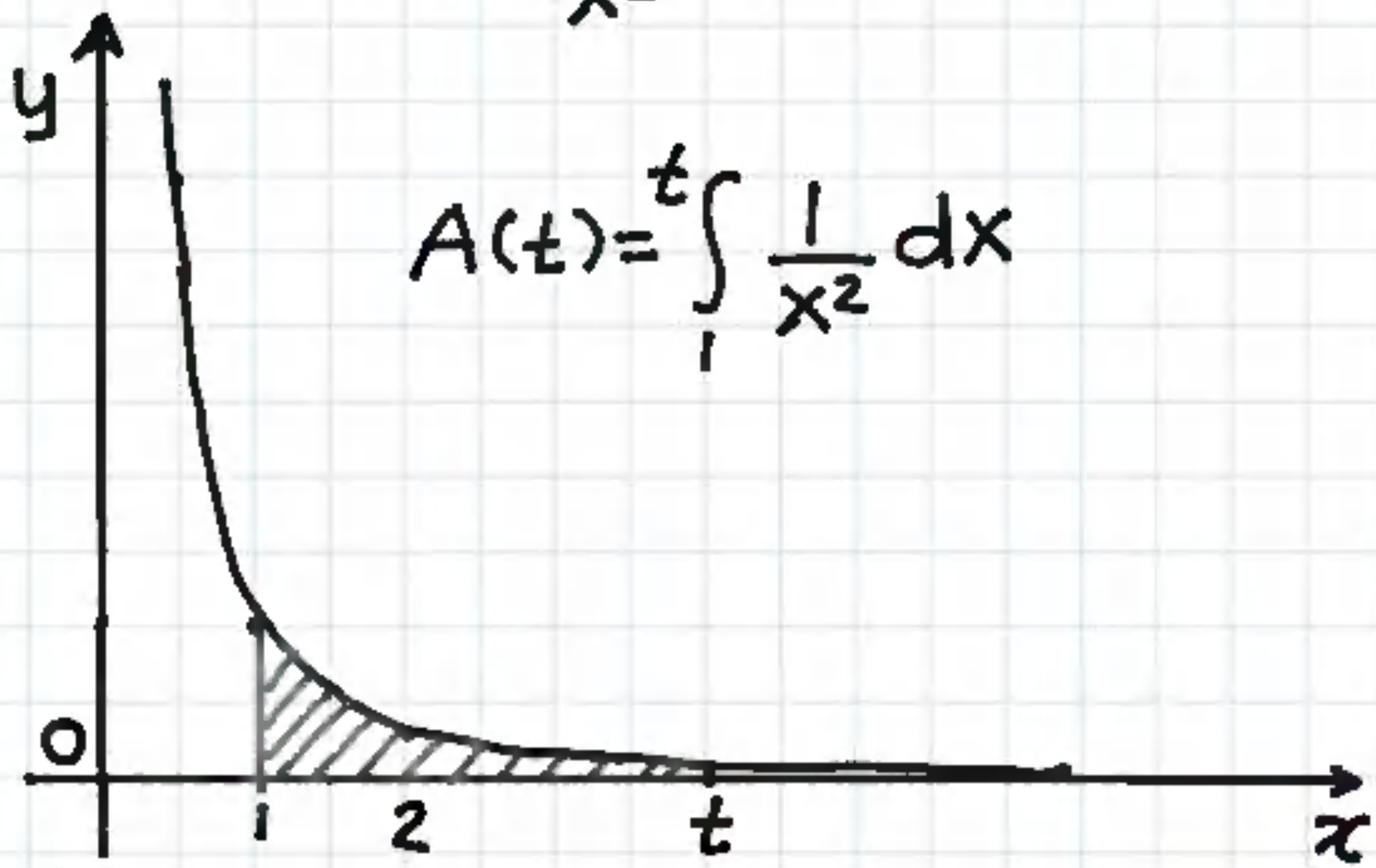
## Improper Integrals I

Type I Infinite Limits of Integration

Consider the definite integral $\int_1^t \frac{1}{x^2}\,dx$ for any real number $t>1$, this definite integral gives the area of the region bounded by $y=\frac{1}{x^2}$, the $x$ axis, $x=1$ and $x=t$



Let's evaluate $A(t) = \int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx = -x^{-1}\Big|_1^t = -\frac{1}{x}\Big|_1^t$

$= -\frac{1}{t} - -\frac{1}{1} = 1 - \frac{1}{t}$

$A(t) = 1 - \frac{1}{t}$

Let's look at values for $A(t)$ as $t$ increases

| $t$ | 1 | 2 | 3 | ... | 100 | |
|---|---|---|---|---|---|---|
| $A(t)$ | 0 | 1/2 | 2/3 | ... | 0.99 | |

Notice as $t$ increases Area $A(t)$ also increases. As $t \to \infty$

$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$

Area of shaded region approaches 1 as $t \to \infty$

therefore $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = 1$

Key concept : Even though the region of integration is infinite as $t \to \infty$ the Area bounded under the curve is finite and has value 1.

Now let's look at $A(t) = \int_1^t \frac{1}{\sqrt{x}}\,dx = \int_1^t x^{-1/2}\,dx = 2x^{1/2}\Big|_1^t$

$= 2t^{1/2} - 2(1)^{1/2} = 2\sqrt{t} - 2$

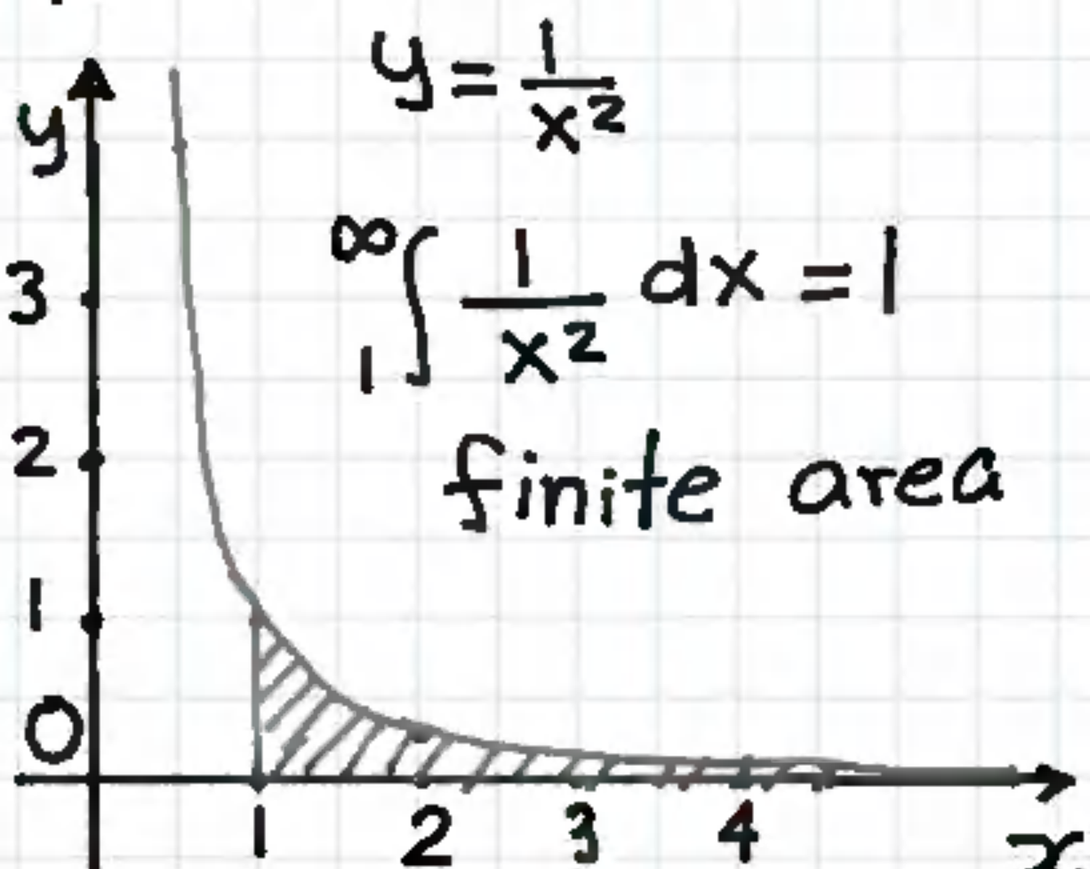
Notice as $t$ increases the Area $A(t) = 2\sqrt{t} - 2$ also increases but in this case:

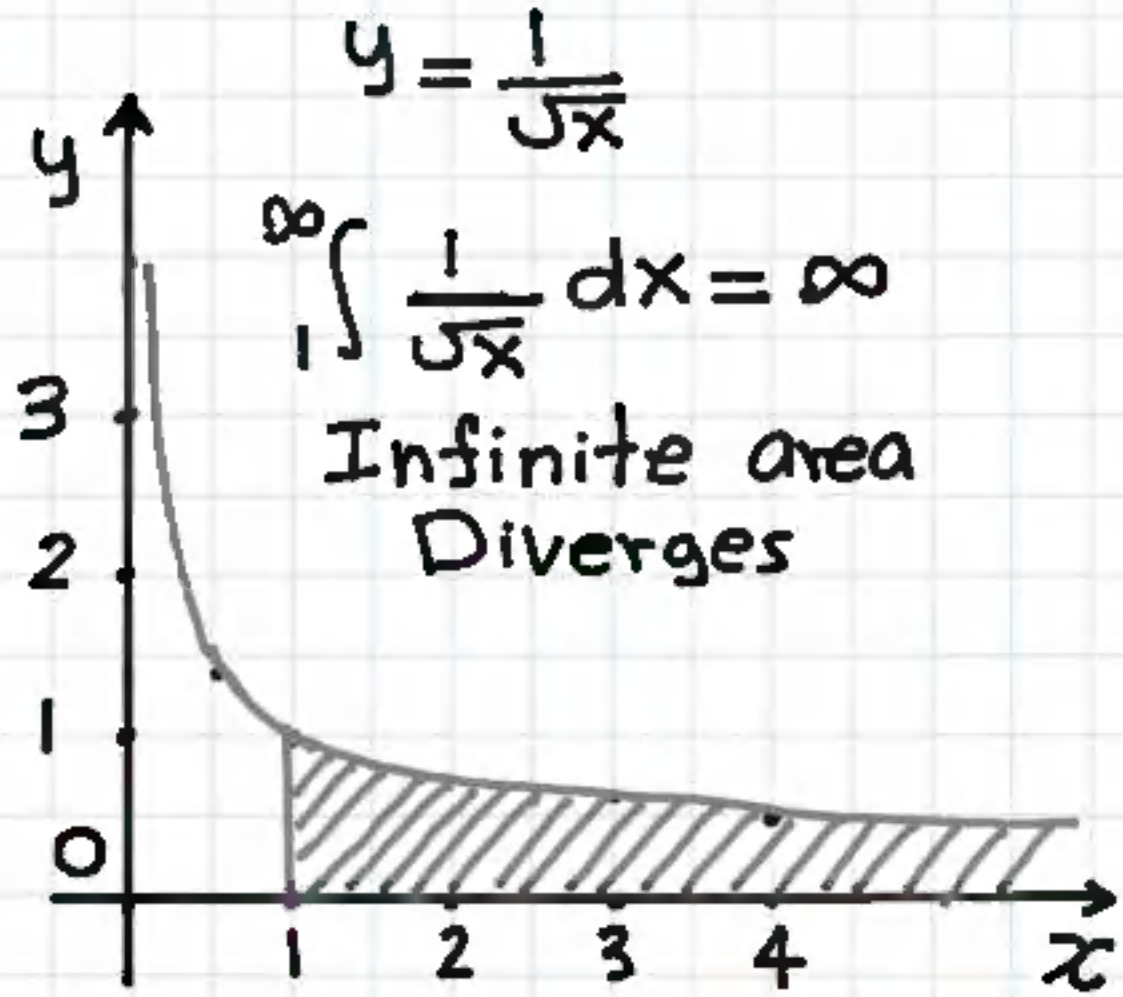
$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 2\sqrt{t} - 2 = 2\sqrt{\infty} - 2 = 2(\infty) - 2 = \infty$$

Therefore the shaded region under $y = 1/\sqrt{x}$ has infinite area. Therefore $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}}\,dx = \infty$

Notice from the graphs below $y = \frac{1}{x^2}$ and $y = \frac{1}{\sqrt{x}}$ both approach 0 as $x \to \infty$ but the values of $y = \frac{1}{\sqrt{x}}$ dont decrease fast enough for the improper integral to have a finite value and for this reason

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \infty \quad \text{Diverges}$$



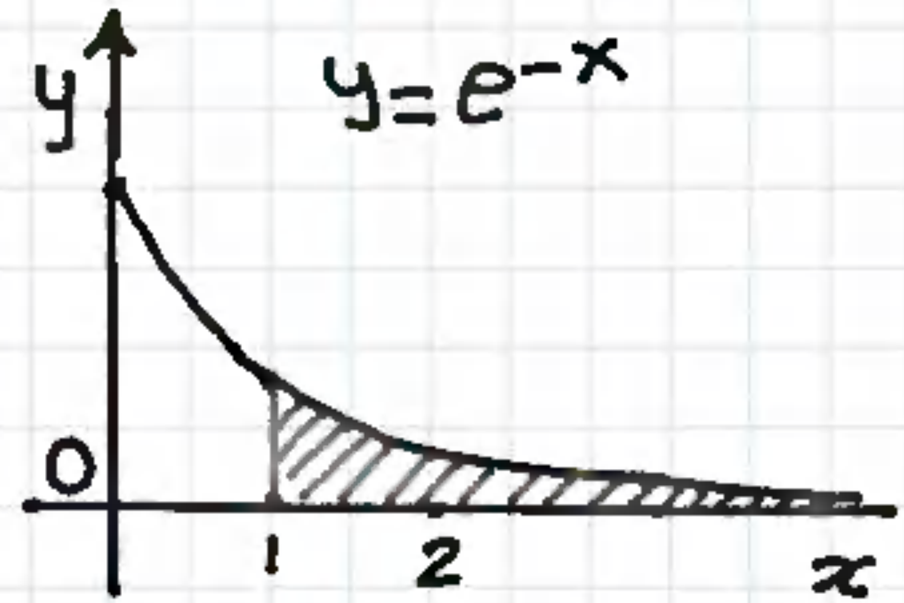
# Improper Integrals 2

## Improper Integrals Definitions and Examples

1. If $f(x)$ is continuous on $[a, \infty)$ then:

$$\int_a^{\infty} f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx \quad \text{provided limit exists}$$

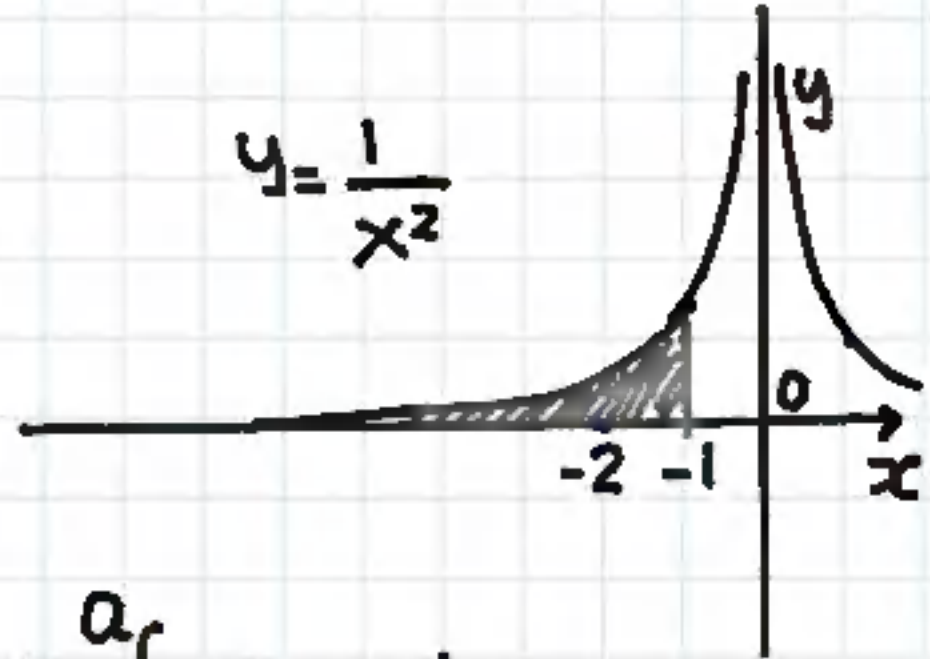
ex. $$\int_1^{\infty} e^{-x}\,dx = \lim_{t \to \infty} \int_1^t e^{-x}\,dx$$




2. If $f(x)$ is continuous on $(-\infty, a)$ then:

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t \to -\infty} \int_t^a f(x)\,dx$$

ex. $\int_{-\infty}^{-1} \frac{1}{x^2}\,dx = \lim_{t \to -\infty} \int_{t}^{-1} \frac{1}{x^2}\,dx$



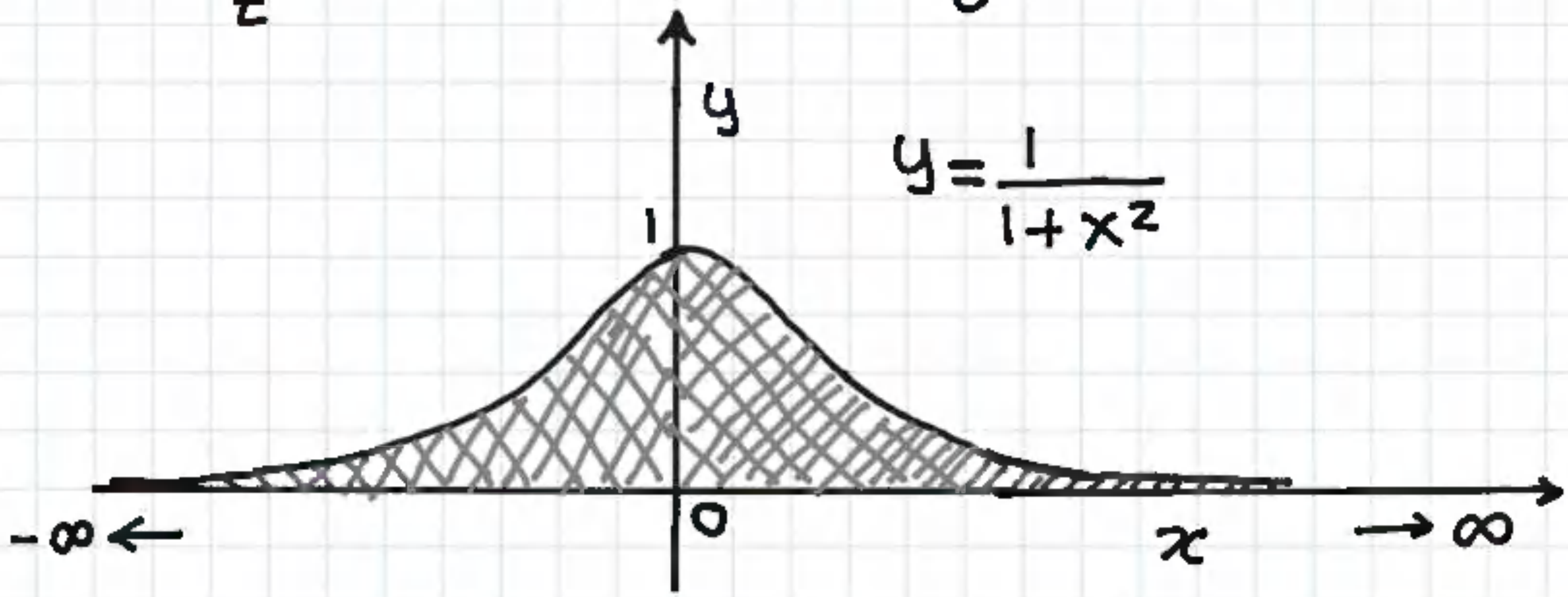

The improper Integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are called convergent if the limit exists or equivalently the improper integral has finite value and improper integral is called divergent if the limit does not exist ie) $\infty$ or $-\infty$

3) If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent

then we define: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$

ex.) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$= \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$



Evaluate: $\int_{-\infty}^{0} xe^{-x^2}dx$

We apply U- Substitution method $u=-x^2 \quad du=-2x\,dx$

$$\frac{du}{-2} = x\,dx$$

$x \to -\infty \quad u=-x^2 \Rightarrow u \to -\infty$

Change limits

$x \to 0 \quad u=-x^2 \Rightarrow u \to 0$

$$\int_{-\infty}^{0} xe^{-x^2}dx = \int_{-\infty}^{0} e^u \frac{du}{-2} = -\frac{1}{2}\lim_{t\to-\infty}\int_{t}^{0} e^u du = -\frac{1}{2}\lim_{t\to-\infty} e^u\Big|_t^0$$

$$= \lim_{t\to-\infty} -\frac{1}{2}e^0 - -\frac{1}{2}e^t = \lim_{t\to-\infty} -\frac{1}{2}+\frac{1}{2}e^t = -\frac{1}{2}+0 = -\frac{1}{2}$$

$$\int_{-\infty}^{0} xe^{-x^2}dx = -\frac{1}{2}$$

Recall: $\lim_{t\to-\infty} e^t = 0$

Evaluate $\int_1^{\infty} x.\ln x \, dx$

let's apply Integration by Parts: $\int u\,dv = uv - \int v\,du$

choose u easy to differentiate $u = \ln x$

choose dv easy to integrate $dv = x\,dx$

$u = \ln x \quad du = \frac{1}{x} dx \quad dv = x\,dx \quad v = \frac{x^2}{2}$

$$\int_1^{\infty} x.\ln x \, dx = \lim_{t \to \infty} \int_1^{t} x.\ln x \, dx = \lim_{t \to \infty} \frac{\ln x . x^2}{2} \Big|_1^t - \int_1^t \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \lim_{t \to \infty} \frac{\ln x . x^2}{2} \Big|_1^t - \frac{1}{2} \int_1^t x \, dx = \lim_{t \to \infty} \frac{\ln x . x^2}{2} - \frac{1}{2} \frac{x^2}{2} \Big|_1^t$$

$$= \lim_{t \to \infty} \frac{\ln t . t^2}{2} - \frac{t^2}{4} - \left\{ \frac{\ln 1 . 1}{2} - \frac{1}{4} \right\}$$

$$= \lim_{t\to\infty} \frac{\ln t \cdot t^2}{2} - \frac{t^2}{4} - \left\{ \frac{\overset{0}{\ln 1} \cdot 1}{2} - \frac{1}{4} \right\}$$

$$= \lim_{t\to\infty} \frac{\ln t \cdot t^2}{2} - \frac{t^2}{4} + \frac{1}{4}$$

$$= \lim_{t\to\infty} \frac{t^2}{2}\left[\ln t - \frac{1}{2}\right] + \frac{1}{4}$$

Factor out $\frac{t^2}{2}$

$$= \lim_{t\to\infty} \frac{t^2}{2} \cdot \lim_{t\to\infty}\left[\ln t - \frac{1}{2}\right] + \frac{1}{4}$$

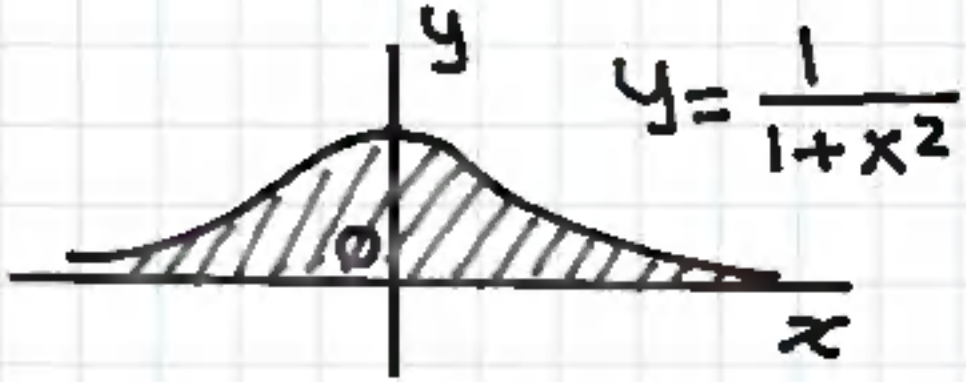
Apply Limit laws

$$= \infty \cdot \infty + \frac{1}{4} = \infty$$

$$\int_1^\infty x \cdot \ln x \, dx = \infty$$ Diverges

## Improper Integrals 3

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$




$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

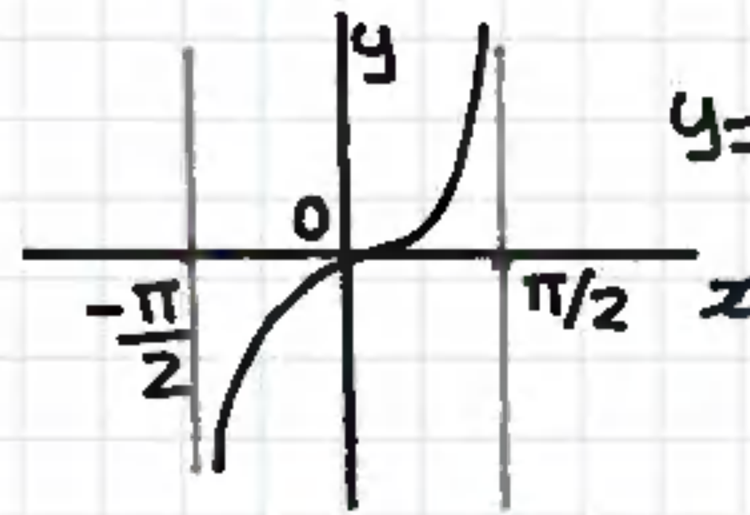
Since the integrand $y=\frac{1}{1+x^2}$ is an even function symmetric with respect to y axis ie) $f(-x)=f(x)$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 2\lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 2\lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= 2\lim_{t\to\infty} \tan^{-1}x\Big|_{0}^{t} = 2\lim_{t\to\infty}\left[\tan^{-1}t - \overset{0}{\overset{\parallel}{\tan^{-1}0}}\right] = 2\cdot\frac{\pi}{2} = \pi$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$




$y = \tan x$

**2]** Evaluate $\displaystyle\int_{0}^{\infty} \sin^2 x\,dx$

$\sin^2 x = \frac{1}{2}(1-\cos(2x))$

$$= \int_{0}^{\infty} \frac{1}{2}(1-\cos(2x))\,dx = \frac{1}{2}\lim_{t\to\infty}\int_{0}^{t}(1-\cos(2x))\,dx$$

$$\int_0^\infty \sin^2 x\,dx = \frac{1}{2}\lim_{t\to\infty}\int_0^t (1-\cos(2x))\,dx$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[x - \frac{\sin(2x)}{2}\right]\Big|_0^t = \frac{1}{2}\lim_{t\to\infty}\left[t - \frac{\sin(2t)}{2} - \left(0 - \frac{\overset{0}{\overset{\|}{\sin 0}}}{2}\right)\right]$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[t - \frac{\sin(2t)}{2}\right]$$

$$-1 \le \sin(2t) \le 1$$

$$-\frac{1}{2} \le \frac{\sin(2t)}{2} \le \frac{1}{2}$$

$$t - \frac{1}{2} \le t - \frac{\sin(2t)}{2} \le t + 1/2$$

$$\lim_{t\to\infty} \Downarrow \infty \qquad \lim_{t\to\infty} \Downarrow \infty \qquad \lim_{t\to\infty} \Downarrow \infty$$

By Squeeze law

$$= \frac{1}{2}\lim_{t\to\infty}\left[t - \frac{\sin(2t)}{2}\right] = \infty \quad \text{Diverges}$$

Squeeze law

$$h(x) \le f(x) \le g(x)$$

$$\lim_{x\to a} h(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

$$\Downarrow L \qquad \Downarrow \text{SQZ law } L \qquad \Downarrow L$$

Evaluate $\int_1^{\infty} \frac{1}{x^2+x}\,dx$

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{1}{x} - \frac{1}{x+1}$$ Partial Fractions

$$\int_1^{\infty} \frac{1}{x^2+x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+x}\,dx = \lim_{t\to\infty} \int_1^{t} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \lim_{t\to\infty} \Big[ \ln|x| - \ln|x+1| \Big]\Big|_1^t = \lim_{t\to\infty} \Big[ \ln\left|\frac{x}{x+1}\right| \Big]\Big|_1^t$$

$$= \lim_{t\to\infty} \left[ \ln\left|\frac{t}{t+1}\right| - \ln\frac{1}{2} \right]$$

since $t > 1$

$\ln\left|\frac{t}{t+1}\right| = \ln\left(\frac{t}{t+1}\right)$

$$= \lim_{t\to\infty} \left[ \ln\left(\frac{t}{t+1}\right) - \ln\frac{1}{2} \right]$$

$$= \lim_{t\to\infty} \left[ \ln\left(\frac{t}{t+1}\right) - (\overset{=0}{\ln 1} - \ln 2) \right]$$

$$= \lim_{t\to\infty} \ln\left(\frac{t}{t+1}\right) + \ln 2$$

$$= \ln\left(\lim_{t\to\infty} \frac{t}{t+1}\right) + \ln 2 = \overset{\overset{0}{\shortparallel}}{\ln 1} + \ln 2 = \ln 2$$

$$\int_1^\infty \frac{1}{x^2+x}\,dx = \ln 2$$

u- substitution

$$\int \cos(2x)\,dx = \frac{\sin(2x)}{2} + C$$

$u = 2x \quad du = 2dx$

Review

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$\ln 1 = 0$

$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ and $\tan(\pi/2) \to \infty$

# Improper Integrals 4 P integrals

**Ex1** Determine the values of P for which $\int_1^\infty \frac{1}{x^P}\,dx$ Converge and Diverge?

Case 1 : P=1 $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x \Big|_1^t$

$= \lim_{t\to\infty} \ln t - \underbrace{\ln 1}_{0} = \lim_{t\to\infty} \ln t = \infty$ Diverges

Case 2 : $P\neq 1$ $\int_1^\infty \frac{1}{x^P}\,dx = \lim_{t\to\infty}\int_1^t x^{-P}\,dx = \lim_{t\to\infty} \frac{x^{-P+1}}{-P+1}\Big|_1^t$

$= \lim_{t\to\infty} \frac{1}{1-P}\left(t^{1-P} - 1^{1-P}\right) = \lim_{t\to\infty} \frac{1}{1-P}\left(t^{1-P} - 1\right)$

$= \lim_{t\to\infty} \frac{1}{1-P}\left(\frac{1}{t^{P-1}} - 1\right)$

## Improper Integrals finding the values of P for which the integral converges

$$= \lim_{t \to \infty} \frac{1}{1-P}\left(\frac{1}{t^{P-1}} - 1\right) \qquad \text{idea} \Rightarrow \frac{1}{1-P}\left(\frac{1}{\infty^{P-1}} - 1\right)$$

If $P-1 > 0 \Rightarrow P > 1$ then $\infty^{P-1} \to \infty \Rightarrow \frac{1}{\infty^{P-1}} \to 0$

$$\therefore \lim_{t \to \infty} \frac{1}{t^{P-1}} = 0$$

$$\int_1^\infty \frac{1}{x^P}\,dx = \lim_{t \to \infty} \frac{1}{1-P}\left(\overset{\to 0}{\frac{1}{t^{P-1}}} - 1\right) = \frac{1}{1-P}(0-1) = \frac{-1}{1-P} = \frac{1}{P-1}$$

$$\int_1^\infty \frac{1}{x^P}\,dx = \frac{1}{P-1} \qquad P > 1 \quad \text{converges}$$

Now if $P-1 < 0 \Rightarrow P < 1$ then $\infty^{P-1} \to 0$ and $\frac{1}{\infty^{P-1}} \to \infty$

$$\therefore \lim_{t \to \infty} \frac{1}{t^{P-1}} \to \infty \qquad \infty^{-1} = \frac{1}{\infty} = 0$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right] = \frac{1}{1-p}(\infty - 1) = \infty$$

$p<1$ $\quad \int_1^\infty \frac{1}{x^p}\,dx$ diverges

$p=1$ $\quad \int_1^\infty \frac{1}{x}\,dx$ diverges

Summary: $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$ if $p>1$ converges

$\int_1^\infty \frac{1}{x^p}\,dx = \infty$ if $p\le 1$ Diverges

Some examples of P integrals $\int_1^\infty \frac{1}{x^P}\,dx$

$\int_1^\infty \frac{1}{x}\,dx$ Diverges $P=1$

$\int_1^\infty \frac{1}{x^{1.1}}\,dx$ Converges $p=1.1>1$

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty \frac{1}{x^{1/2}}\,dx$ Diverges $P=\frac{1}{2}<1$

$\int_1^\infty \frac{1}{x^2}\,dx$ Converges $p=2>1$

$\int_1^\infty \frac{1}{x^{0.9}}\,dx$ Diverges $P=0.9<1$

**Find the values of P for which ∫ 1/(x)*(Lnx)^p dx converges , x goes from e to infinity**

Ex2 Find the values of P for which $\int_e^{\infty} \frac{1}{x(\ln x)^P}\,dx$ converges?

$\int_e^{\infty} \frac{1}{x(\ln x)^P}\,dx$ Apply U-Substitution

$U = \ln x \qquad du = \frac{1}{x}\,dx$

$x = e \qquad u = \ln x \Rightarrow u = \ln e = 1$

$x \to \infty \qquad u = \ln x \Rightarrow u = \ln \infty = \infty$

$$\int_e^{\infty} \frac{1}{x(\ln x)^P}\,dx = \int_e^{\infty} \frac{1}{(\ln x)^P} \cdot \frac{1}{x}\,dx = \int_1^{\infty} \frac{1}{u^P}\,du$$

$\int_e^{\infty} \frac{1}{x(\ln x)^P}\,dx = \int_1^{\infty} \frac{1}{u^P}\,du$ converges if $P > 1$

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \int_1^\infty \frac{1}{u^p}\,du \quad \text{converges if } p > 1$$

This result follows from P integral theory we developed earlier $\int_1^\infty \frac{1}{u^p}\,du$ converges $p > 1$

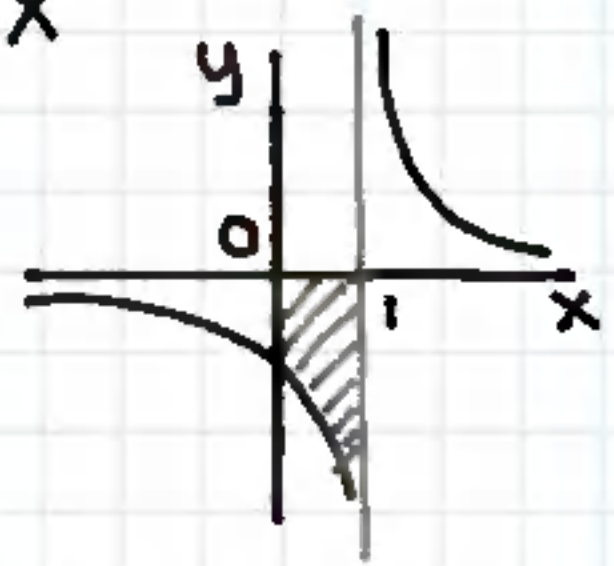
$\int_1^\infty \frac{1}{u^p}\,du$ Diverges $p \le 1$

# Improper Integrals 5 Discontinuous Integrand

1) Improper Integral Type 2 Discontinuous Integrand

If $f(x)$ is continuous on $[a,b)$ and is discontinuous at $x=b$ then $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$

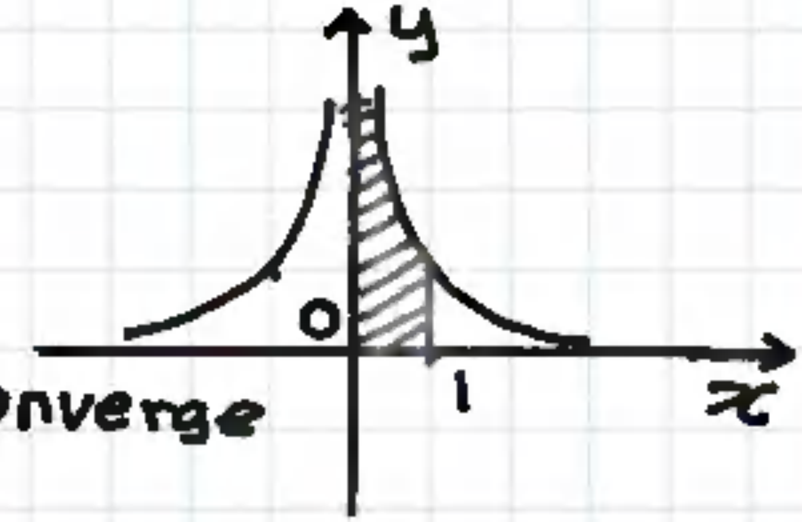
example) $\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$




2) If $f(x)$ is continuous on $(a,b]$ and is discontinuos at $x=a$ then $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

example) $\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$

Example) $\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$

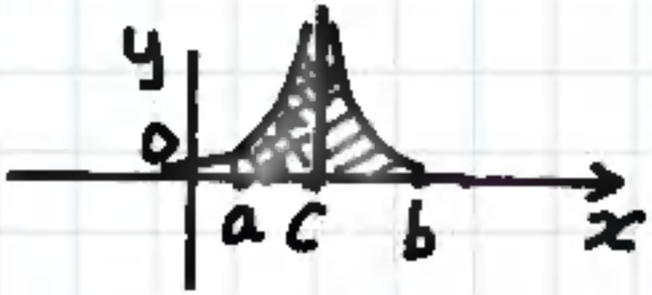



These improper integrals $\int_a^b f(x)\,dx$ converge if the limit exists and is finite and diverges if the limit does not exist.

3) If $f(x)$ has a discontinuity at $x=c$ where $a<c<b$ then we can split up $\int_a^b f(x)\,dx$ into the sum of 2 separate integrals: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$\int_a^b f(x)\,dx = \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$




If both integrals converge then the whole integral $\int_a^b f(x)\,dx$ converges, But if one of the integrals diverges then the whole integral $\int_a^b f(x)\,dx$ diverges.

Example)

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^2}\,dx + \lim_{t\to 1^+}\int_t^2 \frac{1}{(x-1)^2}\,dx$$

**Ex 1]** Evaluate $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$

This is a Type 2 Improper integral since the integrand $\frac{1}{\sqrt{3-x}}$ is not continuous at $x=3$ (vertical asymptote at $x=3$). Since the discontinuity is at the right endpoint of $[2,3]$ we write:

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\int_2^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\int_2^t (3-x)^{-1/2}\,dx$$

$$= \lim_{t\to 3^-}\left[\frac{(3-x)^{1/2}}{(1/2)(-1)}\right]\Bigg|_2^t = \lim_{t\to 3^-} -2\left[(3-x)^{1/2}\right]\Big|_2^t$$

$$= \lim_{t\to 3^-} -2(3-t)^{1/2} - -2(3-2)^{1/2} = -2(3-3^-)^{1/2} - -2(3-2)^{1/2}$$

$$= -2(0^+)^{1/2} + 2(1) = 2$$

$$= -2\sqrt{0} + 2 = 0 + 2 = 2$$

Converges answer $= 2$

Ex 2 Evaluate $\int_0^1 \frac{1}{2-3x}\,dx$

This is a Type 2 improper integral since $y = \frac{1}{2-3x}$ has a vertical asymptote at $x = 2/3$ since $2-3x=0$ $\Rightarrow 2 = 3x \Rightarrow x = 2/3$. So we split up $\int_0^1 \frac{1}{2-3x}\,dx$

into the sum of 2 separate improper integrals.

$$\int_0^1 \frac{1}{2-3x}\,dx = \int_0^{\frac{2}{3}} \frac{1}{2-3x}\,dx + \int_{2/3}^1 \frac{1}{2-3x}\,dx$$

$$= \lim_{t \to (2/3)^-} \int_0^t \frac{1}{2-3x}\,dx + \lim_{t \to (2/3)^+} \int_t^1 \frac{1}{2-3x}\,dx$$

let's consider the first integral $\lim_{t \to (2/3)^-} \int_0^t \frac{1}{2-3x}\,dx$

$$\lim_{t \to (2/3)^-} \int_0^t \frac{1}{2-3x}\,dx = \lim_{t \to (2/3)^-} \frac{\ln|2-3x|}{-3}\Bigg|_0^t$$

$$= \lim_{t \to (2/3)^-} -\frac{1}{3}\ln|2-3t| - \frac{\ln 2}{-3} = -\frac{1}{3}\ln\left|2-3(2/3)^-\right| + \frac{\ln 2}{3}$$

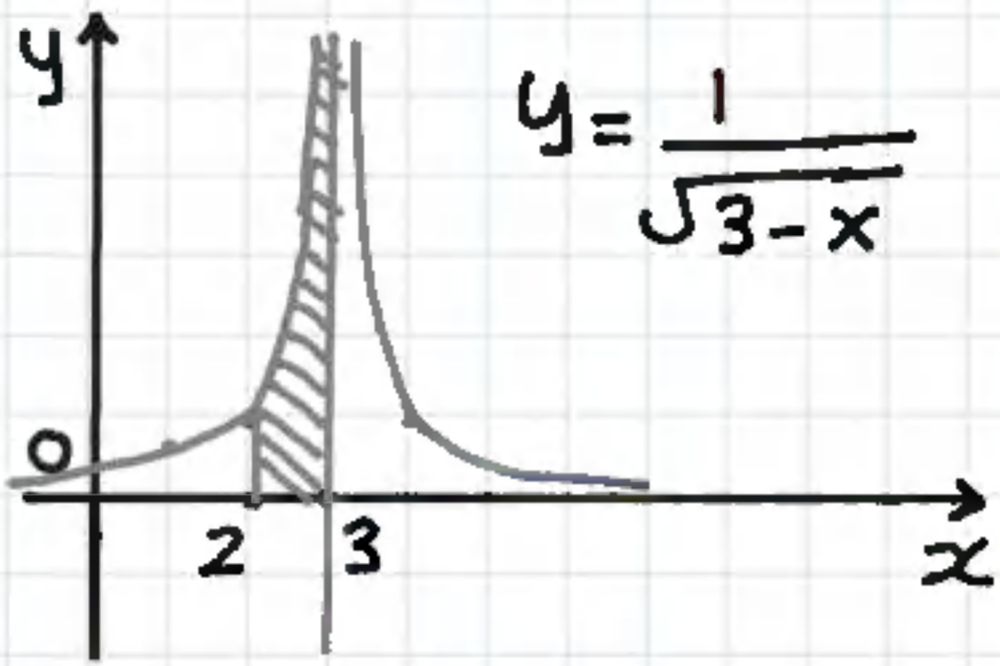
$$= -\frac{1}{3}\ln\left|2-2^-\right| + \frac{\ln 2}{3} = -\frac{1}{3}\ln|0^+| + \frac{\ln 2}{3} = -\frac{1}{3}(-\infty) + \frac{\ln 2}{3}$$

$= \infty$ Diverges

Since the first integral $\int_0^{2/3} \frac{1}{2-3x}\,dx$ diverges we dont need to evaluate the second integral $\int_{\frac{2}{3}}^1 \frac{1}{2-3x}\,dx$ and we conclude that the whole integral $\int_0^1 \frac{1}{2-3x}\,dx$ diverges.

# Diagrams and Integration Review

**Ex1** $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = 2$ convergent Improper Integral




$y = \dfrac{1}{\sqrt{3-x}}$

Integration Review
U- Substitution

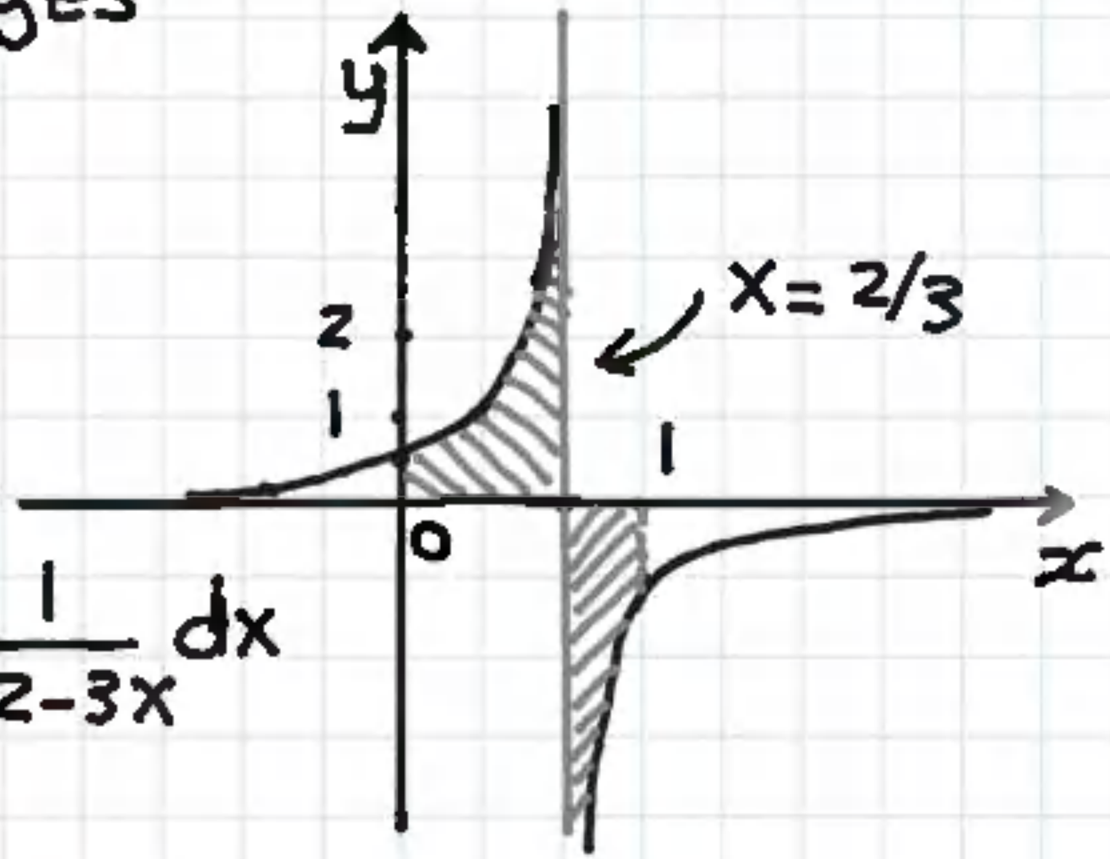
$$\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + C$$

$u = ax+b$

$du = a\,dx$

# Diagrams and U substitution Review continued

**Ex2** $\int_0^1 \frac{1}{2-3x}\,dx$ Diverges



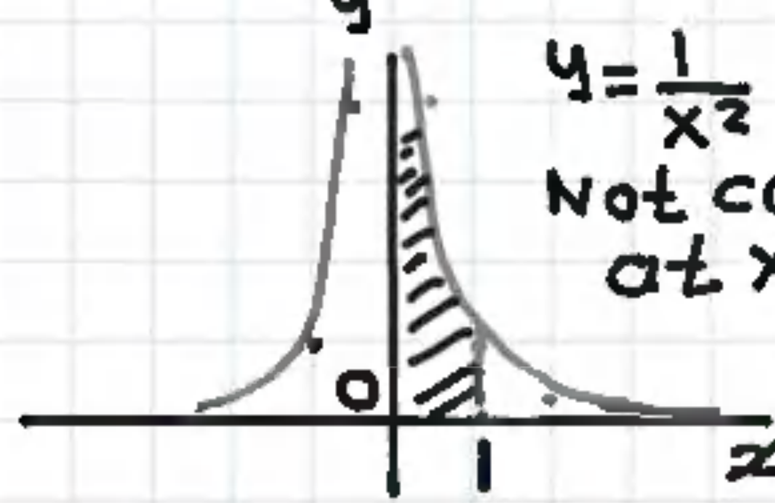

$$\int_0^1 \frac{1}{2-3x}\,dx = \int_0^{2/3} \frac{1}{2-3x}\,dx + \int_{2/3}^1 \frac{1}{2-3x}\,dx$$

## U-Substitution Review

$$\int \frac{1}{\sqrt{ax+b}}\,dx = \int (ax+b)^{-1/2}\,dx = \frac{(ax+b)^{1/2}}{(1/2)(a)} + C = \frac{2}{a}\sqrt{ax+b} + C$$

$$u = ax+b \quad du = a\,dx \quad dx = \frac{1}{a}\,du$$

# Improper Integrals 6




Ex1 Evaluate $\int_0^1 \frac{1}{x^2}\,dx$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-2}\,dx$$

$$= \lim_{t\to 0^+} \left.\frac{x^{-1}}{-1}\right|_t^1 = \lim_{t\to 0^+} \left.\frac{-1}{x}\right|_t^1 = \lim_{t\to 0^+} -\frac{1}{1}+\frac{1}{t}$$

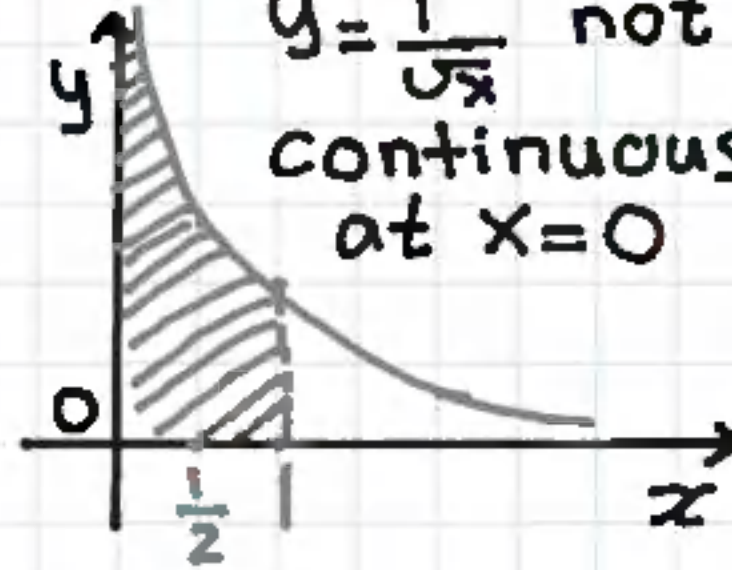
$= -1+\infty = \infty$ Diverges

$\int_0^1 \frac{1}{x^2}\,dx = \infty$ Diverges

**Ex 2** Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx = \lim_{t\to 0^+} 2x^{1/2}\Big|_t^1$$

$$= \lim_{t\to 0^+} 2(1)^{1/2} - 2(t)^{1/2} = 2 - 2(0)^{1/2} = 2$$




$y = \frac{1}{\sqrt{x}}$ not continuous at $x = 0$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 \quad \text{converges}$$

Why Does $\int_0^1 \frac{1}{x^2}\,dx = \infty$ Diverges while $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$ Converges? Let's Investigate!!

P integral $\int_0^1 \frac{1}{x^P}\,dx$ Find values of P that make this integral converge?

$$\int_0^1 \frac{1}{x^P}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-P}\,dx = \lim_{t\to 0^+} \left.\frac{x^{-P+1}}{-P+1}\right|_t^1$$

$$= \lim_{t\to 0^+} \frac{1^{1-P}}{1-P} - \frac{t^{1-P}}{1-P} = \lim_{t\to 0^+} \frac{1}{1-P} - \frac{t^{1-P}}{1-P} = \lim_{t\to 0^+} \frac{1}{1-P}\left(1 - t^{1-P}\right)$$

$$= \lim_{t\to 0^+} \frac{1-t^{1-P}}{1-P}$$

Key concept:
If $1-P>0 \Rightarrow 0^{1-P}=0$
ex. $0^2=0$ or $0^1=0$

Lets consider $1-P>0 \Rightarrow P<1$

$$\lim_{t\to 0^+} \frac{1-t^{1-P}}{1-P} = \frac{1}{1-P} \quad \text{since} \quad \frac{1-0^{1-P}}{1-P} = \frac{1}{1-P}$$

$\int_0^1 1/x^P\,dx = 1/(1-P)$ Converges if $P<1$

Let's consider $1-P<0 \Rightarrow P>1$

$$\int_0^1 \frac{1}{x^P}\,dx = \lim_{t\to 0^+} \frac{1-t^{1-P}}{1-P} = \frac{1-0^{1-P}}{1-P}$$

If $1-P<0 \Rightarrow 0^{1-P} \to \infty$ since $0^{-1} = \frac{1}{0} \to \infty$

or $0^{-2} = \frac{1}{0} \to \infty$

$\therefore$ If $1-P<0$ or $P>1$

$$\int_0^1 \frac{1}{x^P}\,dx = \lim_{t\to 0^+} \frac{1-t^{1-P}}{1-P} = \infty \quad \text{Diverges if } P>1$$

Now that we have considered $P<1$ and $P>1$

Let's consider $P=1$

$$\int_0^1 \frac{1}{x^p}\,dx \qquad \text{case } p=1$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln x \Big|_t^1$$

$$= \lim_{t\to 0^+} \overset{\,=0}{\ln 1} - \ln t = \lim_{t\to 0^+} -\ln t = --\infty = \infty \quad \text{Diverges}$$

Recall $\lim_{t\to 0^+} \ln t = -\infty$

P integral Summary

$$\int_0^1 \frac{1}{x^p}\,dx \begin{cases} = \dfrac{1}{1-p} & \text{if } p<1 \quad \text{converges} \\ = \infty & \text{if } p \geq 1 \quad \text{Diverges} \end{cases}$$

We found earlier:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^{1/2}}\,dx = 2 \quad \text{Converges} \quad p = \frac{1}{2} < 1$$

$$\int_0^1 \frac{1}{x^2}\,dx = \infty \quad \text{Diverges} \quad p = 2 > 1$$

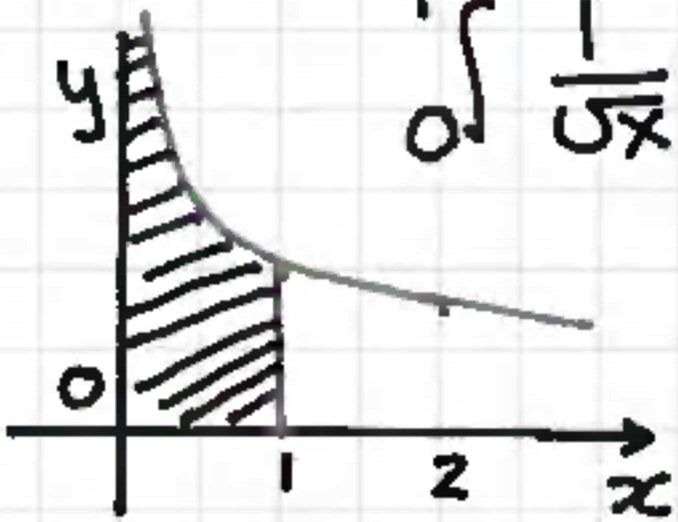
<u>Summary</u>

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} \quad \text{if } p < 1 \quad \text{Converges}$$

$$\int_0^1 \frac{1}{x^p}\,dx = \infty \quad \text{if } p \geq 1 \quad \text{Diverges}$$

**Improper P integrals with discontinuous integrand convergence explained with a diagram**

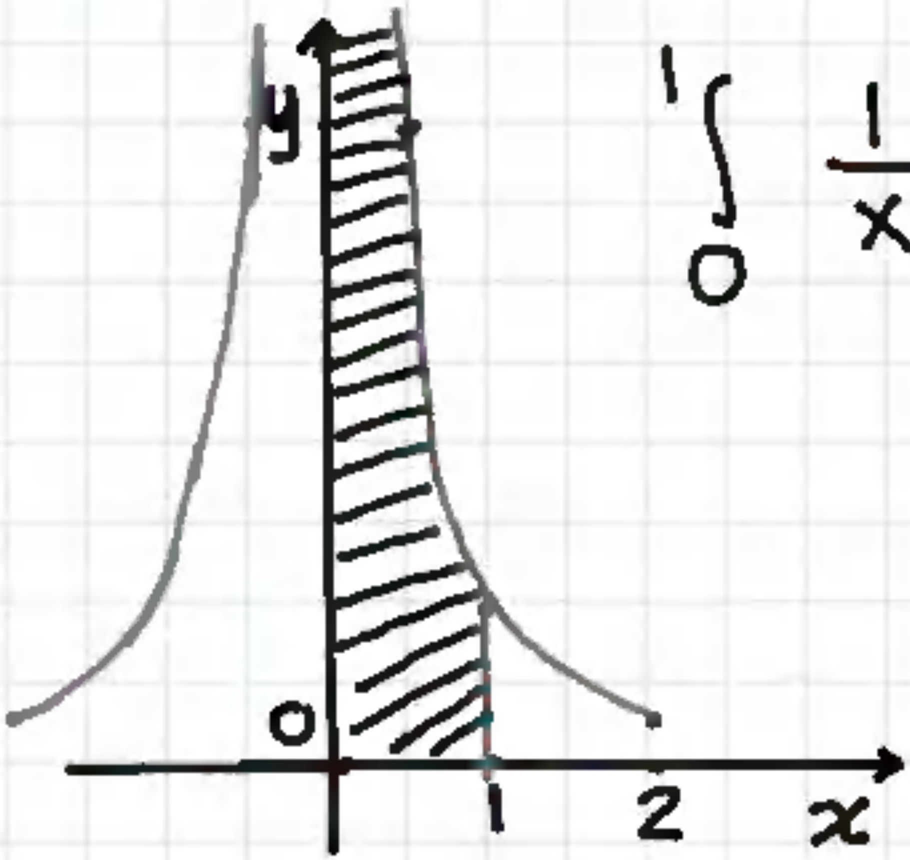
## Diagrams




$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 \quad \text{converges}$$

$$P = \frac{1}{2} < 1$$

Notice how $y = \frac{1}{\sqrt{x}}$ approaches the y axis fast and $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges.



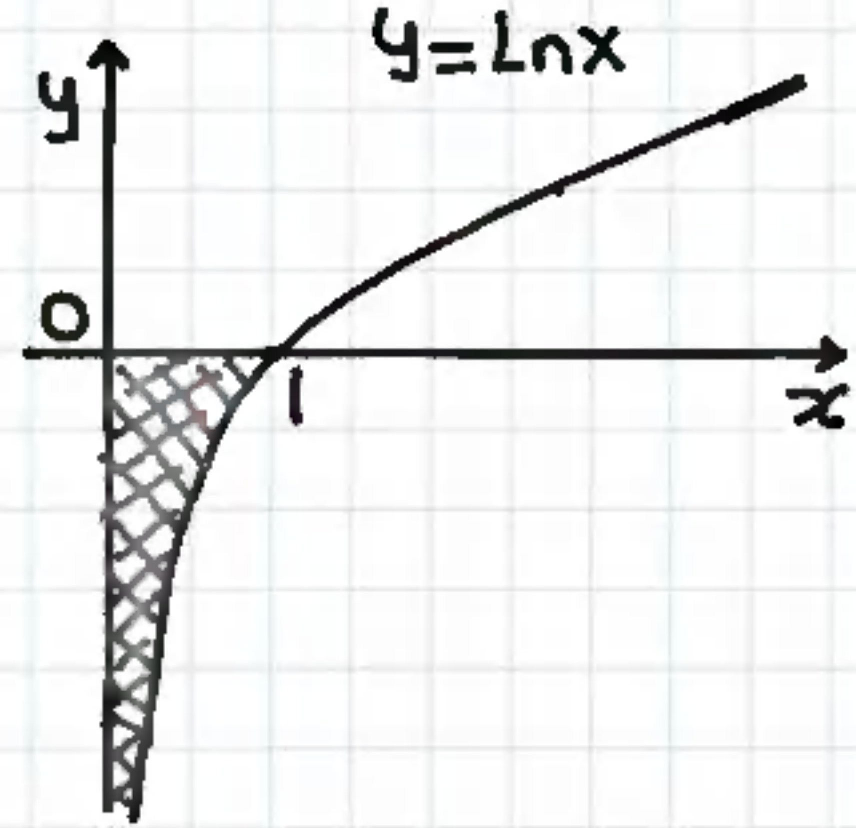

$$\int_0^1 \frac{1}{x^2}\,dx = \infty \quad \text{Diverges}$$

$$P = 2 > 1$$

Notice how $y = \frac{1}{x^2}$ approaches the y axis slow and $\int_0^1 \frac{1}{x^2}\,dx$ Diverges and goes to $\infty$.

# Improper Integrals 7

Ex 1 Evaluate $\int_0^1 \ln x \, dx$




We know that $y = \ln x$ has an infinite discontinuity at $x=0$ or a vertical asymptote at $x=0$ since $\lim_{x \to 0^+} \ln x = -\infty$

Therefore $\int_0^1 \ln x \, dx$ is improper

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

Lets Integrate by Parts: $\int u \, dv = uv - \int v \, du$

$u = \ln x \quad du = \frac{1}{x} dx \quad dv = dx \quad v = x$

$u = \ln x \quad du = \frac{1}{x}dx \quad dv = dx \quad v = x$ Integrate by Parts

$$\int_t^1 \ln x\, dx = x\ln x\Big|_t^1 - \int_t^1 x \cdot \frac{1}{x}\, dx = x\ln x - x\Big|_t^1$$

$$= 1\overset{0}{\ln 1} - 1 - (t\ln t - t) = -1 - t\ln t + t$$

$$\int_0^1 \ln x\, dx = \lim_{t \to 0^+} -t\ln t + t - 1$$

To find $\lim_{t \to 0^+} -t\ln t$ we need to apply L'Hopital's rule

$\lim_{t \to 0^+} -t\ln t = 0(-\infty)$ Indeterminate form

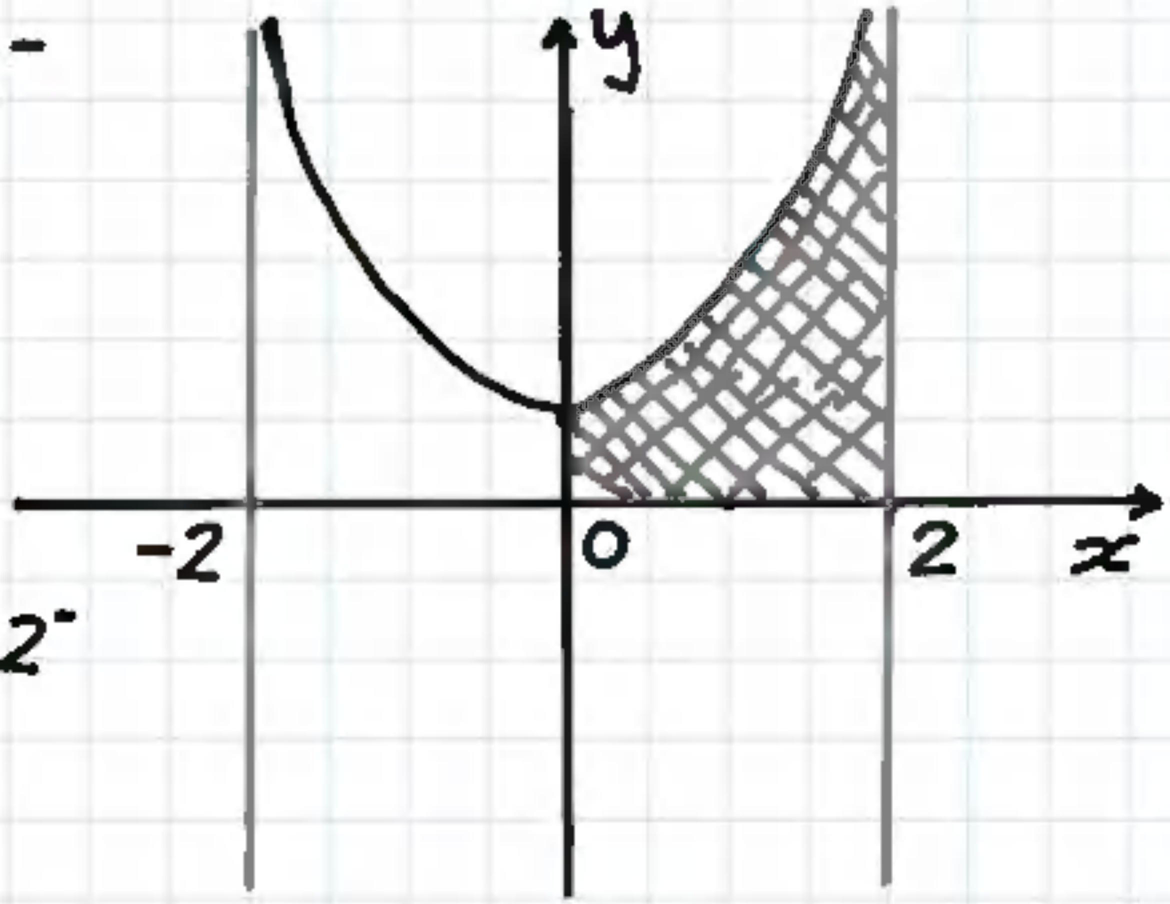
$$\lim_{t \to 0^+} -t\ln t = \lim_{t \to 0^+} \frac{-\ln t}{1/t} \overset{H}{=} \lim_{t \to 0^+} \frac{-1/t}{-1/t^2} = \lim_{t \to 0^+} t = 0$$

$$\int_0^1 \ln x\, dx = \lim_{t \to 0^+} -t\ln t + t - 1 = 0 + 0 - 1 = -1$$

Ex2| Evaluate $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

This is an improper integral. Because the integrand

$y = \frac{1}{\sqrt{4-x^2}} \to \infty$ as $x \to 2^-$




we apply improper integrals
when $x \to \pm\infty$ or $y \to \pm\infty$

Since $y = \frac{1}{\sqrt{4-x^2}} \to \infty$ as $x \to 2^-$

This is a Type 2 Improper

Integral (Discontinuous Integrand)

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{t \to 2^-} \int_0^t \frac{1}{\sqrt{4-x^2}}\,dx$$

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{t\to 2^-} \int_0^t \frac{1}{\sqrt{4-x^2}}\,dx$$

This integral has the same pattern as:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(x/a) + C$$

Table of Integrals
or U-Substitution
+ Algebra

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \int_0^2 \frac{1}{\sqrt{2^2-x^2}}\,dx$$

Apply formula
with $a=2$

$$\lim_{t\to 2^-} \int_0^t \frac{1}{\sqrt{2^2-x^2}}\,dx = \lim_{t\to 2^-} \sin^{-1}(x/2)\Big|_0^t$$

$$= \lim_{t\to 2^-} \sin^{-1}(t/2) - \sin^{-1} 0$$

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{t\to 2^-} \sin^{-1}(t/2) - \sin^{-1}0$$

$$= \sin^{-1}(2/2) - \sin^{-1}0 = \sin^{-1}(1) - \sin^{-1}(0)$$

$\sin^{-1}(1) = \frac{\pi}{2}$ since $\sin\frac{\pi}{2} = 1$

$\sin^{-1}(0) = 0$ since $\sin 0 = 0$

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{2}$$ converges

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# Improper Integrals 8

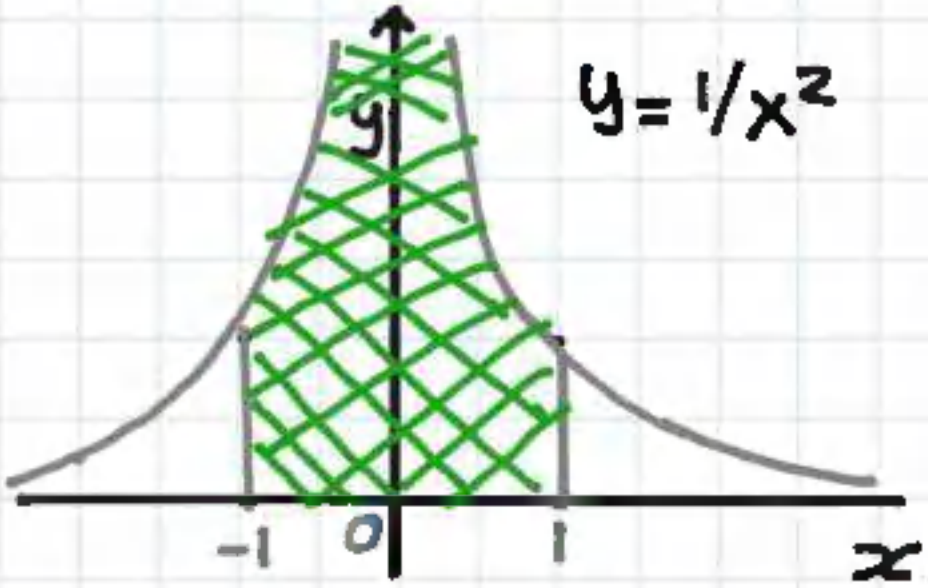
Question: What is wrong with the following computation?

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx = \left.\frac{-1}{x}\right|_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -1-1 = -2$$

Solution: Intuitively we know the answer $\int_{-1}^{1} \frac{1}{x^2}\,dx = -2$ must be wrong because the integrand $y = \frac{1}{x^2}$ is positive for $-1 < x < 1$ therefore the integral $\int_{-1}^{1} \frac{1}{x^2}\,dx$ must be positive!

## Lets illustrate with a diagram

$\int_{-1}^{1} \frac{1}{x^2}\,dx$ represents the shaded green Area and since $y = \frac{1}{x^2}$ lies above the




$x$ axis the integral $\int_{-1}^{1} \frac{1}{x^2}\,dx$ must be positive.

and therefore $\int_{-1}^{1} \frac{1}{x^2}\,dx = -2$ must be wrong!

This example shows that the conditions of the fundamental theorem of Calculus must be satisfied; $f(x)$ must be continuous and bounded on $[a,b]$. Clearly $y = \frac{1}{x^2}$ is not bounded at $x=0$

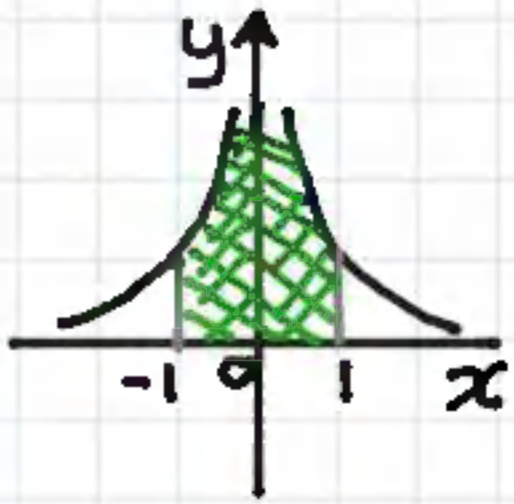
$\int_{-1}^{1} \frac{1}{x^2}\,dx$ is an improper integral because the integrand $f(x)=\frac{1}{x^2}$ is unbounded and has a vertical asymptote at $x=0$ since $\lim_{x\to 0} \frac{1}{x^2} \to \infty$

Correct way!

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

Notice that $f(x)=\frac{1}{x^2}$ is an even function since $f(-x)=f(x)$ or $f(x)=\frac{1}{x^2}$ is symmetric with respect to y axis.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx = 2\int_{0}^{1} \frac{1}{x^2}\,dx$$




Since $f(x) = \frac{1}{x^2}$ is an even function (symmetric with respect to y axis)

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = 2\int_{0}^{1} \frac{1}{x^2}\,dx$$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = 2\int_{0}^{1} \frac{1}{x^2}\,dx = \lim_{t\to 0^+} 2\int_{t}^{1} x^{-2}\,dx = \lim_{t\to 0^+} \frac{2x^{-1}}{-1}\Big|_{t}^{1}$$

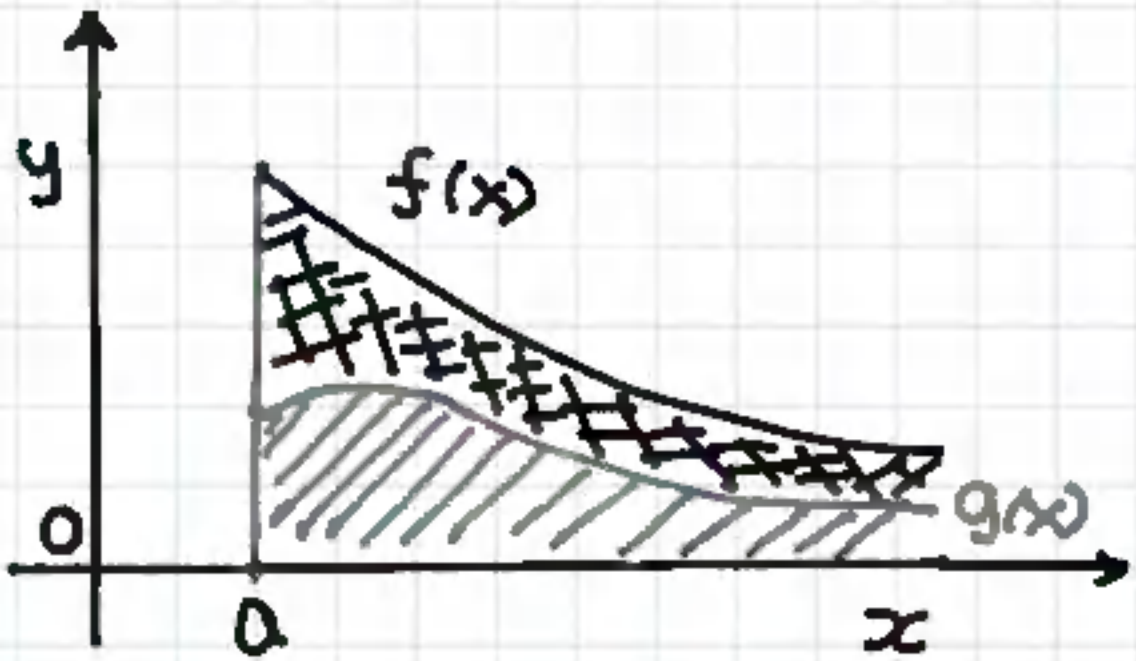
$$= \lim_{t\to 0^+} \frac{-2}{x}\Big|_{t}^{1} = \lim_{t\to 0^+} \frac{-2}{1} - -\frac{2}{t} = \lim_{t\to 0^+} -2 + \frac{2}{t}$$

$$= \lim_{t\to 0^+} -2 + \lim_{t\to 0^+} \frac{2}{t} = -2 + \infty = \infty$$ Diverges

Therefore we have shown that:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \infty \quad \text{Diverges}$$

## Improper Integrals 9




Key Concept:
Since $f(x) \geq g(x)$ for $x \geq a$ then area under $f(x)$ and above the $x$ axis is larger than the area below $g(x)$ and above $x$ axis

Comparison Test for Improper Integrals

Assume that $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, \infty)$

a) If $\int_a^\infty f(x)\,dx$ is convergent $\Rightarrow \int_a^\infty g(x)\,dx$ is also

and $\int_a^\infty g(x)\,dx \leq \int_a^\infty f(x)\,dx$ **CONVERGENT**

b) If $\int_a^\infty g(x)\,dx$ Diverges $\Rightarrow \int_a^\infty f(x)\,dx$ also Diverges

and $\int_a^\infty f(x)\,dx \geq \int_a^\infty g(x)\,dx$

Key Concept: When applying comparison test for Improper integrals the goal is to prove that $\int_a^\infty g(x)\,dx \leq$ <u>Convergent Integral</u>

so that we can conclude that $\int_a^\infty g(x)\,dx$ converges

Otherwise if $\int_a^\infty g(x)\,dx \leq$ Divergent Integral

No conclusion can be made.

Alternatively if you are trying to prove that $\int_a^\infty f(x)\,dx$

Diverges the goal is to prove that:

$\int_a^\infty f(x)\,dx \geq$ Divergent Integral

otherwise if $\int_a^\infty f(x)\,dx \geq$ Convergent Integral

No conclusion can be made.

Key Concept: When applying comparison theorem make sure to compare $\int_a^\infty f(x)\,dx$ to a <u>simpler integral</u> whose convergence or Divergence can be <u>easily Determined.</u>

Ex| Prove that $\int_0^\infty e^{-x^2}\,dx$ converges.

It is impossible to integrate $f(x)=e^{-x^2}$ because the antiderivative of $f(x)=e^{-x^2}$ cannot be expressed in terms of elementary functions.
Let's split up the integral into the sum of two separate integrals.

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

Consider $\int_0^1 e^{-x^2}\,dx$ ; since $e^{-x^2}$ is a continuous function on $[0,1]$ the definite integral $\int_0^1 e^{-x^2}\,dx$ converges and has finite value.

$$\int_0^{\infty} e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^{\infty} e^{-x^2}dx$$

Lets now consider $\int_1^{\infty} e^{-x^2}dx$

Since $x \geq 1 \Rightarrow x^2 \geq x \Rightarrow -x^2 \leq -x \Rightarrow e^{-x^2} \leq e^{-x}$

$$\Rightarrow \int_1^{\infty} e^{-x^2}dx \leq \int_1^{\infty} e^{-x}dx$$

$$\int_1^{\infty} e^{-x}dx = \lim_{t\to\infty} \int_1^{t} e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t = \lim_{t\to\infty} -e^{-t} - -e^{-1}$$

$$= \lim_{t\to\infty} -e^{-t} + \frac{1}{e} = 0 + \frac{1}{e} = \frac{1}{e}$$

Recall: $\lim_{t\to\infty} e^{-t} = 0$

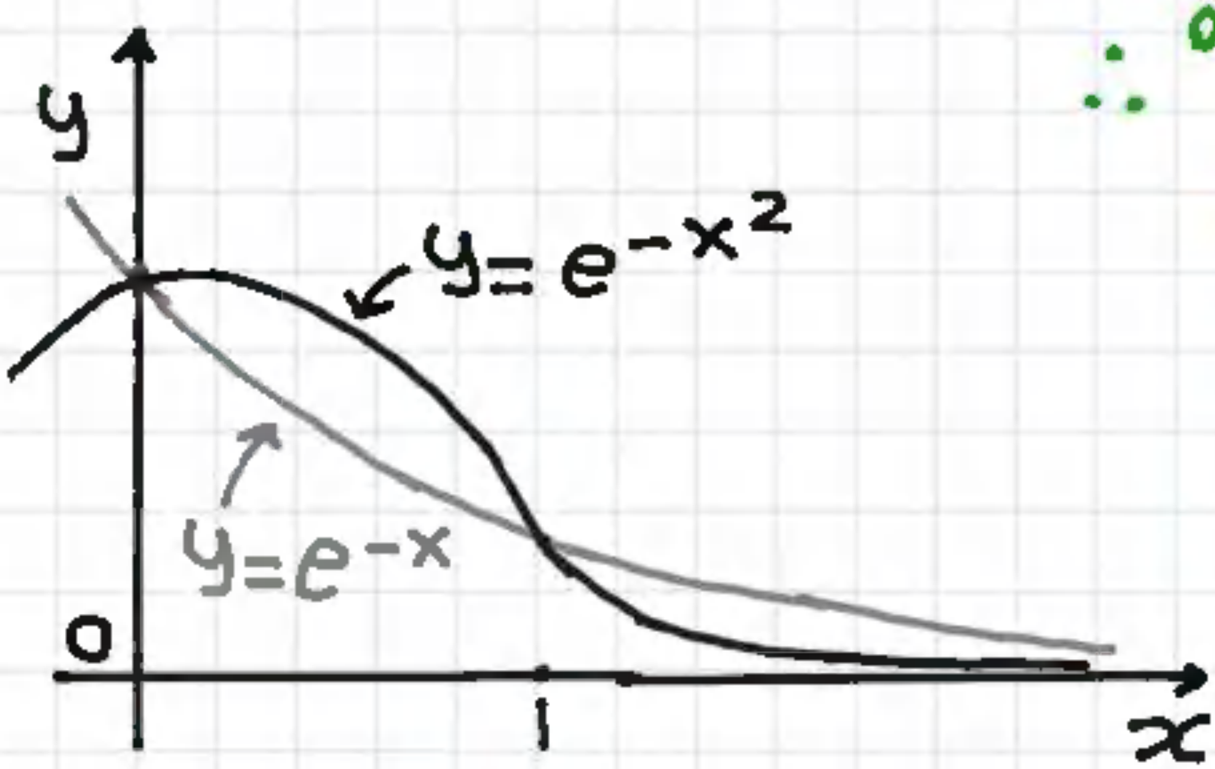
$$\therefore \int_1^{\infty} e^{-x^2}dx \leq \int_1^{\infty} e^{-x}dx = \frac{1}{e} \quad \text{Converges}$$

$\therefore$ By comparison theorem $\int_1^\infty e^{-x^2}dx$ also Converges

Summary: $$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{Convergent}} + \underbrace{\int_1^\infty e^{-x^2}dx}_{\text{Convergent}}$$

Since $\int_0^\infty e^{-x^2}dx$ is the sum of two convergent integrals we conclude that $\int_0^\infty e^{-x^2}dx$ converges.

## Diagrams




## Key Concept:

When $x \geq 1$ $e^{-x^2} \leq e^{-x}$

$\therefore \int_1^{\infty} e^{-x^2} dx \leq \int_1^{\infty} e^{-x} dx = \frac{1}{e}$

Converges

$\therefore$ By Comparison Theorem $\int_1^{\infty} e^{-x^2} dx$ also converges.

# Improper Integral proving convergence by comparison theorem solved example

## Improper Integrals 10

Ex 1| Does $\int_1^\infty \frac{1-e^{-x^2}}{x^4}\,dx$ converge or Diverge?

Let's apply comparison theorem for improper integrals.

$x>1 \Rightarrow x^2>1 \Rightarrow -x^2<-1 \Rightarrow e^{-x^2}<e^{-1}$

we also know that $e^{-x^2}>0 \Rightarrow 0<e^{-x^2}<\frac{1}{e}$

$0>-e^{-x^2}>-\frac{1}{e}$

$1>1-e^{-x^2}>1-\frac{1}{e} \Rightarrow 1-\frac{1}{e}<1-e^{-x^2}<1$

$0<\frac{1-1/e}{x^4}<\frac{1-e^{-x^2}}{x^4}<\frac{1}{x^4}$

$$0 < \frac{1-e^{-x^2}}{x^4} < \frac{1}{x^4} \Rightarrow \int_1^\infty \frac{1-e^{-x^2}}{x^4}\,dx \le \int_1^\infty \frac{1}{x^4}\,dx$$

Since $\int_1^\infty \frac{1}{x^4}\,dx$ is a convergent P integral with $P=4$ therefore $\int_1^\infty \frac{1-e^{-x^2}}{x^4}\,dx$ Converges

by the comparison test for improper integrals.

Big Picture : $\int_1^\infty \frac{1-e^{-x^2}}{x^4}\,dx <$ Convergent Improper Integral

Therefore $\int_1^\infty \frac{1-e^{-x^2}}{x^4}\,dx$ Converges

**Comparison test for improper integrals solved example**

Ex. 2 Does $\int_1^\infty \frac{2+\sin x}{x^2}\,dx$ converge or Diverge ?

Let's Apply comparison test for improper integrals.

We will start by building up an upper bound for the integrand $\frac{2+\sin x}{x^2}$

$$-1 \le \sin x \le 1 \implies 1 \le 2+\sin x \le 3$$

$$\frac{1}{x^2} \le \frac{2+\sin x}{x^2} \le \frac{3}{x^2}$$

$$\int_1^\infty \frac{1}{x^2}\,dx \le \int_1^\infty \frac{2+\sin x}{x^2}\,dx \le \int_1^\infty \frac{3}{x^2}\,dx = 3\int_1^\infty \frac{1}{x^2}\,dx$$

$$\int_1^\infty \frac{1}{x^2}\,dx \le \int_1^\infty \frac{2+\sin x}{x^2}\,dx \le 3\int_1^\infty \frac{1}{x^2}\,dx$$

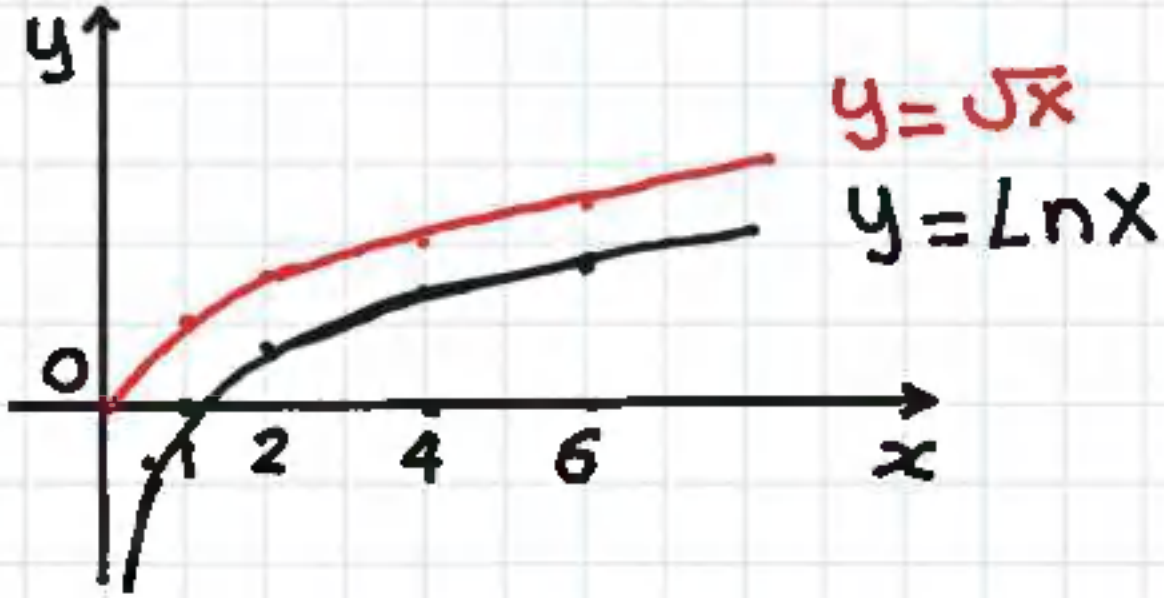
Since $\int_1^\infty \frac{1}{x^2}\,dx$ is a convergent P integral with $P=2$ therefore $\int_1^\infty \frac{2+\sin x}{x^2}\,dx$ converges by the comparison test for improper integrals.

Big Picture: $\int_1^\infty \frac{2+\sin x}{x^2}\,dx <$ Convergent Improper Integral

therefore $\int_1^\infty \frac{2+\sin x}{x^2}\,dx$ converges

Ex.3 Does $\int_3^\infty \frac{1}{\sqrt{x}\,\ln x}\,dx$ Converge or Diverge ?

Lets start by analyzing the Denominator $\sqrt{x}\,\ln x$




It is clear from the graphs of $\sqrt{x}$ and $\ln x$ that $\sqrt{x} > \ln x$ for $x \geq 1$

Let's build up inequality for the integrand $y = 1/(\sqrt{x}\,\ln x)$ so we can apply comparison test for integrals.

$\sqrt{x} > \ln x$ for $x > 1$ $\Rightarrow$ $\sqrt{x} > \ln x$ for $x \geq 3$

$\sqrt{x}\,\boxed{\ln x} < \sqrt{x}\cdot\boxed{\sqrt{x}}$ $\Rightarrow$ $\sqrt{x}\ln x < x \Rightarrow \dfrac{1}{\sqrt{x}\ln x} > \dfrac{1}{x}$

$$\int_3^{\infty} \frac{1}{\sqrt{x}\ln x}\,dx > \int_3^{\infty} \frac{1}{x}\,dx$$

Since $\int_3^{\infty} \frac{1}{x}\,dx$ is a Divergent P integral with $P=1$

therefore $\int_3^{\infty} \frac{1}{\sqrt{x}\ln x}\,dx$ Diverges by the comparison test for improper Integrals.

Big Picture: $\int_3^{\infty} \frac{1}{\sqrt{x}\ln x}\,dx >$ Divergent Improper Integral

$\therefore$ Diverges

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