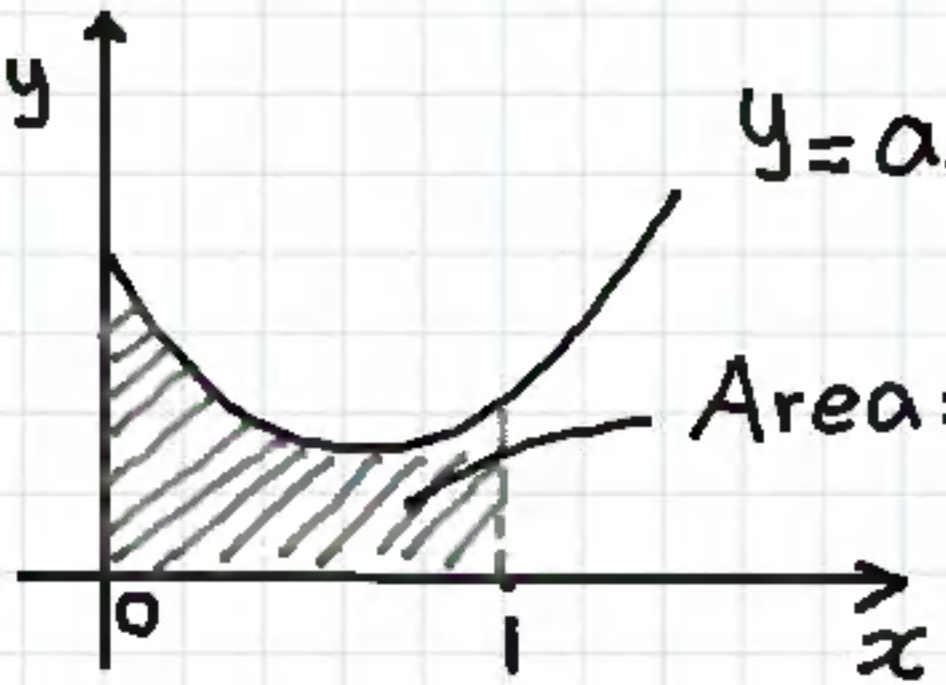
Solution: $f(x)=ax^2+bx+C$ is a polynomial function and is continuous on $(-\infty,\infty)$ therefore it is continuous on $[0,1]$

$$\int_0^1 [ax^2+bx+C]\,dx = \frac{ax^3}{3}+\frac{bx^2}{2}+Cx\Big|_0^1$$

$$= \frac{a}{3}(1)^3+\frac{b}{2}(1)^2+C(1)-\left[\frac{a}{3}(0)^3+\frac{b}{2}(0)^2+C(0)\right]$$

$$= \boxed{\frac{a}{3}+\frac{b}{2}+C}$$

Assume quadratic polynomial $y = ax^2 + bx + C$ is above x axis for $0 \leq x \leq 1$



$y = ax^2 + bx + C$

$\text{Area} = \frac{a}{3} + \frac{b}{2} + C$

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Fundamental Theorem of Calculus 2 (Theory + Examples)

Motivation: Differentiation and Integration are inverse processes. F.T.C gives the relationship between these two processes in terms of a formula.

Theorem: Let $f(t)$ be continuous on $[a, b]$, If $a \leq c \leq b$ then $\frac{d}{dx} \int_c^x f(t)\,dt = f(x)$

Ex. Find the derivative of $g(x) = \int_x^1 \sqrt[3]{1+t}\,dt$

Solution: $f(t) = \sqrt[3]{1+t}$ is continuous on $(-\infty, \infty)$ so we can apply F.T.C

$$g(x) = \int_x^1 \sqrt[3]{1+t}\ dt$$

Let's interchange limits of Integration so upper Limit is x

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$$

$$g(x) = -\int_1^x \sqrt[3]{1+t}\ dt$$

$$g'(x) = -\frac{d}{dx}\int_1^x \sqrt[3]{1+t}\ dt = -\sqrt[3]{1+x}$$

Theorem 1 Extension

Let $f(t)$ be continuous on $[a,b]$, If $a \leq C \leq b$

then $\frac{d}{dx}\int_C^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$

Proof using Chain Rule

Let $y = \frac{d}{dx}\int_c^{g(x)} f(t)\,dt$

$$\frac{dy}{dx} = \frac{d}{du}\left[\int_c^{u} f(t)\,dt\right]\frac{du}{dx}$$

$$\frac{dy}{dx} = f(u)\,\frac{du}{dx}$$

$$\frac{dy}{dx} = f(g(x))\cdot g'(x)$$

Apply Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$u = g(x)$$

$$\frac{du}{dx} = g'(x)$$

$$\frac{d}{dx}\int_{c}^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$ F.T.C + Chain Rule

Ex Find the derivative of $h(x)=\int_{1}^{x^4}\sqrt{1+t^2}\,dt$

$\sqrt{1+t^2}$ is continuous on $(-\infty,\infty)$ so apply F.T.C

Notice that the upper limit x^4 is a function of x

we must apply chain Rule + F.T.C

$$h'(x)=\frac{d}{dx}\int_{1}^{x^4}\sqrt{1+t^2}\,dt=\sqrt{1+(x^4)^2}\cdot 4x^3$$

$$\boxed{h'(x)=\sqrt{1+x^8}\cdot 4x^3}$$

F.T.C Review

Assume $f(t)$ is continuous on $[a,b]$

1] $$\int_a^b f(t)\,dt = F(t)\Big|_a^b = F(b) - F(a)$$

2] $$\frac{d}{dx}\int_c^x f(t)\,dt = f(x)$$

3] $$\frac{d}{dx}\int_c^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$$

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Fundamental Theorem of Calculus 3

Ex) Find the derivative of $f(x)$

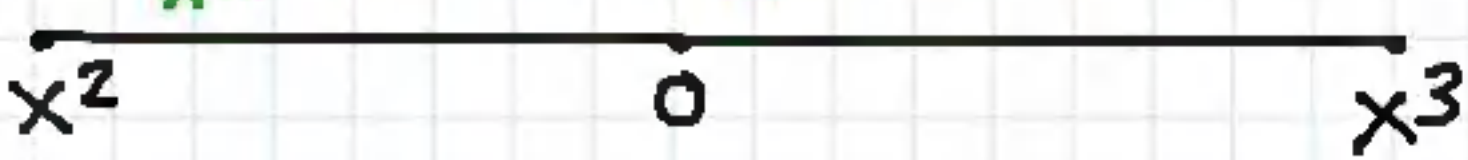
$$f(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt$$

Solution: Apply F.T.C + Chain Rule

$$\frac{d}{dx}\int_{c}^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

let's split up integral so we can apply above rule!

$$\int_{x^2}^{0} e^{-t^2}\,dt + \int_{0}^{x^3} e^{-t^2}\,dt$$



$$\int_{x^2}^{x^3} e^{-t^2}\,dt = \int_{x^2}^{0} e^{-t^2}\,dt + \int_{0}^{x^3} e^{-t^2}\,dt$$

$$\int_{x^2}^{x^3} e^{-t^2}\,dt = -\int_{0}^{x^2} e^{-t^2}\,dt + \int_{0}^{x^3} e^{-t^2}\,dt$$

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}\,dt = -\frac{d}{dx}\int_{0}^{x^2} e^{-t^2}\,dt + \frac{d}{dx}\int_{0}^{x^3} e^{-t^2}\,dt$$

$$= -e^{-(x^2)^2}\cdot(x^2)' + e^{-(x^3)^2}\cdot(x^3)'$$

$$= -e^{-x^4}\cdot 2x + e^{-x^6}\cdot 3x^2$$

$$= \boxed{-2x\,e^{-x^4} + 3x^2 e^{-x^6}}$$

Fundamental theorem of Calculus example with product rule.

Ex Given $F(x) = \int_0^x x \cdot t^3 \sin t \, dt$ Find $F'(x)$

$$F(x) = \int_0^x x \cdot t^3 \sin t \, dt = x \int_0^x t^3 \sin t \, dt$$

$$F'(x) = \frac{d}{dx}[x] \int_0^x t^3 \sin t \, dt + x \cdot \frac{d}{dx} \int_0^x t^3 \sin t \, dt$$ Product Rule

$$F'(x) = 1 \int_0^x t^3 \sin t \, dt + x \cdot x^3 \sin x$$ F.T.C

$$F'(x) = \int_0^x t^3 \sin t \, dt + x^4 \sin x$$ simplify

Theory

$$\frac{d}{dx}[u \cdot v] = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$ Product Rule

$$\frac{d}{dx} \int_c^x f(t) \, dt = f(x)$$ F.T.C

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Fundamental theorem of Calculus (Derivative of an Integral example)

Fundamental Theorem of Calculus 4

Ex] Determine values of X for which $\int_0^{x^2-4x} e^{-t^2}\,dt$ is increasing?

Solution: Let $F(x) = \int_0^{x^2-4x} e^{-t^2}\,dt$

Recall F.T.C $\frac{d}{dx}\int_c^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$

Strategy: To find intervals of increase and decrease for $F(x)$ we must set $F'(x)=0$ to find critical points and apply First Derivative test to find values of X for which $F(x)$ increases or decreases.

Since $F(x) = \int_0^{x^2-4x} e^{-t^2}\,dt$

$$F'(x) = \frac{d}{dx}\int_0^{x^2-4x} e^{-t^2}\,dt$$

APPLY F.T.C + Chain Rule

$$F'(x) = e^{-(x^2-4x)^2} \cdot \frac{d}{dx}(x^2-4x)$$

$$F'(x) = e^{-(x^2-4x)^2} \cdot [2x-4]$$

$$0 = e^{-(x^2-4x)^2} \cdot [2x-4]$$

Set $F'(x) = 0$ to find critical points of $F(x)$

since $e^{-(x^2-4x)^2} > 0$ for all x

Set $2x-4=0 \Rightarrow \boxed{x=2}$ only critical point

Now Let's find regions of increase and decrease for $F(x)$ by Applying First Derivative test.

$$F'(x) = e^{-(x^2-4x)^2}[2x-4]$$

$x = 2$ critical point of $F(x)$

First Derivative test

$- - - - - \quad + + + + + + + +$

$x < 2 \qquad x = 2 \qquad x > 2$

For $x < 2$ say $x = 1$

$$F'(1) = e^{-(-3)^2}[2-4] = e^{-9}[-2] = -2/e^9 < 0$$

$F'(x) < 0$ hence $F(x) \downarrow$

For $x > 2$ say $x = 4$

$$F'(4) = e^{-(0)^2} \cdot [4] = 4 > 0$$

$F'(x) > 0$ hence $F(x) \uparrow$

Therefore $F(x)=\int_0^{x^2-4x} e^{-t^2}\,dt$ is increasing

on $(2,\infty)$

Key Concept when $x>2$ $F'(x)>0$

therefore $F(x)$ is increasing.

Review

1| Find $F'(x)$ by applying F.T.C

2| Set $F'(x)=0$ to find critical x value

3| Find where $F(x)$ is increasing by applying first Derivative test.

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Fundamental Theorem of Calculus theoretical example pdf

Fundamental Theorem of Calculus 5

Ex| Given $f(1) = 10$, $f'(x)$ is continuous, and

$\int_5^1 f'(x)\,dx = 3$, Find the value of $f(5)$?

Solution: $\int_a^b f'(t)\,dt = f(t)\Big|_a^b = f(b) - f(a)$

F.T.C

Given $\int_5^1 f'(x)\,dx = 3 \Rightarrow -\int_1^5 f'(x)\,dx = 3$

$\int_1^5 f'(x)\,dx = -3$ $\quad \int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$

$\int_1^5 f'(x)\,dx = f(x)\Big|_1^5 = f(5) - f(1)$

$-3 = f(5) - f(1) \Rightarrow -3 = f(5) - 10 \Rightarrow \boxed{f(5) = 7}$

Ex] Determine a function $f(x)$ and a constant C such that $100 + \int_C^x t^3 f(t)\,dt = x^{2/3}$

Solution: $\frac{d}{dx}\left[100 + \int_C^x t^3 f(t)\,dt\right] = \frac{d}{dx}\left[x^{2/3}\right]$

$$0 + \frac{d}{dx}\int_C^x t^3 f(t)\,dt = \frac{2}{3}x^{-1/3}$$

Diff. both sides + F.T.C

$$x^3 f(x) = \frac{2}{3x^{1/3}} \Rightarrow \boxed{f(x) = \frac{2}{3x^{1/3}x^3} = \frac{2}{3x^{10/3}}}$$

Now Find C

Strategy: plug in $x = C$ into $100 + \int_C^x t^3 f(t)\,dt = x^{2/3}$ so that integral vanishes and solve for C.

$$100 + \int_C^x t^3 f(t)\,dt = x^{2/3}$$

Subst. $x = C$

$$100 + \int_C^C t^3 f(t)\,dt = C^{2/3}$$

$$\int_a^a g(t)\,dt = 0$$

$$100 + 0 = C^{2/3}$$

$$C^{2/3} = 100 \Rightarrow [C^{1/3}]^2 = 100$$

$$\left[[C^{1/3}]^{\cancel{2}}\right]^{1/\cancel{2}} = [100]^{1/2}$$

$$C^{1/3} = \sqrt{100} \Rightarrow C^{1/3} = 10 \Rightarrow (C^{1/\cancel{3}})^{\cancel{3}} = (10)^3$$

$$\boxed{C = 1000 \qquad f(x) = \frac{2}{3x^{10/3}}}$$

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Fundamental Theorem of Calculus 6

a) Show that $\sin(x^2) \leq \sin x$ for $0 \leq x \leq 1$

b) Prove that $\int_0^{\pi/6} \sin(x^2)\, dx \leq 1 - \frac{\sqrt{3}}{2}$

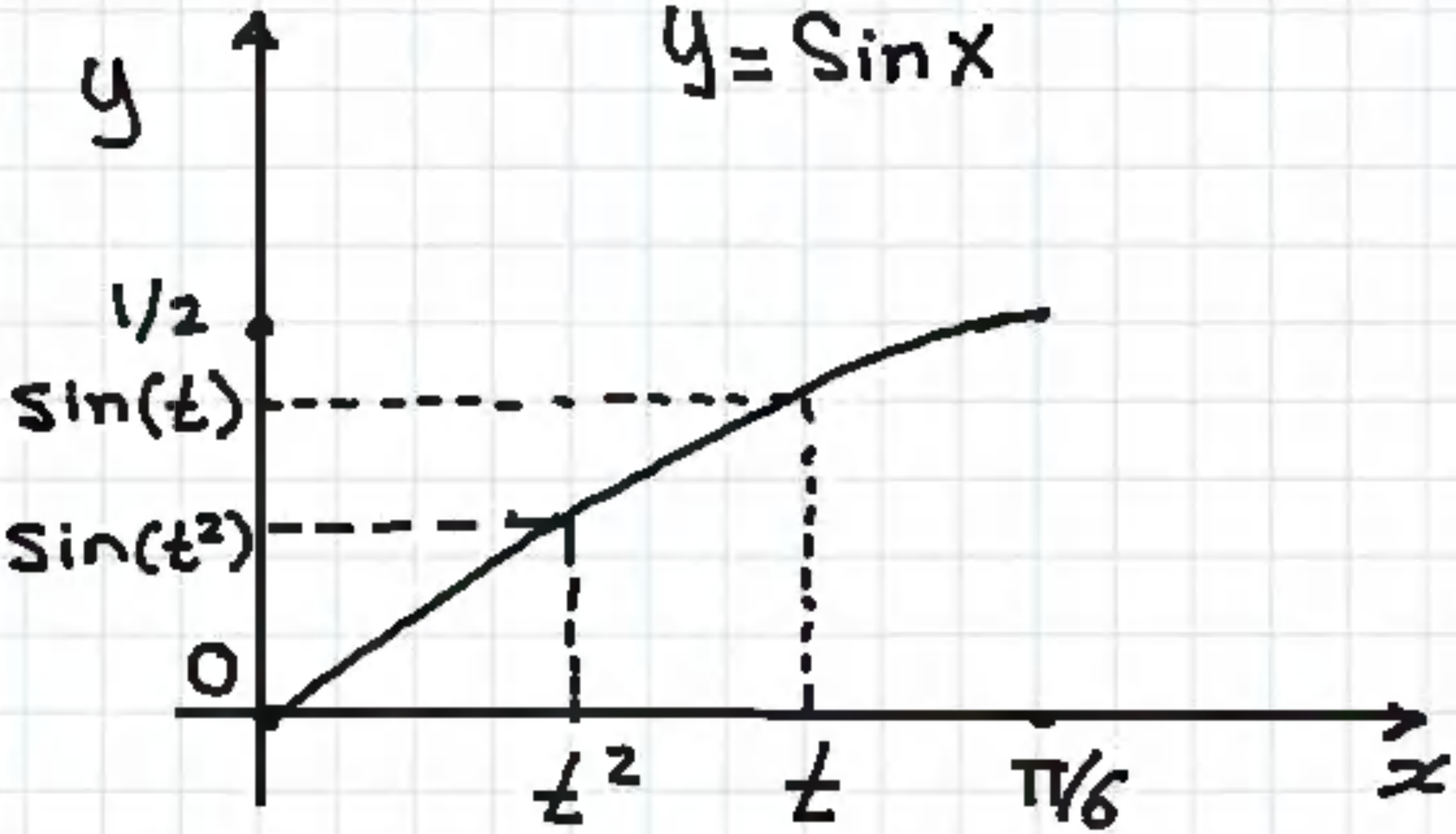
Solution:

a) For $0 \leq x \leq \pi/6$, $x^2 \leq x$ since $0 \leq x \leq \pi/6 \leq 1$

Note: $\sin x$ is an increasing function on $[0,1]$

$\sin(x^2) \leq \sin x$ in $[0,1]$

Let's illustrate this inequality with a diagram.



$0 \leq t \leq 1 \Rightarrow$ $t^2 \leq t$ and since $\sin x$ is increasing on $[0,1]$

$\sin(t^2) \leq \sin t$ on $[0,1]$

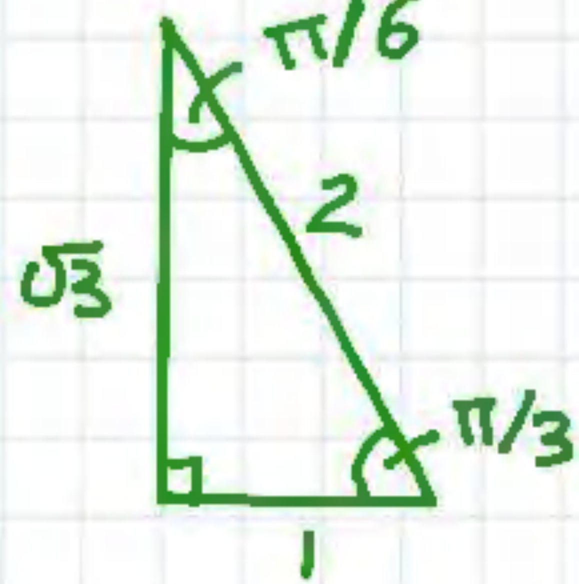
$\sin(x^2) \leq \sin x$ on $[0,1]$

b) Prove that $\int_0^{\pi/6} \sin(x^2)\,dx \le 1 - \sqrt{3}/2$

Solution: We have already proven that

$\sin(x^2) \le \sin x$ on $[0,1]$, therefore

$\sin(x^2) \le \sin x$ on $[0, \pi/6]$



$$\int_0^{\pi/6} \sin(x^2)\,dx \le \int_0^{\pi/6} \sin x\,dx$$

$$= -\cos x \Big|_0^{\pi/6}$$

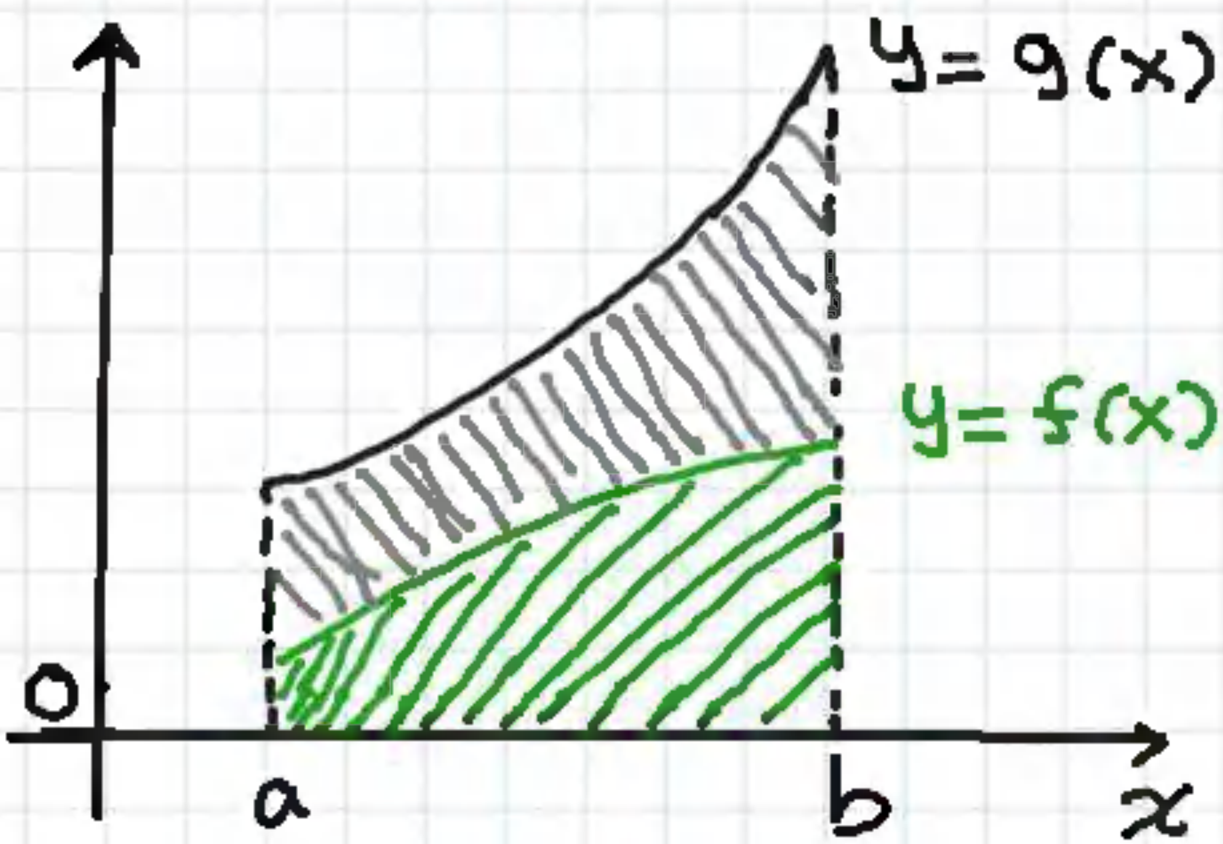
$$= -[\cos \pi/6 - \cos 0]$$

$$= -[\sqrt{3}/2 - 1] = 1 - \frac{\sqrt{3}}{2}$$ Proved

Theory : Comparison property of Integral

If $f(x) \le g(x)$ for $a \le x \le b$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

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Applying Fundamental theorem of Calculus to evaluate a definite integral example

Fundamental Theorem of Calculus 7

Evaluate $\int_1^2 \frac{x^3+8}{x+2}\,dx$

Solution:

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$x^3+8=x^3+2^3=(x+2)(x^2-2x+4)$

$$\int_1^2 \frac{x^3+8}{x+2}\,dx=\int_1^2 \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}}\,dx$$

$$\int_1^2 [x^2 - 2x + 4]\,dx = \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + 4x \Big|_1^2$$

$$= \frac{2^3}{3} - 4 + 8 - \left\{\frac{1}{3} - 1 + 4\right\}$$

$$= \frac{8}{3} - 4 + 8 - \frac{1}{3} + 1 - 4 = 1 + \frac{7}{3} = \boxed{\frac{10}{3}}$$

Before doing next Question here are some trig. identities you need to memorize.

$\sin^2\theta + \cos^2\theta = 1$

$\boxed{1 + \tan^2\theta = \sec^2\theta}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

Evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos\theta + \cos\theta\tan^2\theta}{\sec^2\theta}\, d\theta$

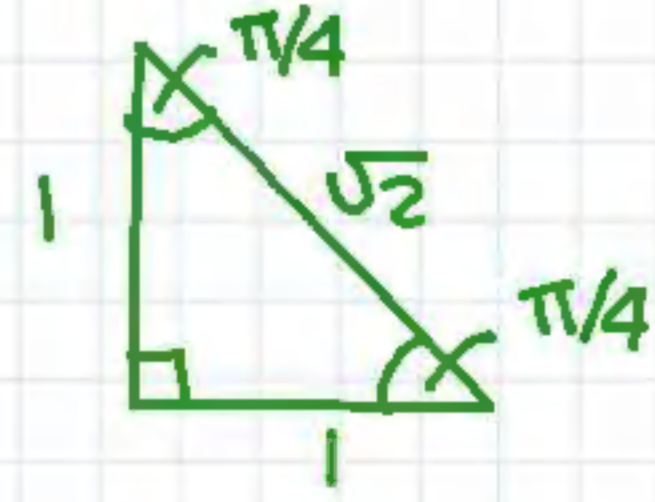
$$\int_0^{\frac{\pi}{4}} \frac{\cos\theta\,\cancel{[1+\tan^2\theta]}}{\cancel{\sec^2\theta}}\, d\theta$$

Trig. Identity
$1+\tan^2\theta = \sec^2\theta$

$$\int_0^{\frac{\pi}{4}} \cos\theta\, d\theta = \sin\theta \Big|_0^{\pi/4}$$

$$= \sin\frac{\pi}{4} - \sin 0$$

$$= \frac{1}{\sqrt{2}} - 0 = \boxed{\frac{1}{\sqrt{2}}}$$



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Fundamental Theorem of Calculus 8

Evaluate the integral $\int_1^2 \frac{2x^{1/3}+x^{1/2}}{x}\,dx$

$$\int_1^2 \frac{2x^{1/3}+x^{1/2}}{x}\,dx = \int \frac{2x^{1/3}}{x^1} + \frac{x^{1/2}}{x^1}\,dx$$

$$= \int_1^2 2x^{-2/3} + x^{-1/2}\,dx = \left[\frac{2x^{1/3}}{1/3} + \frac{x^{1/2}}{1/2} \right]\Bigg|_1^2$$

$$= 6x^{1/3} + 2x^{1/2}\Big|_1^2$$

$$= 6(2)^{1/3} + 2(2)^{1/2} - [6(1)^{1/3} + 2(1)^{1/2}]$$

$$= 6(2)^{1/3} + 2(2)^{1/2} - [6+2]$$

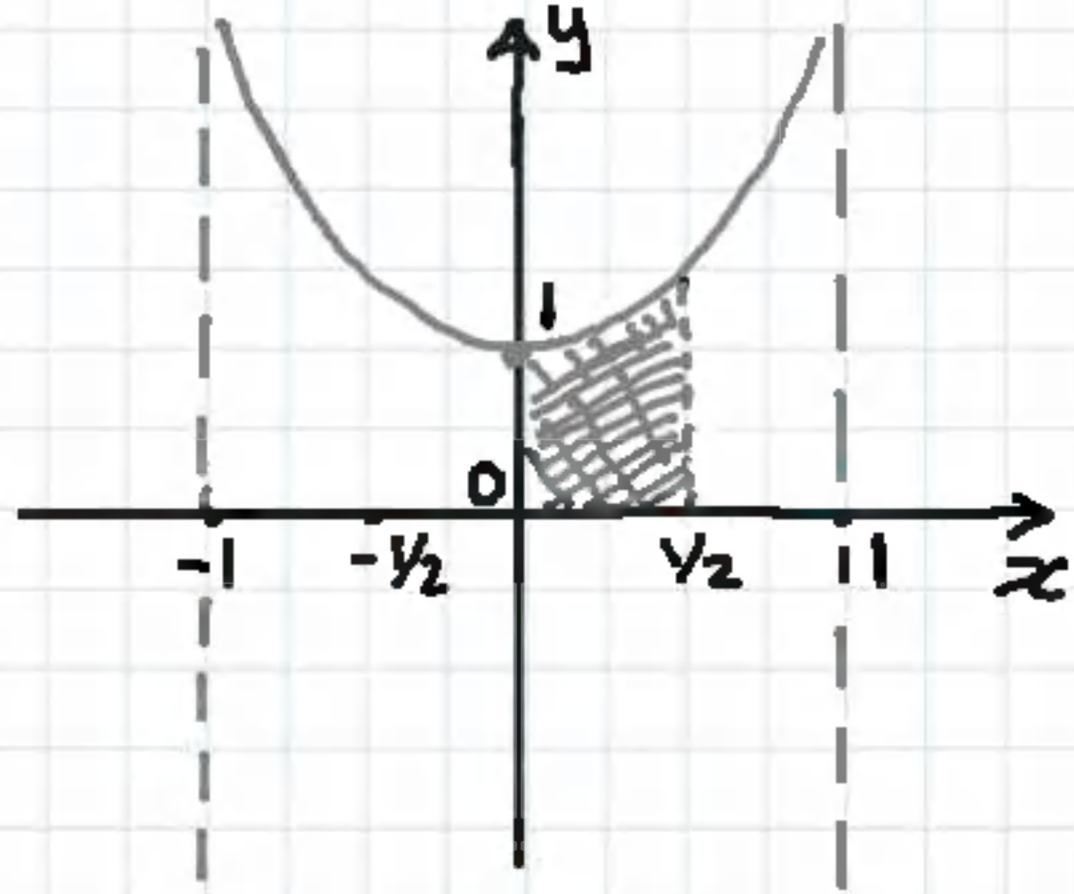
$$= \boxed{6\sqrt[3]{2} + 2\sqrt{2} - 8}$$

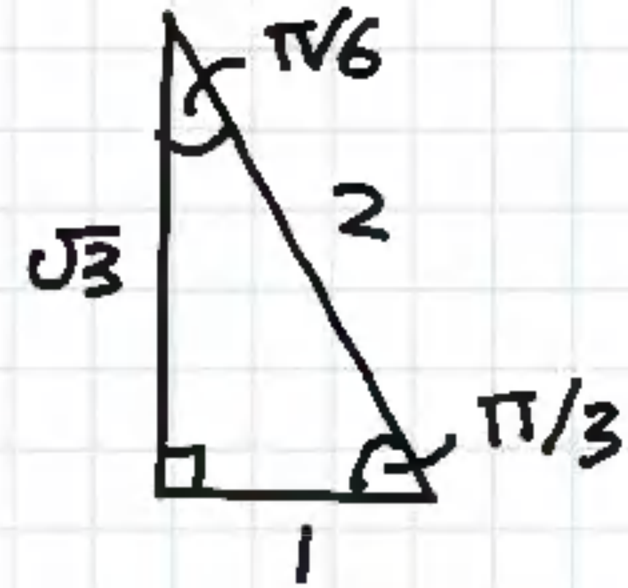
Evaluate $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x \Big|_0^{1/2}$$

$$= \sin^{-1}(1/2) - \sin^{-1}(0)$$

$$= \pi/6 - 0 = \boxed{\pi/6}$$





Fundamental theorem of Calculus tricky example

What is wrong with this computation?

$$\int_{-2}^{1} \frac{1}{x^4}\,dx = \int_{-2}^{1} x^{-4}\,dx = \left.\frac{x^{-3}}{-3}\right|_{-2}^{1} = \left.\frac{-1}{3x^3}\right|_{-2}^{1}$$

$$= \frac{-1}{3}\left[\frac{1}{1^3} - \frac{1}{(-2)^3}\right] = \frac{-1}{3}\left[1 - \frac{1}{-8}\right] = \frac{-1}{3}\left[1 + \frac{1}{8}\right]$$

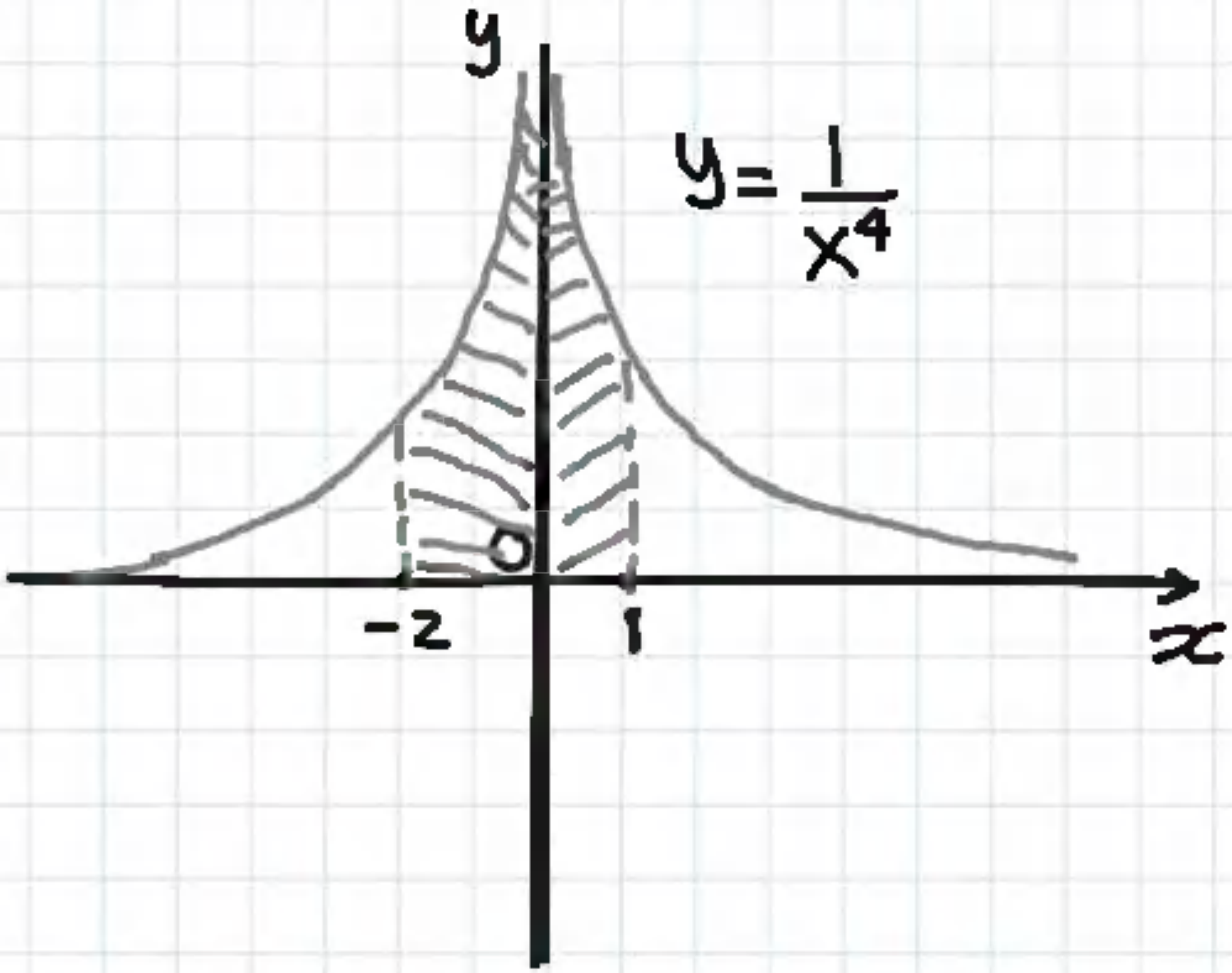
$$= \frac{-1}{3}\left[\frac{9}{8}\right] = \boxed{\frac{-3}{8}}$$

Solution: This answer has to be incorrect since $f(x) = 1/x^4 > 0$ for all x, therefore $\int_{-2}^{1} 1/x^4\,dx > 0$ and above answer is negative!

F.T.C can only be applied to continuous

F.T.C can only be applied to continuous functions and $f(x) = 1/x^4$ is discontinuous at $x = 0$.

$$\lim_{x \to 0} \frac{1}{x^4} = \infty$$



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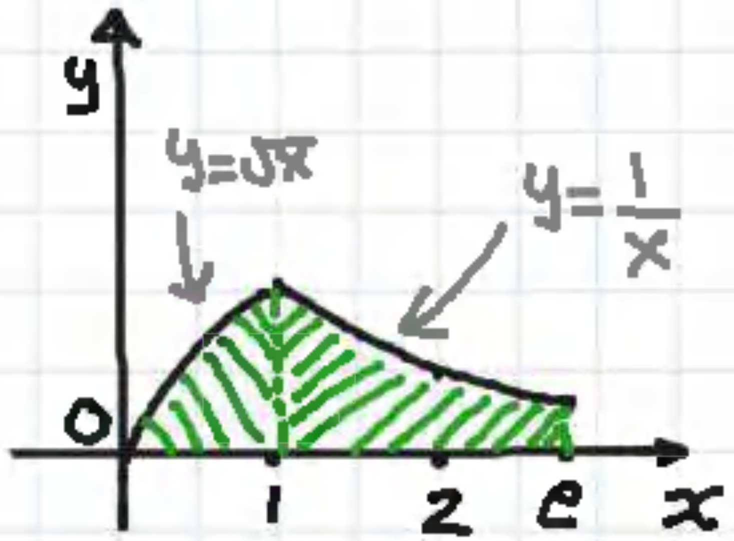
Applying FTC to evaluate definite integral of a piecewise continuous functon

Fundamental Theorem of Calculus 9

EX| Evaluate the integral $\int_0^e f(x)\,dx$

where $f(x) = \begin{cases} \sqrt{x} & 0 \le x \le 1 \\ 1/x & 1 < x \le e \end{cases}$



Solution: since $f(x)$ is made up of two piecewise continuous functions on $[0,1] \cup [1,e]$ we can apply F.T.C to compute $\int_0^e f(x)\,dx$

$$\int_0^e f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^e f(x)\,dx = \int_0^1 \sqrt{x}\,dx + \int_1^e \frac{1}{x}\,dx$$

$$= \int_0^1 \sqrt{x}\,dx + \int_1^e \frac{1}{x}\,dx$$

$$= \frac{x^{3/2}}{3/2}\Big|_0^1 + \ln|x|\Big|_1^e$$

$$= \frac{2}{3}x^{3/2}\Big|_0^1 + \ln x\Big|_1^e$$

$$= \frac{2}{3}(1^{3/2} - 0^{3/2}) + \underbrace{\ln e}_{=1} - \underbrace{\ln 1}_{=0}$$

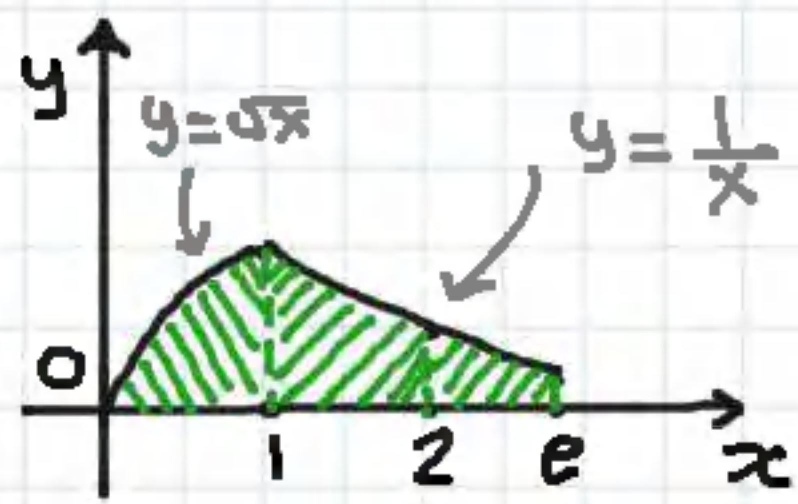
$$= \frac{2}{3} + 1 = \boxed{5/3}$$

Note: $|x| = x \quad x > 0$

$\ln|x| = \ln x \quad x > 0$

$\ln e = 1$

$\ln 1 = 0$



Applying FTC to find definite integral of absolute value of a function

Ex| Evaluate the integral $\int_0^2 |x^2 - \sqrt{x}|\, dx$

Solution: To evaluate above integral we need to find out the intervals on which $x^2 > \sqrt{x}$ or $x^2 < \sqrt{x}$ so that we can split up the integral into the sum of two separate integrals over two separate intervals.

STEP 1| Set $x^2 - \sqrt{x} = 0$ and solve for x

$x^2 - \sqrt{x} = 0 \Rightarrow x^2 = \sqrt{x} \Rightarrow (x^2)^2 = (\sqrt{x})^2$

$x^4 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$

$x(x^3 - 1) = 0 \Rightarrow x(x-1)(x^2 + x + 1) = 0$

$$x(x-1)(x^2+x+1)=0$$

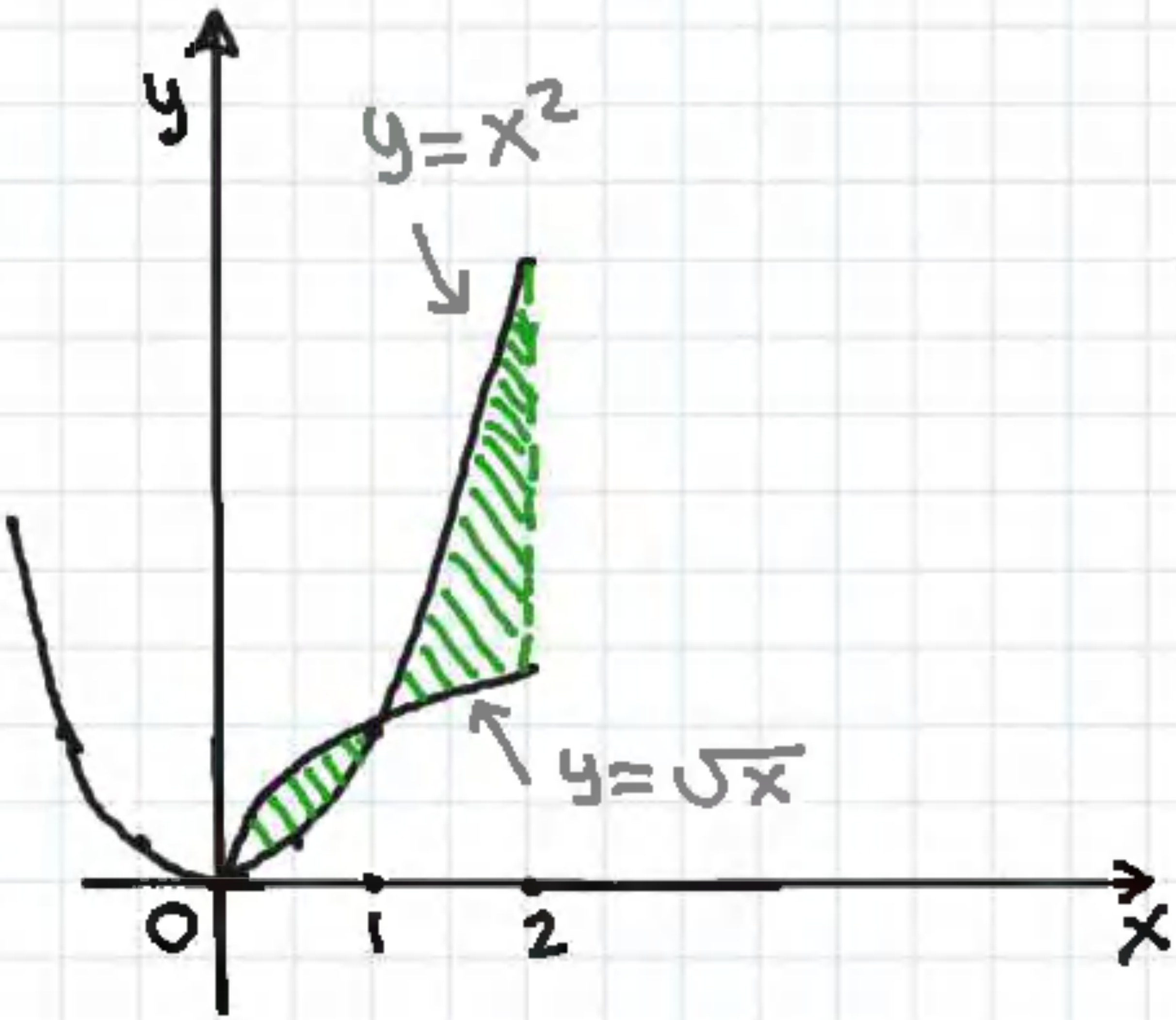
$x=0 \qquad x-1=0 \Rightarrow x=1$

$x^2-\sqrt{x}=0 \Rightarrow x=0,\ x=1$

$x^2+x+1=0$ apply quadratic formula

$$x=\frac{-1\pm\sqrt{1-4(1)(1)}}{2}=\frac{-1\pm\sqrt{-3}}{2}$$

Not possible



$$|x^2-\sqrt{x}|=\begin{cases}-(x^2-\sqrt{x}) & 0\le x\le 1\\ x^2-\sqrt{x} & 1\le x\le 2\end{cases}$$

$$\int_0^2 |x^2-\sqrt{x}|\,dx = \int_0^1 -(x^2-\sqrt{x})\,dx + \int_1^2 x^2-\sqrt{x}\ \,dx$$

$$= -\left[\frac{x^3}{3} - \frac{x^{3/2}}{3/2}\right]\Big|_0^1 + \frac{x^3}{3} - \frac{x^{3/2}}{3/2}\Big|_1^2$$

$$= -\left[\frac{1}{3} - \frac{2}{3} - (0-0)\right] + \left[\frac{8}{3} - \frac{2}{3}\cdot 2^{3/2} - \left[\frac{1}{3} - \frac{2}{3}\right]\right]$$

$$= \frac{1}{3} + \left[\frac{8}{3} - \frac{2^{5/2}}{3} - -\frac{1}{3}\right]$$

$$= \frac{1}{3} + \frac{8}{3} + \frac{1}{3} - \frac{2^{5/2}}{3} = \frac{10}{3} - \frac{2^{5/2}}{3} = \frac{10-2^{5/2}}{3} \cong 1.448$$

$$\int_0^2 |x^2-\sqrt{x}|\,dx = \frac{10-2^{5/2}}{3}$$

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Fundamental theorem of calculus word problem example

Fundamental Theorem of Calculus 10

Ex| At $t=0$, an oil reservoir contains 10000 Litres of oil, to raise the level of oil a pipe delivers $500t$ Litres/hour into the reservoir. Find the time t such that the volume of oil in reservoir is 110,000 Litres.

$V(t)$ is volume of oil at time t in Litres

Solution: Apply F.T.C

$$\int_0^T V'(t)\,dt = V(t)\Big|_0^T = V(T) - V(0)$$

$$V(T) = V(0) + \int_0^T V'(t)\,dt$$

$$V(T) = V(0) + \int_0^T V'(t)\,dt$$

$$110000 = 10000 + \int_0^T 500t\,dt$$

$$\underbrace{110{,}000}_{\text{Final Vol.}} = \underbrace{10\,000}_{\text{Initial Volume}} + \underbrace{\int_0^T 500t\,dt}_{\text{Change in Volume}}$$

$$110{,}000 = 10000 + \frac{500t^2}{2}\Big|_0^T$$

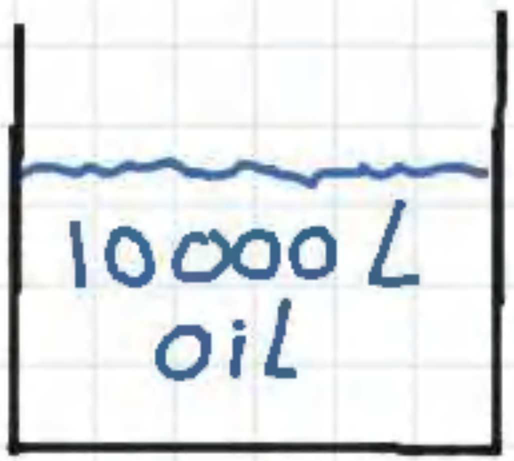
$$100{,}000 = 250t^2\Big|_0^T$$

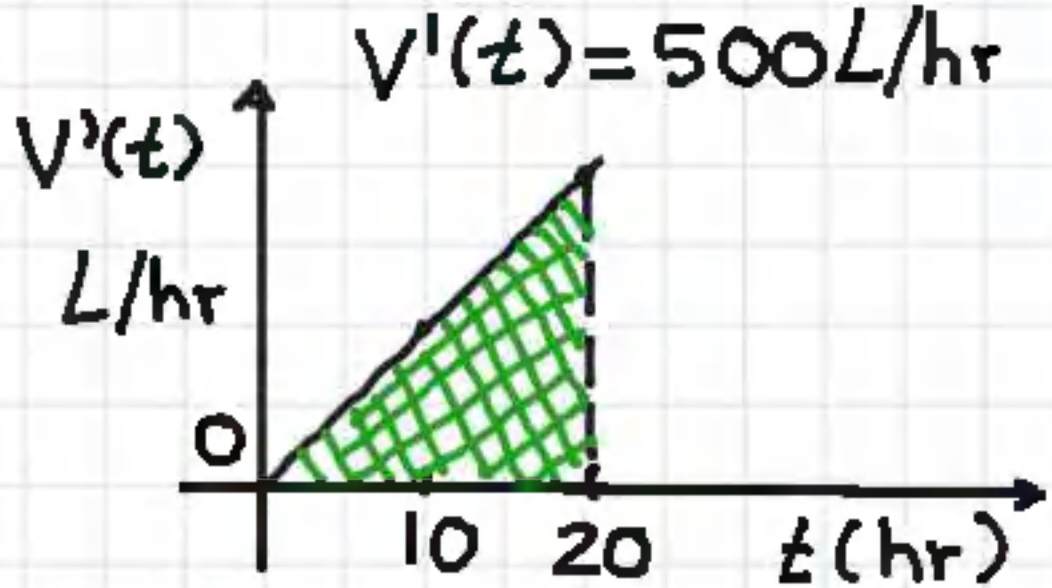
$$\frac{100{,}000}{250} = \frac{\cancel{250}}{\cancel{250}}(T^2 - 0) \Rightarrow 400 = T^2 \Rightarrow T = 20 \text{ hrs}$$

$$\sqrt{400} = \sqrt{T^2}$$

$$20 = T$$

Therefore it takes 20 hours to raise the volume of the reservoir from 10,000 Litres to 110,000 Litres.





BIG Picture

$$10000 + \int_0^{20} 500t \, dt = 110{,}000$$

Fundamental theorem of calculus theoretical example

Ex| Find value of c such that $\int_0^c \frac{1}{1+t^2}\,dt = \pi/4$

Solution: Apply F.T.C

$$\int_0^c \frac{1}{1+t^2}\,dt = \pi/4$$

$$\tan^{-1} t \Big|_0^c = \pi/4$$

$$\tan^{-1} c - \tan^{-1} 0 = \pi/4$$

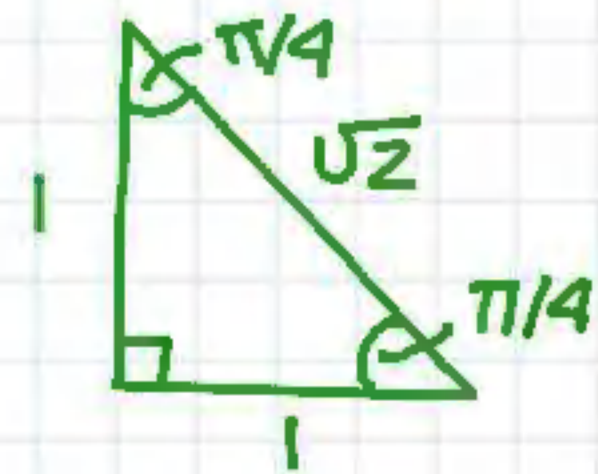
$$\tan^{-1} c - 0 = \pi/4$$

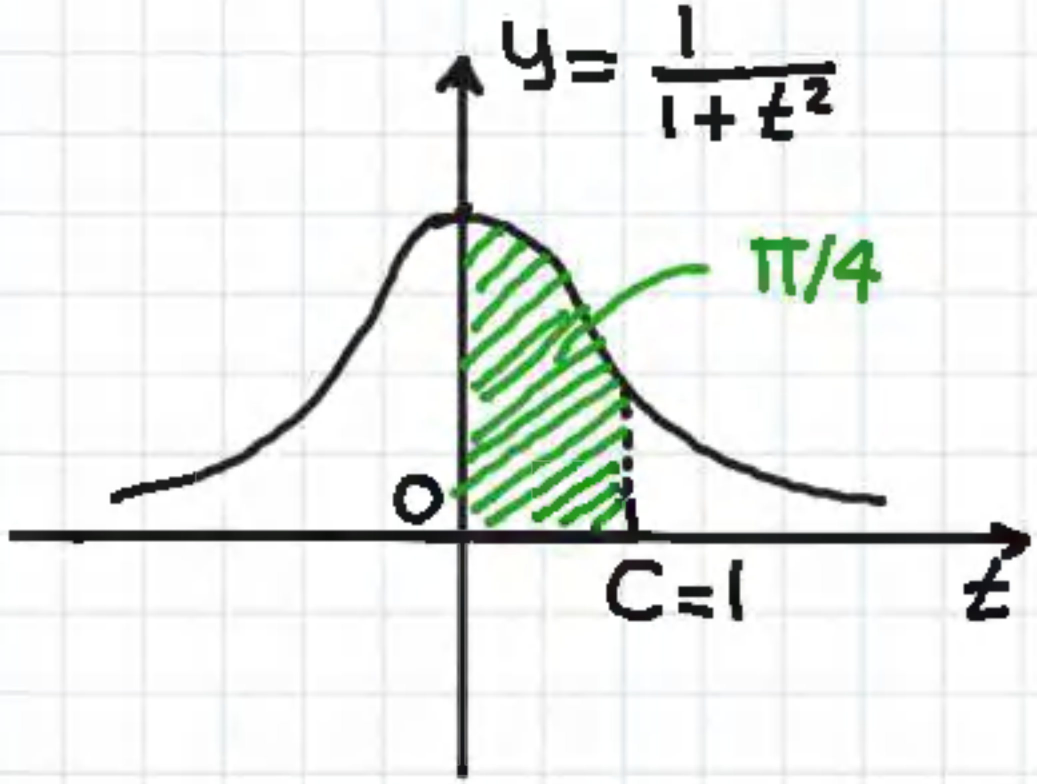
$$\tan^{-1} c = \pi/4 \Rightarrow \cancel{\tan}\,\cancel{\tan^{-1}} c = \tan \pi/4$$

$$\Rightarrow c = \tan \pi/4 \Rightarrow \boxed{c = 1}$$

Notes:

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$





$$\int_0^c \frac{1}{1+t^2}\,dt = \pi/4$$

$$C = 1$$

$$\int_0^1 \frac{1}{1+t^2}\,dt = \pi/4$$

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Fundamental Theorem of Calculus II

Ex] A Skateboarder travels along a road such that its velocity at time t is given by $V(t) = t^2 - 4t + 3$ (m/sec) during $1 \leq t \leq 4$ seconds

a) Find the displacement of the skateboarder over the time interval $[1,4]$

b) Find the total distance travelled during this time interval.

a) Solution: displacement $= \displaystyle\int_{t_1}^{t_2} V(t)\,dt$

a) $\text{displ.} = \int_{t_1}^{t_2} V(t)\, dt$

$$\text{displ.} = \int_1^4 \left(t^2 - 4t + 3\right) dt = \frac{t^3}{3} - \frac{4t^2}{2} + 3t \Big|_1^4$$

$$\text{displ.} = \frac{4^3}{3} - 2(4)^2 + 3(4) - \left[\frac{1}{3} - \frac{4}{2} + 3\right]$$

$$= \frac{64}{3} - 32 + 12 - \frac{1}{3} + 2 - 3$$

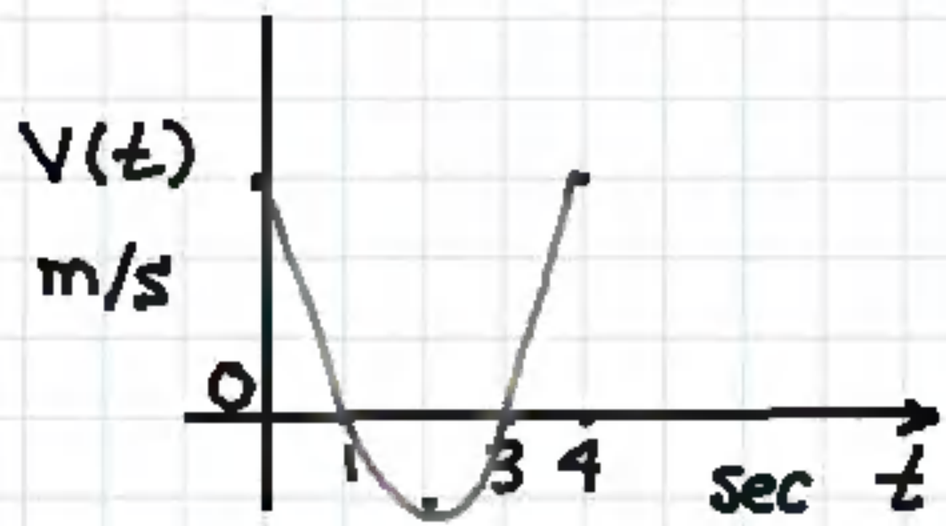
$$= \frac{64}{3} - \frac{1}{3} - 20 - 1 = \frac{63}{3} - \frac{21}{1} = \frac{63 - 63}{3} = 0 \text{ m}$$

Therefore the skateboarder has a net change in position of 0 metres.

b) Find the total distance travelled

$$V(t) = t^2 - 4t + 3 = (t-1)(t-3)$$

$$\text{distance} = \int_1^4 |v(t)|\, dt$$



$$|v(t)| = \begin{cases} -(t^2-4t+3) & 1 \le t \le 3 \\ t^2-4t+3 & 3 \le t \le 4 \end{cases}$$

$$\text{distance} = -\int_1^3 (t^2-4t+3)\,dt + \int_3^4 (t^2-4t+3)\,dt$$

$$\text{dist.} = -\left[\frac{t^3}{3} - \frac{4t^2}{2} + 3t \Big|_1^3 \right] + \frac{t^3}{3} - \frac{4t^2}{2} + 3t \Big|_3^4$$

$$\text{dist.} = -\left[\frac{27}{3} - 18 + 9 - \left(\frac{1}{3} - 2 + 3\right)\right] + \left[\frac{64}{3} - 18 + 12 - (9 - 18 + 9)\right]$$

$$d = -\left[\frac{27}{3} - 18 + 9 - \left(\frac{1}{3} - 2 + 3\right)\right] + \left[\frac{64}{3} - 32 + 12 - (9 - 18 + 9)\right]$$

$$d = -\left[0 - 4/3\right] + \frac{64}{3} - 20 - 0 = \frac{4}{3} + \frac{64}{3} - 20 = \frac{68}{3} - 20$$

$$d = \frac{68}{3} - \frac{20}{1} = \frac{68-60}{3} = \frac{8}{3} \text{ metres}$$

Review

a) $\text{displ.} = \int_1^4 v(t)\,dt = 0 \text{ metres}$

b) $\text{distance} = \int_1^4 |v(t)|\,dt = \frac{8}{3} \text{ metres}$

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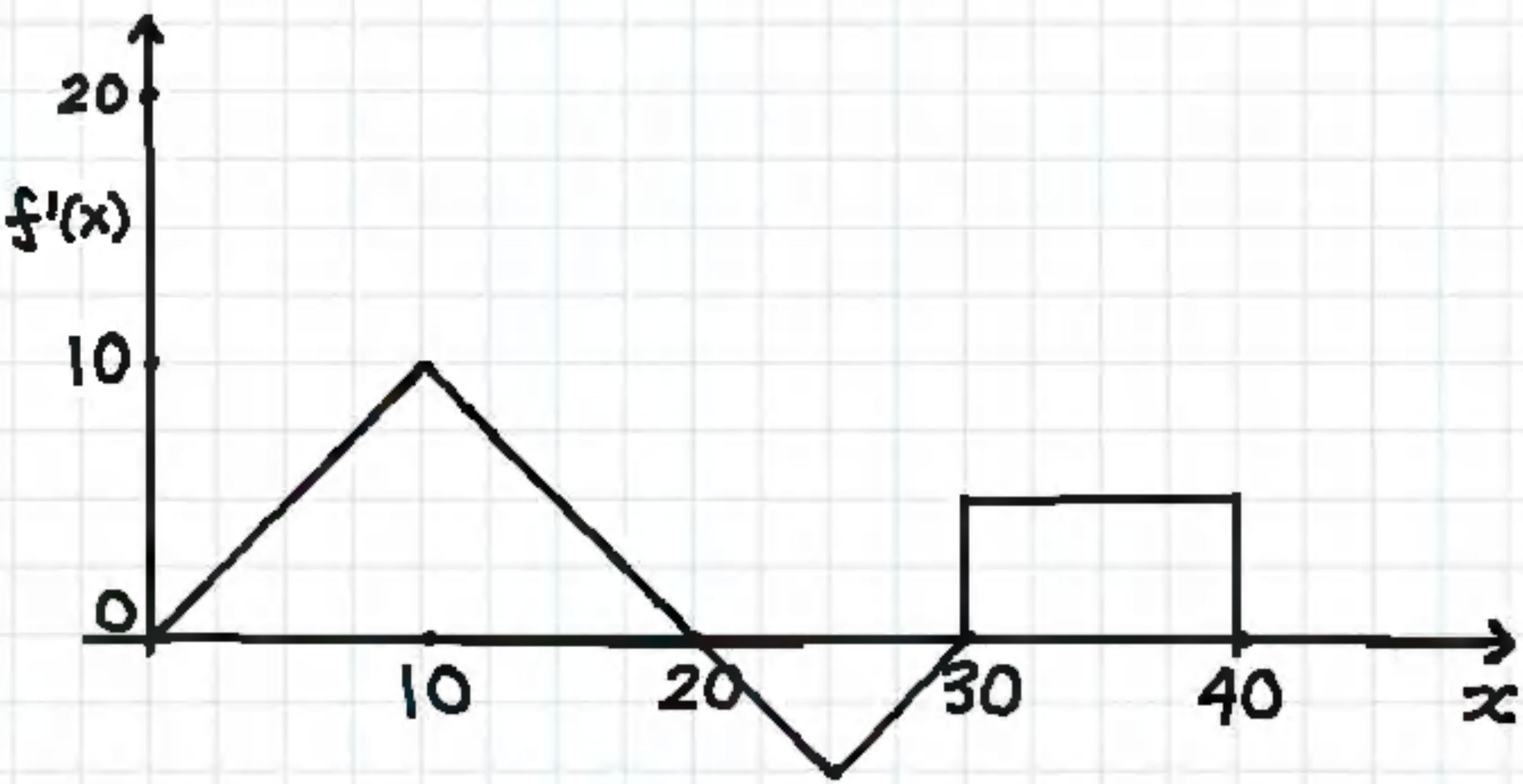
Fundamental Theorem of Calculus Sketching antiderivative example

Fundamental Theorem of Calculus 12 (sketch antiderivatives)

Ex] Given the graph of the derivative $f'(x)$ and $F(0)=50$ Sketch the graph of the antiderivative.

where $F(x) = F(0) + \int_0^x f'(t)\,dt$



Note $f'(x)$ is piecewise continuous on [0,40]

Solution: $F(x) = F(0) + \int_0^x f'(t)\,dt$

Critical points of $F(x)$ occur at $x=20$ and $x=30$

Since $F'(x) = \frac{d}{dx}\int_0^x f'(t)\,dt = f'(x)$ Apply F.T.C

$\therefore F'(x) = 0 \Rightarrow f'(x) = 0$ where $f'(x)$ cuts thru the x axis we will have maximum or minimum for $F(x)$

Inflection points of $F(x)$ occur at $x=10$ and $x=25$ where $f'(x)$ has a maximum or minimum.

Apply F.T.C $\int_a^b f'(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

$F(b) = F(a) + \int_a^b f'(x)\,dx$

Since $F(0) = 50$

$$F(10) = F(0) + \int_0^{10} f'(x)\,dx = 50 + \frac{1}{2}(10)(10) = 100$$

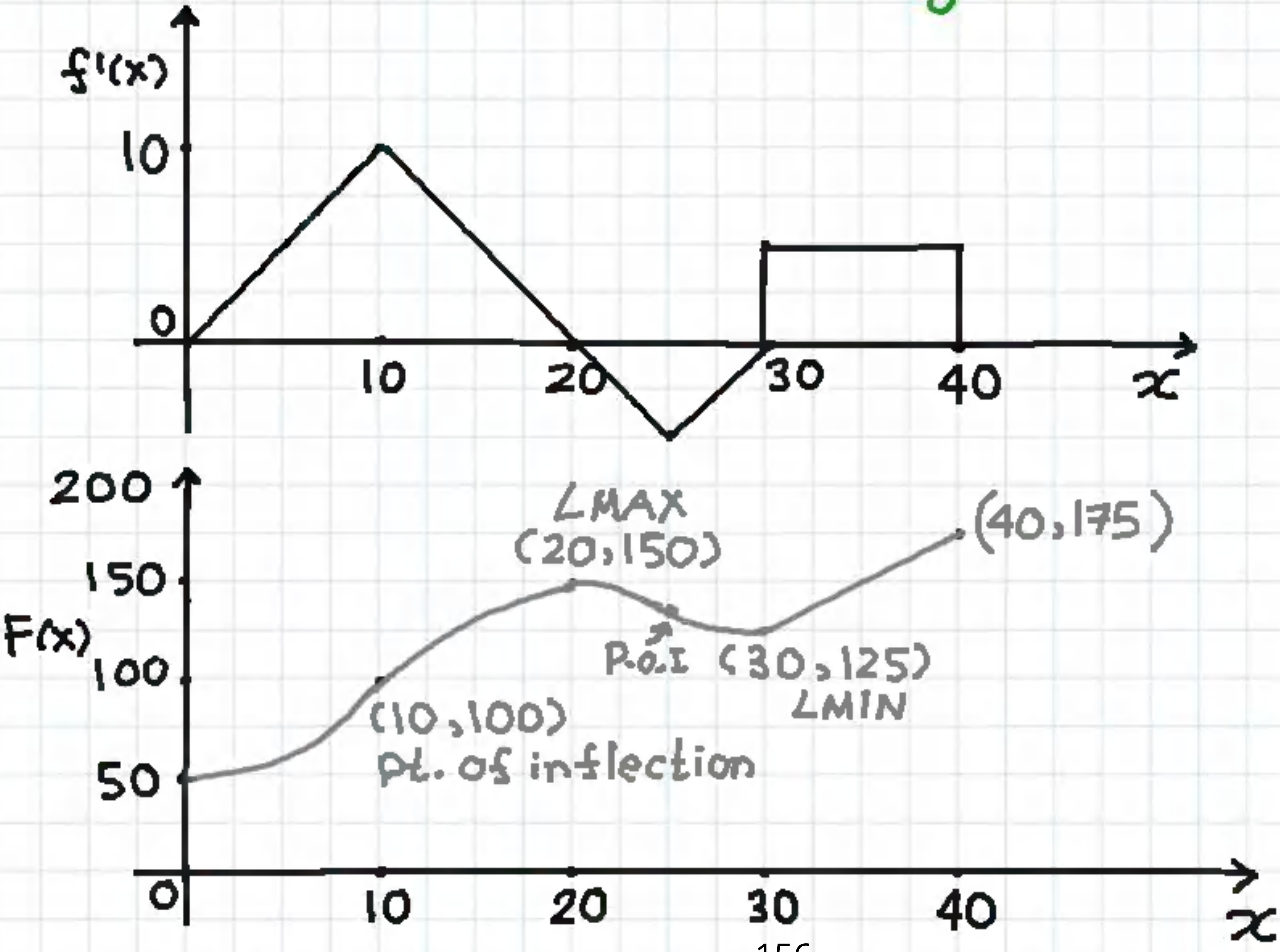
$$F(20) = F(10) + \int_{10}^{20} f'(x)\,dx = 100 + \frac{1}{2}(10)(10) = 150$$

$$F(25) = F(20) + \int_{20}^{25} f'(x)\,dx = 150 - \underbrace{\frac{1}{2}(5)(5)}_{\text{Area below } x \text{ axis subtracted}} = 137.5$$

$$F(30) = F(25) + \int_{25}^{30} f'(x)\,dx = 137.5 - \frac{1}{2}.5.5 = 125$$

$$F(40) = F(30) + \int_{30}^{40} f'(x)\,dx = 125 + 10 \times 5 = 175$$

$$F(x) = 50 + \int_0^x f'(t)\,dt$$



Review

$$F(x) = F(0) + \int_0^x f'(t)\,dt \qquad \boxed{\text{F.T.C}}$$

$F(0) = 50$ (Initial value of $F(x)$)

$F(10) = 100$ (Point of inflection)

$F(20) = 150$ (Local maximum)

$F(25) = 137.5$ (Point of inflection)

$F(30) = 125$ (Local minimum)

$F(40) = 175$ (Final value of $F(x)$)

$$\int_a^b f'(t)\,dt = F(t)\Big|_a^b = F(b) - F(a) \qquad \boxed{\text{F.T.C}}$$

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