Infinite Sequences I

Basic idea: Ordered list of numbers with a pattern:

$\{a_1, a_2, a_3, \dots, a_n, a_{n+1}, \dots\}$

For every positive integer n, there is a corresponding number a_n so we can define $a_n = f(n)$, that is a sequence can be defined as a special function $f(n)$ whose domain is the set of positive integers n.

ex] $f(n) = a_n = n \qquad n \geq 1$

$$\{a_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty} = \{\underset{a_1}{1}, \underset{a_2}{2}, 3, 4, \underset{a_5}{5}, \dots, \underset{a_n}{n}, \dots\}$$

ex] $f(n) = a_n = \dfrac{(-1)^n \cdot n}{2^n}, \quad n \geq 1$

$$\left\{ \frac{(-1)^n n}{2^n} \right\}_{n=1}^{\infty} = \left\{ \underset{\substack{\downarrow \\ a_1 = f(1)}}{\frac{-1}{2}}, \frac{2}{4}, \underset{\substack{\downarrow \\ a_3 = f(3)}}{\frac{-3}{8}}, \frac{4}{16}, \ldots, \underset{\substack{\downarrow \\ a_n = f(n)}}{\frac{(-1)^n \cdot n}{2^n}}, \ldots \right\}$$

ex) $a_n = \sin(n\pi/6)$, $n \geq 0$

$$\{a_n\}_{n=0}^{\infty} = \{\sin(n\pi/6)\}_{n=0}^{\infty} = \left\{ \underset{\substack{\downarrow \\ a_0 = f(0)}}{0}, 1/2, \underset{\substack{\downarrow \\ a_2 = f(2)}}{\frac{\sqrt{3}}{2}}, 1, \ldots, \underset{\substack{\downarrow \\ a_n = f(n)}}{\sin(n\pi/6)}, \ldots \right\}$$

Finding a formula for the general term a_n of a sequence requires us to find a pattern such that $a_n = f(n)$.

Ex| Find a formula for the general term a_n of the infinite sequence given by $\{2/4, 3/16, 4/64, 5/256, \ldots\}$

Notice that the Numerator of all the fractions starts with 2 and increases by 1, as we advance to the next term, so we can write the numerators as:

$\frac{1+1}{\square}, \frac{2+1}{\square}, \frac{3+1}{\square}, \frac{4+1}{\square}, \ldots, \frac{n+1}{\square}$; so numerator of $a_n = n+1$

The denominators are powers of 4, so we can write the denominator as: $\frac{\square}{4^1}, \frac{\square}{4^2}, \frac{\square}{4^3}, \frac{\square}{4^4}, \ldots, \frac{\square}{4^n}$; so we can write the denominator as 4^n. Therefore we can write the general term $f(n) = a_n = \frac{n+1}{4^n}$ or $\left\{\frac{n+1}{4^n}\right\}_{n=1}^{\infty}$

Note: Not all sequences have explicit formulas $f(n)=a_n$

Ex) The sequence whose nth term is the nth decimal digit of the number π; $\pi = 3.14159265358 9\ldots$

$$\{a_n\}_{n=1}^{\infty} = \{1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, \ldots\}$$

Ex) The sequence $\{a_n\}_{n=1}^{\infty}$ of prime integers

$$\{a_n\}_{n=1}^{\infty} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots\}$$

A sequence $a_n = f(n) = \{a_n\}_{n=1}^{\infty}$ is said to converge

If $\lim\limits_{n\to\infty} a_n = L$, that is as n becomes arbitarily large, the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ get closer and closer to L.

Ex] Consider $a_n = \frac{n}{n+2}$, $n \geq 1$; Find the limit of this sequence.

$a_n = \frac{n}{n+2}$, $n \geq 1$

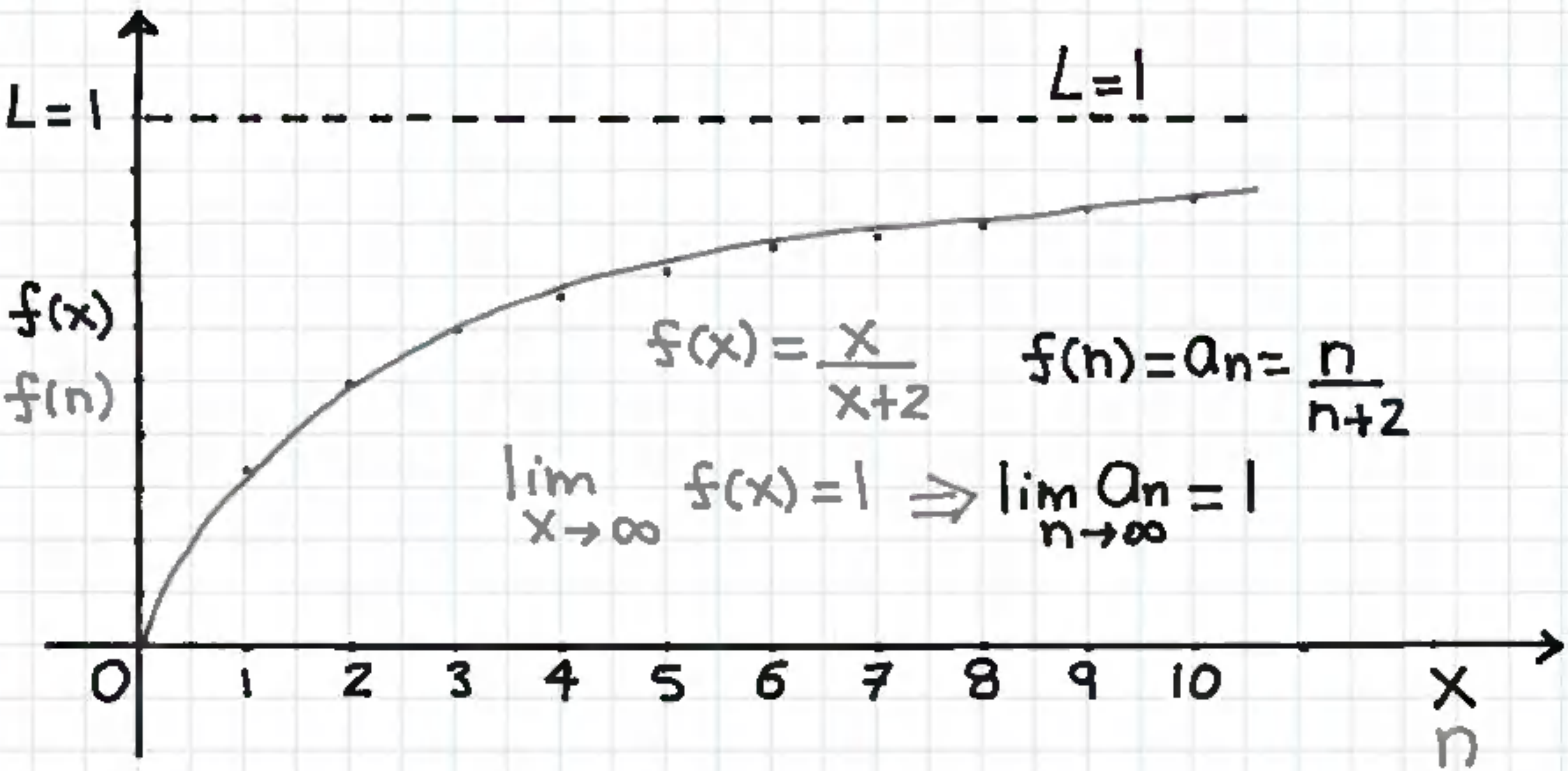
$a_1 = 1/3$, $a_2 = 2/4$, $a_3 = 3/5$, $a_4 = 4/6$, ..., $a_{100} = 100/102$

It appears that the terms of the sequence approach the limit 1 as n becomes large, more formally

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{n/n}{\frac{n}{n}+\frac{2}{n}} = \lim_{n\to\infty} \frac{1}{1+\underbrace{2/n}_{\searrow 0}} = 1$$

Theory: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, n is an integer then $\lim_{n\to\infty} a_n = L$; treat sequence $a_n = f(n)$ as $f(x)$ and take $\lim_{x\to\infty} f(x) = L$ to find $\lim_{n\to\infty} f(n) = L$

let's graph $a_n = n/(n+2)$; graph (n, a_n) ; $(n, n/(n+2))$



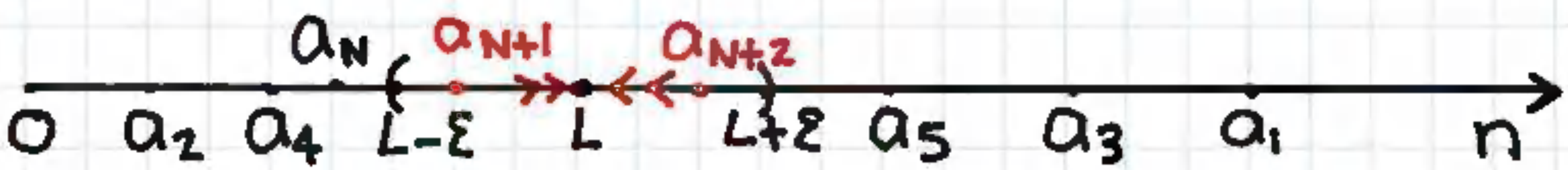
Graphically it is clear as n becomes large, the terms of the sequence $a_n = n/(n+2)$ approach a limiting value of 1 ; $\lim_{n\to\infty} a_n = 1$

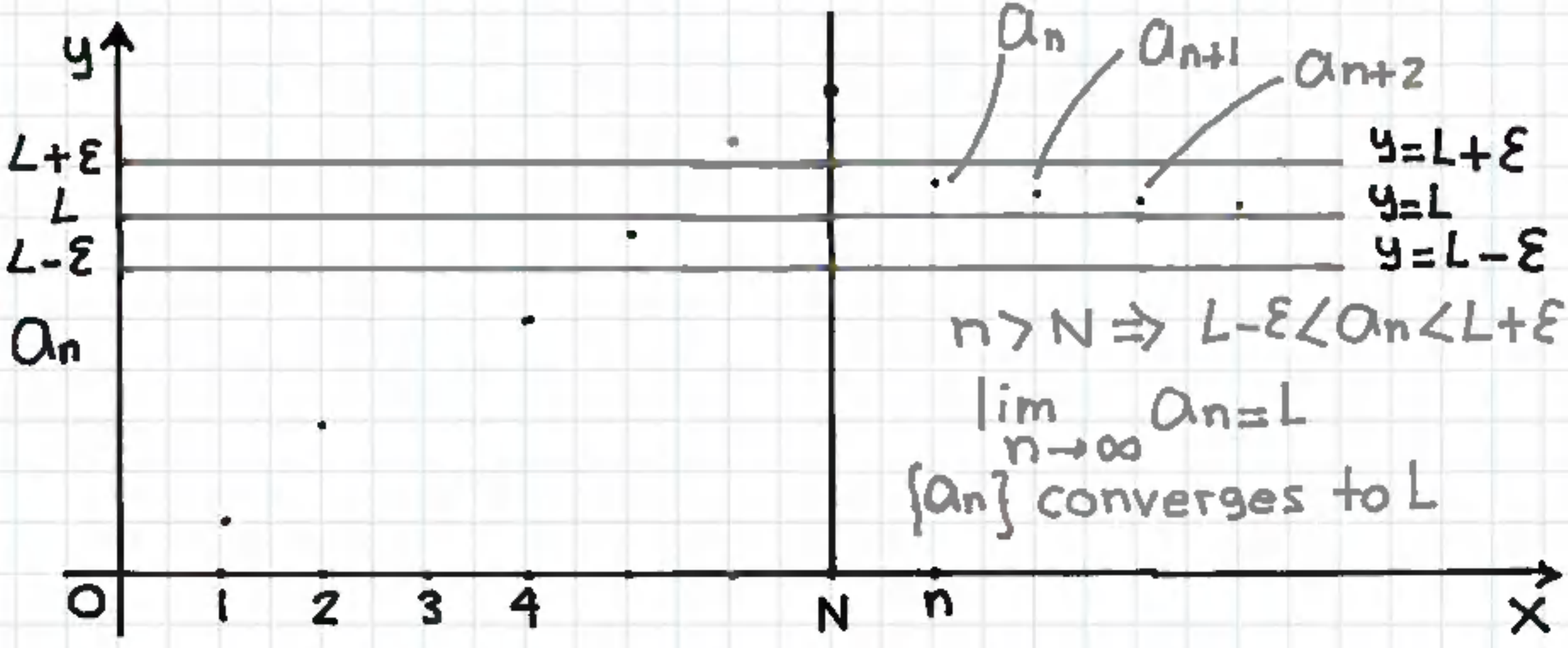
Infinite Sequences 2

Theory: Formal definition of limit of a sequence

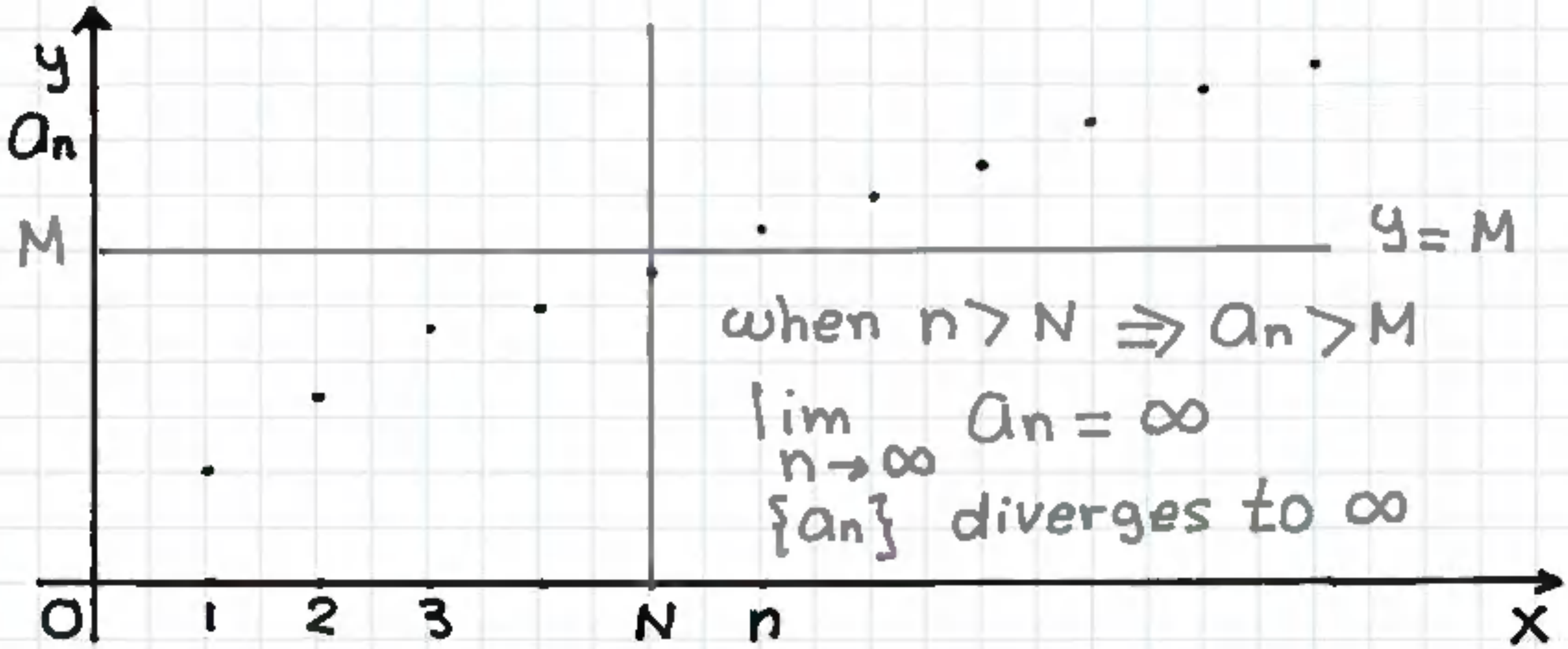
Sequence $a_n = f(n)$, $\{a_n\}_{n=1}^{\infty}$ converges to L if the terms of a_n can be made arbitarily close to L by choosing n large enough, more precisely, $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ provided that for every $\varepsilon > 0$, there exists an integer N such that if $n > N \Rightarrow |a_n - L| < \varepsilon$

$$-\varepsilon < a_n - L < \varepsilon \Rightarrow L - \varepsilon < a_n < L + \varepsilon$$





Alternatively, we say $\lim_{n\to\infty} a_n = \infty$ if for each positive number M there is an integer N such that if $n > N \Rightarrow a_N > M$



Ex] Consider $f(n) = a_n = \dfrac{n}{n+2}$, $n \geq 1$

a) Given $\varepsilon = 0.03$, find an integer N such that $|a_n - L| < \varepsilon$ when $n > N$

b) Prove that $\lim_{n \to \infty} a_n = 1$ by applying formal definition of limit of a sequence.

a) Soln: We proved in the previous lesson that

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n+2} = 1$, we need to find an integer N

such that $|a_n - 1| < \varepsilon = 0.03$ when $n > N$

$$|a_n - 1| = \left|\frac{n}{n+2} - 1\right| = \left|\frac{n-(n+2)}{n+2}\right| = \left|\frac{-2}{n+2}\right| = \left|\frac{2}{n+2}\right| < 0.03$$

since $n \geqslant 1 \Rightarrow \left|\frac{2}{n+2}\right| = \frac{2}{n+2} \Rightarrow \frac{2}{n+2} < 0.03$

$$\Rightarrow \frac{n+2}{2} > \frac{1}{0.03} \Rightarrow n+2 > \frac{2}{0.03} \Rightarrow n+2 > 66.\overline{66}$$

$\Rightarrow n > 64.\overline{66} \Rightarrow n = 65$ (round up to next integer)

$\therefore$ $N = 64$ will suffice since $n = 65 > N = 64$

b) Soln: For every $\varepsilon > 0$, we have to find an integer N (dependent on ε) such that $|a_n - 1| < \varepsilon$ when $n > N$ so that we can prove that $\lim_{n\to\infty} a_n = 1$

$$\left| a_n - 1 \right| = \left| \frac{n}{n+2} - 1 \right| = \left| \frac{2}{n+2} \right| = \frac{2}{n+2} \Rightarrow \left| a_n - 1 \right| = \frac{2}{n+2} < \varepsilon$$

let's solve inequality for n:

$$\frac{2}{n+2} < \varepsilon \Rightarrow \frac{n+2}{2} > \frac{1}{\varepsilon} \Rightarrow n+2 > \frac{2}{\varepsilon} \Rightarrow n > \frac{2}{\varepsilon} - 2$$

$\therefore$ For every $\varepsilon > 0$ we can find $N \geq \frac{2}{\varepsilon} - 2$ such that the terms of $a_n = \frac{n}{n+2}$ for $n > N$ lie within $(1-\varepsilon, 1+\varepsilon)$, and since we can find an integer N

for all $\varepsilon > 0$ the limit of $a_n = \dfrac{n}{n+2}$ exists and equals 1

So we have proven that $|a_n - L| < \varepsilon$ when $n > N$

or equivalently $\left|\dfrac{n}{n+2} - 1\right| < \varepsilon$ when $n > \dfrac{2}{\varepsilon} - 2$

Remark: $N = \dfrac{2}{\varepsilon} - 2$ may not be guaranteed to be an integer, simply round up to the next positive integer to find n.

Let's check calculation in part a with $\varepsilon = 0.03$

$$N = \frac{2}{\varepsilon} - 2 = \frac{2}{0.03} - 2 = 64.\overline{66} \Rightarrow \text{round up to } n = 65$$

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Infinite Sequences 3

limit laws for sequences $a_n = f(n)$

Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ and C is a constant then :

1] $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$

2] $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$

3] $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n = LM$

4] $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n = L/M$

if $\lim_{n\to\infty} b_n \neq 0$ (can't divide by 0)

5] $\lim_{n\to\infty} [a_n]^p = [\lim_{n\to\infty} a_n]^p = L^p$ assuming $p>0$

and $a_n > 0$

Squeeze Law For limits of Sequences :

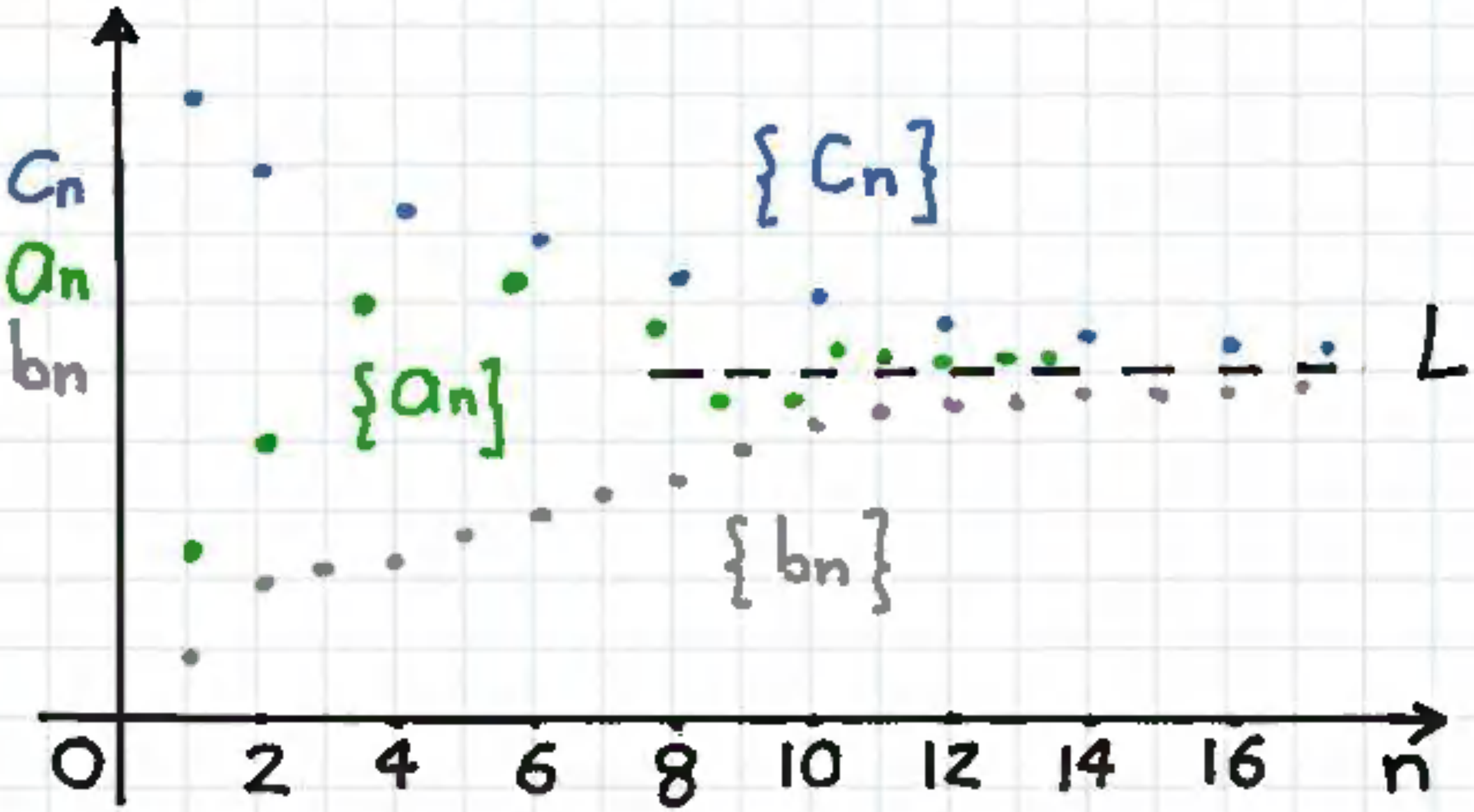
Assume $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences

with $b_n \le a_n \le c_n$ for some $n > N$.

If $\lim_{n\to\infty} b_n = L$ and $\lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} a_n = L$

$$\underbrace{\lim_{n\to\infty} b_n}_{\Downarrow L} \le \underbrace{\lim_{n\to\infty} a_n}_{\Downarrow L} \le \underbrace{\lim_{n\to\infty} c_n}_{\Downarrow L}$$

SQZ Law



The sequence $\{a_n\}$ is getting squeezed more and more between the sequences $\{b_n\}$ and $\{c_n\}$ whose converging limit is L as $n \to \infty$, hence $\lim_{n\to\infty} a_n = L$

Ex] Apply Squeeze law to find the limit of the sequence $a_n = \dfrac{\sin n}{n^2+1}$

Our goal is to find two sequences $\{b_n\}$ and $\{c_n\}$ such that $\{a_n\}$ is sandwiched between $\{b_n\}$ and $\{c_n\}$

Let's start with $-1 \le \sin n \le 1$ for all n

$$\underbrace{\frac{-1}{n^2+1}}_{b_n} \le \underbrace{\frac{\sin n}{n^2+1}}_{a_n} \le \underbrace{\frac{1}{n^2+1}}_{c_n} \Rightarrow b_n \le a_n \le c_n$$

Let $b_n = \dfrac{-1}{n^2+1}$; $a_n = \dfrac{\sin n}{n^2+1}$; $c_n = \dfrac{1}{n^2+1}$

and since $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{-1}{n^2+1} = 0$

and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$ then:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin n}{n^2+1} = 0 \text{ by Squeeze Law}$$

Theory: If $\lim_{n\to\infty} a_n = L$ and the function $f(x)$ is continuous at $x=L$ then:

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

Ex] Compute the limit of the sequence $a_n = \cos(\pi/n)$ as $n\to\infty$

Since $f(x)=\cos x$ is continuous for all x in particular at $x=0$ we obtain:

$$\lim_{n\to\infty} \cos(\pi/n) = \lim_{n\to\infty} \cos(\lim_{n\to\infty} \pi/n) = \cos 0 = 1$$

Ex] Calculate the limits of the following sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ as $n \to \infty$

a) $a_n = \dfrac{n^2}{n+1}$

b) $b_n = \sqrt{\dfrac{2n-1}{4n+2}}$

c) $c_n = e^{1/n} + \cos(2/n) + \tan^{-1}(3n)$

a) Soln. $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n^2/n}{\frac{n}{n}+\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{n}{1+\underbrace{1/n}_{\to 0}} = \frac{\infty}{1+0} = \infty$$

since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n+1} = \infty$

$\therefore \{a_n\}_{n=1}^{\infty}$ diverges to ∞ as $n \to \infty$

b) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sqrt{\frac{2n-1}{4n+2}} = \sqrt{\lim_{n\to\infty} \frac{2n-1}{4n+2}}$

$= \sqrt{\lim_{n\to\infty} \frac{2n/n - 1/n}{4n/n + 2/n}} = \sqrt{\lim_{n\to\infty} \frac{2-1/n}{4+2/n}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}}$

$\therefore \{b_n\}$ converges to $\frac{1}{\sqrt{2}}$ as $n \to \infty$

c) $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \left[e^{1/n} + \cos(2/n) + \tan^{-1}(3n)\right]$

$= \lim_{n\to\infty} e^{1/n} + \lim_{n\to\infty} \cos(2/n) + \lim_{n\to\infty} \tan^{-1}(3n)$

$$\lim_{n\to\infty} C_n = e^{\lim_{n\to\infty}(1/n)} + \cos(\lim_{n\to\infty} 2/n) + \tan^{-1}(\lim_{n\to\infty} 3n)$$

$$\lim_{n\to\infty} C_n = e^0 + \cos 0 + \tan^{-1}(\infty)$$

$$\lim_{n\to\infty} C_n = 1 + 1 + \pi/2 = 2 + \pi/2$$

$\therefore \{C_n\}$ converges to $2+\pi/2$ as $n\to\infty$

Infinite Sequences 4

Ex] Find the limit of the following sequences.

a) $a_n = (\ln n)^2 / 2n$

b) $b_n = (1 + 1/n)^n$

c) $c_n = 2^n / n!$

d) $d_n = \dfrac{(-1)^n \, n^{2/3}}{(n^{2/3} + 3n)}$

e) $e_n = n - \sqrt{n^2 - 3n}$

a) Soln. $a_n = (\ln n)^2 / 2n$

Since $(\ln n)^2 \to \infty$ as $n \to \infty$ and $2n \to \infty$ as $n \to \infty$

Since $\lim\limits_{n\to\infty} (\ln n)^2/2n$ has ∞/∞ pattern we apply L'Hopital's rule.

Let $f(x) = (\ln x)^2/2x \Rightarrow \lim\limits_{x\to\infty} (\ln x)^2/2x \overset{H}{=} \dfrac{\cancel{2}\ln x \cdot 1/x}{\cancel{2}}$

$= \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (\ln n)^2/2n = 0$

$\therefore \{a_n\} = \left\{\dfrac{(\ln n)^2}{2n}\right\}$ converges to 0

b) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} (1+1/n)^n$ pattern 1^∞

Apply Log function to change pattern from 1^∞ to $0/0$

Let $y = (1+1/x)^x \Rightarrow \ln y = \ln(1+1/x)^x = x\ln(1+1/x)$

$$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \cdot \ln(1+1/x) = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x}$$

now we have $0/0$ pattern as $\ln(1+1/x) \to 0$

and $1/x \to 0$ as $x \to \infty$ Recall $\ln 1 = 0$

$\therefore$ Apply L'Hopital's rule to $\ln y$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+1/x} \cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{1}{(1+1/x)} = 1$$

$$\therefore \lim_{x\to\infty} \ln y = 1 \Rightarrow \lim_{x\to\infty} e^{\ln y} = e^1 \Rightarrow \lim_{x\to\infty} y = e$$

but $y = (1+1/x)^x \Rightarrow \lim_{x\to\infty} (1+1/x)^x = e \Rightarrow \lim_{n\to\infty} (1+\frac{1}{n})^n = e$

$\therefore \{b_n\} = \{1+\frac{1}{n}\}^n$ converges to e.

C) $C_n = 2^n/n!$ where $n! = 1\times2\times3\times\ldots\times n$

$C_1 = \frac{2^1}{1!}$; $C_2 = \frac{2\times2}{2\times1}$; $C_3 = \frac{2\times2\times2}{3\times2\times1}$; $C_n = \frac{2\times2\times\ldots\times2}{1\times2\times\ldots\times n}$

It looks like the fractions are decreasing as the denominator grows faster than the Numerator

$$C_n = \frac{2}{1}\times\frac{2}{2}\cdot\left[\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{2}{5}\cdot\frac{2}{6}\cdots\times\frac{2}{n-2}\times\frac{2}{n-1}\times\frac{2}{n}\right]$$

$$C_n = 2\times 1\times \frac{2}{n}\left[\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{2}{5}\cdots\times\frac{2}{n-2}\times\frac{2}{n-1}\right]$$

Note $[2/3 \times 2/4 \times \cdots \times 2/(n-2) \times 2/(n-1)] < 1$ since all the fractions are less than 1

$$\therefore C_n = 2\times\frac{2}{n}\Big[\underbrace{\frac{2}{3}\times\frac{2}{4}\times\frac{2}{5}\times\cdots\times\frac{2}{n-2}\times\frac{2}{n-1}}_{\text{Less than } 1}\Big] < \frac{4}{n}$$

$\therefore 0 < C_n < 4/n$ since all terms are positive

Now apply Squeeze Law

$$\underbrace{\lim_{n\to\infty} 0}_{0} < \underbrace{\lim_{n\to\infty} C_n}_{0 \text{ By Squeeze Law}} < \underbrace{\lim_{n\to\infty} (4/n)}_{0} \Rightarrow 0 < \underbrace{\lim_{n\to\infty} C_n}_{0 \text{ Squeeze Law}} < 0$$

$\therefore$ By Squeeze Law $\lim_{n\to\infty} c_n = 0 \Rightarrow \{c_n\}$ converges to 0

d) $\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{(-1)^n\, n^{2/3}}{n^{2/3}+3n}$ Apply Squeeze Law

$$\frac{-n^{2/3}}{(n^{2/3}+3n)} \leq \frac{(-1)^n \cdot n^{2/3}}{(n^{2/3}+3n)} \leq \frac{n^{2/3}}{(n^{2/3}+3n)}$$

$$\lim_{n\to\infty} \frac{-n^{2/3}}{(n^{2/3}+3n)} = \lim_{n\to\infty} \frac{-n^{2/3}/n}{(n^{2/3}/n + 3n/n}$$

$$= \lim_{n\to\infty} \frac{-1/n^{1/3}}{(1/n^{1/3}+3)} = \frac{0}{3} = 0$$

Similiarly $\lim_{n\to\infty} \frac{n^{2/3}}{(n^{2/3}+3n)} = 0$

$\therefore$ By Squeeze Law $\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{(-1)^n n^{2/3}}{(n^{2/3}+3n)} = 0$

e) $\lim_{n\to\infty} e_n = \lim_{n\to\infty} \left(n - \sqrt{n^2-3n}\right)$

let's multiply by conjugate $\left(n+\sqrt{n^2-3n}\right)$

$$\lim_{n\to\infty} \left(n-\sqrt{n^2-3n}\right) \cdot \frac{\left(n+\sqrt{n^2-3n}\right)}{\left(n+\sqrt{n^2-3n}\right.}$$

$$= \lim_{n\to\infty} \frac{\left(n^2-(n^2-3n)\right)}{\left(n+\sqrt{n^2-3n}\right)} = \lim_{n\to\infty} \frac{3n}{\left(n+\sqrt{n^2-3n}\right)}$$

$$= \lim_{n\to\infty} \frac{3n/n}{\left(n/n+\sqrt{n^2/n^2-3n/n^2}\right.}$$

$$= \lim_{n\to\infty} \frac{3}{\left(1+\sqrt{1-\underbrace{3/n}_{0}}\right)} = \frac{3}{(1+1)} = \frac{3}{2}$$

$$\therefore \{e_n\} = \{n - \sqrt{n^2-3n}\} \text{ converges to } 3/2$$

Infinite Sequences 5 (Sequences Definitions)

$\{a_n\}$ is increasing if $a_{n+1} > a_n$, ex. $\{1, 2, 4, 8, 16, \ldots\}$

$\{a_n\}$ is decreasing if $a_{n+1} < a_n$, ex. $\{125, 25, 5, 1, \ldots\}$

$\{a_n\}$ is monotonic if it is either increasing or decreasing

$\{a_n\}$ is bounded above if $a_n \le M$ for all n.

$\{a_n\}$ is bounded below if $a_n \ge L$ for all n.

$\{a_n\}$ is alternating if $a_n a_{n+1} < 0$ for $n = 0, 1, 2, 3, \ldots$

ex. $\{1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{16}, \ldots, (-1)^n\left(\frac{1}{2}\right)^n, \ldots\}$ for $n = 0, 1, 2, \ldots$

Theory: If a sequence $\{a_n\}$ is bounded above and monotonically increasing then the sequence $\{a_n\}$ converges, and similiarly if a sequence is monotonically decreasing and bounded below then the sequence $\{a_n\}$ converges. Therefore every bounded monotonic sequence converges.

Ex) $a_n = 2 + \frac{1}{n}$; $b_n = 2 - \frac{1}{n}$ $\quad n = 1, 2, 3, \ldots$

Show that the sequences $\{a_n\}$ and $\{b_n\}$ are bounded and monotonic sequences and prove that $\{a_n\}$ and $\{b_n\}$ both converge to 2.

$$\{a_n\} = \left\{2+\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \ldots, \frac{21}{10}, \ldots\right\}$$

Clearly $\{a_n\}$ is bounded above by 3, that is $a_n \leq 3$ for all n and $\{a_n\}$ is bounded below by 2, that is $a_n > 2$ for all n, since $\lim_{n\to\infty}\left\{2+\frac{1}{n}\right\} = 2$

Since $2 < a_n \leq 3$, $|a_n| \leq 3$ for $n \geq 1$.

Therefore $\{a_n\}$ is a bounded sequence, and clearly the sequence $\{a_n\}$ is monotonically decreasing since the terms of $\{a_n\}$ are decreasing in size.

Therefore the sequence $\{a_n\} = \left\{2+\frac{1}{n}\right\}_{n=1}^{\infty}$ is a Bounded Monotonic sequence and hence converges.

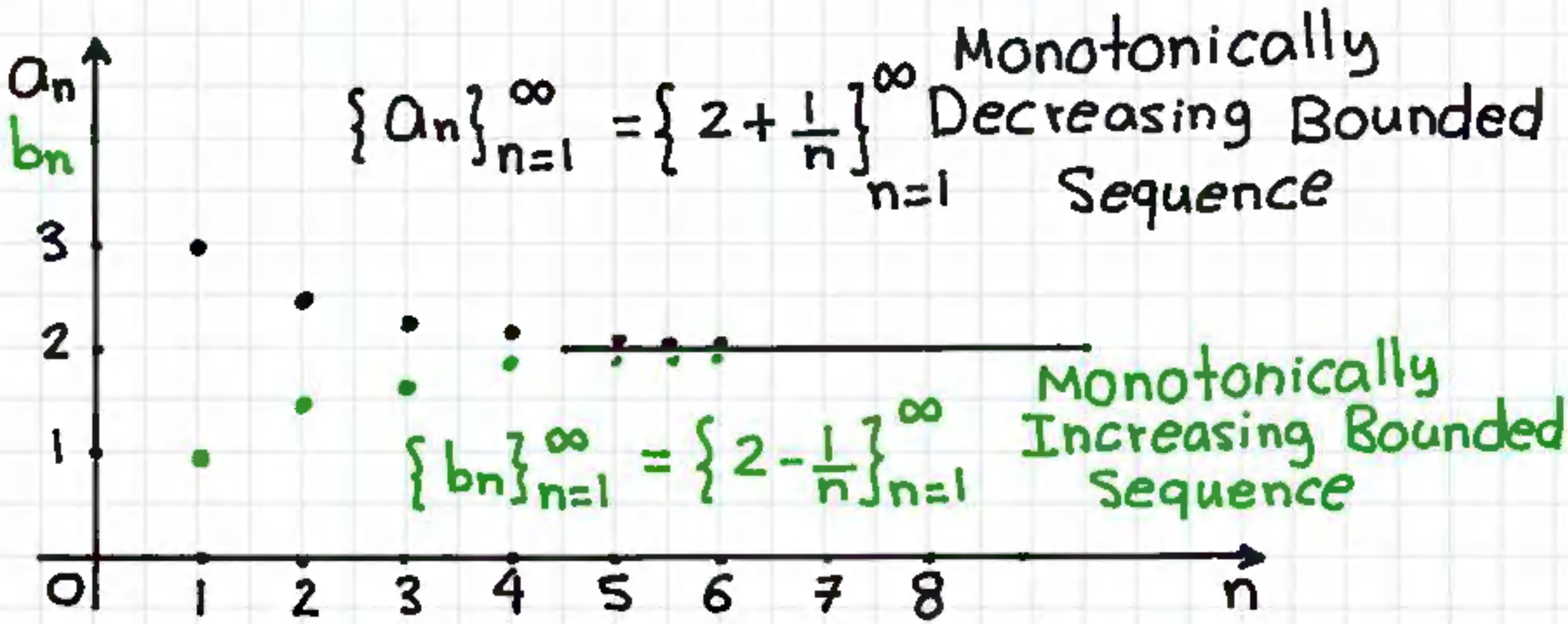
$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(2+\frac{1}{n}) = 2$ $\{a_n\}_{n=1}^{\infty}$ converges to 2.

Similiarly $\{b_n\}_{n=1}^{\infty} = \{2-\frac{1}{n}\}_{n=1}^{\infty} = \{1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots, \frac{19}{10}, \ldots\}$

Clearly $\{b_n\}_{n=1}^{\infty}$ is bounded above by 2 that is $b_n < 2$ for all n since $\lim_{n\to\infty}\{2-\frac{1}{n}\} = 2$ and $\{b_n\}_{n=1}^{\infty}$ is bounded below by 1, that is $b_n \geq 1$ for all n.

Since $1 \leq b_n < 2$, $|b_n| < 2$ for $n \geq 1$, Therefore $\{b_n\}$ is a bounded sequence and clearly the sequence $\{b_n\}$ is monotonically increasing since the terms of b_n are increasing in size, therefore the sequence $\{b_n\} = \left\{2 - \frac{1}{n}\right\}_{n=1}^{\infty}$ is a Bounded Monotonic sequence and hence converges.

$\lim\limits_{n \to \infty} b_n = \lim\limits_{n \to \infty} \left(2 - \frac{1}{n}\right) = 2 \quad \therefore \{b_n\}$ converges to 2



Both Sequences $\{a_n\}$ and $\{b_n\}$ converge to 2

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Infinite Sequences 6

Ex] Show that the sequence $\{a_n\}$ defined recursively as: $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 4)$ converges.

Let's compute a few terms of the sequence

$$a_1 = 2 \quad , \quad a_2 = \frac{1}{2}(a_1 + 4) \Rightarrow a_2 = \frac{1}{2}(2+4) = 3$$

$$a_3 = \frac{1}{2}(a_2 + 4) \Rightarrow a_2 = \frac{1}{2}(3+4) = \frac{7}{2} = 3.5$$

$$a_4 = \frac{1}{2}(a_3 + 4) \Rightarrow a_4 = \frac{1}{2}\left(\frac{7}{2} + 4\right) = 3.75$$

$$a_5 = \frac{1}{2}(a_4 + 4) \Rightarrow a_5 = \frac{1}{2}(3.75 + 4) = 3.875$$

$a_1 = 2$, $a_2 = 3$, $a_3 = 3.5$, $a_4 = 3.75$, $a_5 = 3.875$

The first 5 terms of this sequence suggest that this sequence is increasing and bounded above by $M = 4$, Let's formally verify our intuition by applying mathematical induction.

Let's Prove $a_{n+1} > a_n$ for all $n \geq 1$ by induction.

$a_1 = 2$, $a_2 = 3 \Rightarrow a_2 > a_1$ since $3 > 2$ holds for $n = 1$

Let's assume $a_{n+1} > a_n$ is true for $n = k$, then

$a_{k+1} > a_k$

$a_{k+1} + 4 > a_k + 4$ Add 4 to both sides

$\frac{1}{2}(a_{k+1}+4) > \frac{1}{2}(a_k+4)$ Multiply both sides by $\frac{1}{2}$

Recall $a_{k+1} = \frac{1}{2}(a_k+4) \Rightarrow a_{k+2} = \frac{1}{2}(a_{k+1}+4)$

$\therefore\ a_{k+2} > a_{k+1}$

We have shown that $a_{n+1} > a_n$ is true for $n=k+1$ and since $a_{n+1} > a_n$ is also true for $n=k$, hence by mathematical induction $a_{n+1} > a_n$ for all n.

Now we need to verify that the sequence $\{a_n\}$ is bounded above by $M=4$ that is $a_n < 4$ for all n.

We have already proven that $\{a_n\}$ is increasing so $\{a_n\}$ must have a lower bound : $a_n \geq a_1 = 2$

$a_n \geq a_1 = 2$ for all n. That is $\{a_n\}$ has a lower bound $L = 2$. Let's apply induction to prove $a_n < 4$ for all n.

Recall $a_{n+1} = \frac{1}{2}(a_n + 4)$ and $a_1 = 2$ $n \geq 1$

Lets start with $a_1 < 4$ since $2 < 4$, hence $a_n < 4$ holds for $n = 1$. Suppose $a_n < 4$ holds for $n = k$ then $a_k < 4$

$a_k + 4 < 4 + 4 \Rightarrow a_k + 4 < 8$ Add 4 to both sides

$\frac{1}{2}(a_k + 4) < \frac{1}{2}(8) \Rightarrow \frac{1}{2}(a_k + 4) < 4$

$\therefore a_{k+1} < 4$ so $a_n < 4$ holds for $n = k+1$

Since $a_{k+1} = \frac{1}{2}(a_k + 4)$

We have shown that $a_n < 4$ is true for $n = k+1$ and since $a_n < 4$ is also true for $n = k$, hence by mathematical induction $a_n < 4$ for all n.

$\therefore M = 4$ is an upper bound for the sequence $\{a_n\}$

We have proven that the sequence $\{a_n\}$ is monotonically increasing and is bounded above by 4, therefore the sequence $\{a_n\}$ must converge to a limit P, but what is the limit P?

We know $P = \lim_{n \to \infty} a_n$ exists, Let's use the recurrence relation to find P.

$$a_{n+1} = \frac{1}{2}(a_n + 4) \; ; \; a_1 = 2$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n+4) = \frac{1}{2}\left(\lim_{n\to\infty} a_n + 4\right) = \frac{1}{2}(P+4)$$

Key concept: as $a_n \to P \Rightarrow a_{n+1} \to P$

as $n \to \infty$, $n+1 \to \infty$ as well, hence if

$$\lim_{n\to\infty} a_n = P \Rightarrow \lim_{n\to\infty} a_{n+1} = P$$

Hence $P = \frac{1}{2}(P+4) \Rightarrow 2P = P+4 \Rightarrow P = 4$

$\therefore \{a_n\}$ converges to 4 as we predicted

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Infinite Sequences 7 Geometric Sequences

a Geometric Sequence can be defined recursively

$a_{n+1} = r a_n \quad n \geq 0$

$a_1 = r a_0 \Rightarrow a_2 = r a_1 = r \cdot r a_0 = r^2 a_0$

$a_3 = r a_2 = r \cdot r^2 a_0 = r^3 a_0 \Rightarrow a_4 = r a_3 = r \cdot r^3 a_0 = r^4 a_0$

$\therefore a_1 = r a_0 \,,\, a_2 = r^2 a_0 \,,\, a_3 = r^3 a_0 \,, \ldots, a_n = r^n a_0$

Let's define a geometric sequence as:

$a_n = a r^n \quad n \geq 0 \quad$ or $\quad a_n = a r^{n-1} \quad n \geq 1$

Examples: $a_n = \left(\frac{1}{2}\right)^n$; $a_n = 2^n$; $a_n = \left(\frac{-3}{4}\right)^n$

For $n \geq 0$

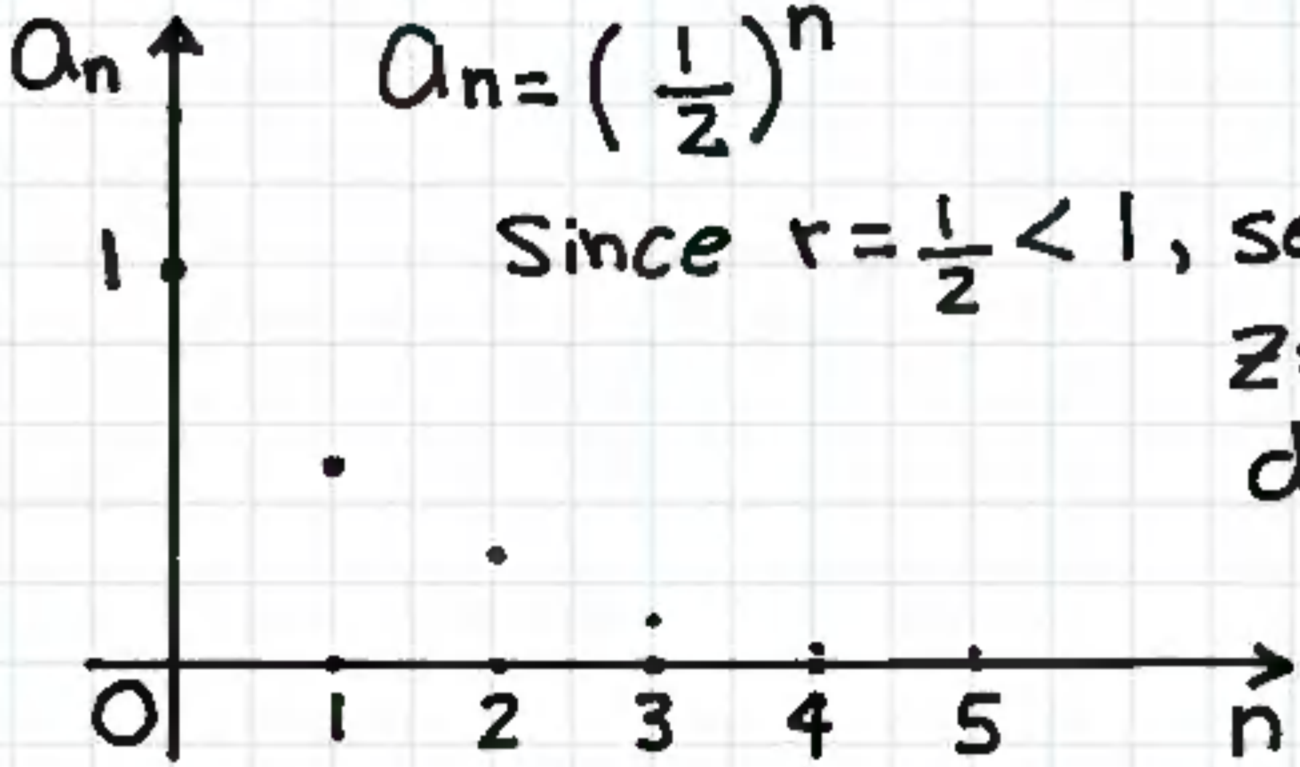
Ex] Graph the following sequences and discuss their convergence:

a) $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=0}^{\infty}$; b) $\left\{(2)^n\right\}_{n=0}^{\infty}$; c) $\left\{\left(\frac{-3}{4}\right)^n\right\}_{n=0}^{\infty}$

a) $a_n = \left(\frac{1}{2}\right)^n$; $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{8}, \ldots,$

Notice common ratio $r = \frac{1}{2} < 1$ and since $r < 1$ is the terms of the geometric sequence are getting smaller and smaller as every term is half the previous term that is $a_{n+1} = \frac{1}{2} a_n \quad n \geq 0$

So the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=0}^{\infty}$ converges to zero and is monotonically decreasing.



$a_n = \left(\frac{1}{2}\right)^n$

Since $r = \frac{1}{2} < 1$, sequence converges to zero and is monotonically decreasing.

b) $b_n = 2^n$; $b_0 = 1$, $b_1 = 2$, $b_2 = 4$, $b_3 = 8$,...

Notice the common ratio $r = 2 > 1$ and since $r > 1$ the terms of the geometric sequence are getting larger and larger as every term is twice the previous term that is $b_{n+1} = 2b_n$ so the terms of the sequence $\{2^n\}_{n=0}^{\infty}$ are positive and increase without bound as $n \to \infty$. This sequence diverges to infinity that is $\lim_{n\to\infty} 2^n = \infty$ and ofcourse the sequence $b_n = 2^n$ is monotonically increasing.

$b_n = 2^n$, since $r = 2 > 1$ the sequence diverges to ∞ and is monotonically increasing.

c) $c_n = \left(\frac{-3}{4}\right)^n$; $c_0 = 1$, $c_1 = -\frac{3}{4}$, $c_2 = \frac{9}{16}$, $c_3 = \frac{-27}{64}$,...

Let's rewrite $c_n = \left(\frac{-3}{4}\right)^n$ as $c_n = (-1)^n \left(\frac{3}{4}\right)^n$

Notice $(-1)^n$ oscillates between -1 and 1 while $\left(\frac{3}{4}\right)^n$ decreases to zero since every term is 3/4 the magnitude of the previous term, so by the squeeze law we can show that $\lim_{n\to\infty} (-1)^n \left(\frac{3}{4}\right)^n = 0$

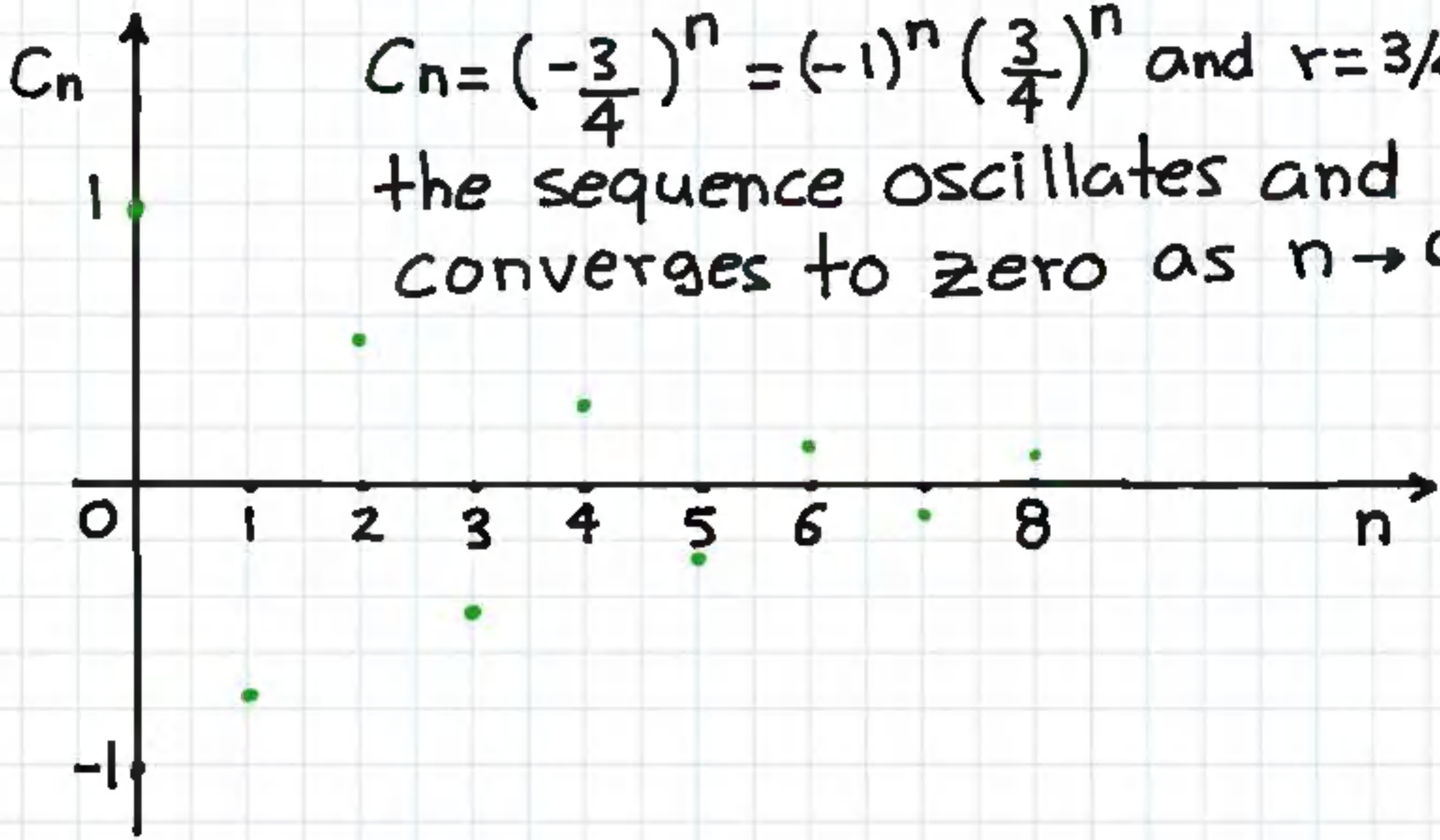
$$-\left(\frac{3}{4}\right)^n < (-1)^n \left(\frac{3}{4}\right)^n < \left(\frac{3}{4}\right)^n$$

$$\lim_{n\to\infty} \qquad \lim_{n\to\infty} \qquad \lim_{n\to\infty}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$0 \qquad 0 \qquad 0$$

SQUEEZE LAW



$C_n = \left(-\frac{3}{4}\right)^n = (-1)^n \left(\frac{3}{4}\right)^n$ and $r = 3/4 < 1$

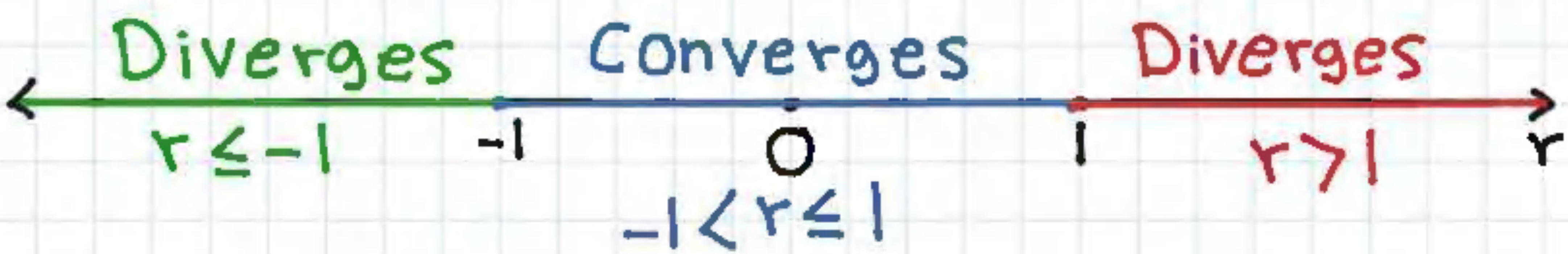
the sequence oscillates and converges to zero as $n \to \infty$

Remark: Above sequence is <u>Not Monotonic</u> but rather the sequence is <u>oscillatory</u>.

Theory : Geometric Sequences

Let r (ratio) be a real number

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \text{ or } -1<r<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{Not exist} & r\le -1 \end{cases}$$



Infinite Sequences 8

Ex] Show that the sequence $a_n = \dfrac{2n+1}{n^2+n+1}$ is decreasing for $n \geq 1$.

Intuitively we know that the sequence is decreasing since $(2n+1) < (n^2+n+1)$ and hence the ratio $a_n = (2n+1)/(n^2+n+1)$ decreases as n increases. Let's prove it formally.

$a_{n+1} < a_n \Rightarrow \{a_n\}$ is decreasing

$$\frac{2(n+1)+1}{(n+1)^2+(n+1)+1} < \frac{2n+1}{n^2+n+1}$$

$$\Rightarrow \frac{2n+3}{n^2+2n+1+n+2} < \frac{2n+1}{n^2+n+1}$$

$$\Rightarrow \frac{2n+3}{n^2+3n+3} < \frac{2n+1}{n^2+n+1}$$

$$\Rightarrow (2n+3)(n^2+n+1) < (2n+1)(n^2+3n+3)$$

$$\Rightarrow (\cancel{2n^3}+2n^2+2n+3n^2+\cancel{3n}+\cancel{3} < \cancel{2n^3}+6n^2+6n+n^2+\cancel{3n}+\cancel{3}$$

$$\Rightarrow 5n^2+2n < 7n^2+6n$$

$$\Rightarrow -2n^2-4n < 0 \quad \text{or} \quad 2n^2+4n > 0$$

Since $n \geq 1$ it is clear that $2n^2+4n > 0$

$\therefore$ $a_{n+1} < a_n$ and $\{a_n\} = \left\{\frac{2n+1}{n^2+n+1}\right\}$ is Decreasing.

Ex] Does the sequence given by $a_n = (-1)^n \dfrac{2n}{3n+5}$ for $n \geq 1$ Converge or Diverge?

Notice that $\{a_n\}$ is bounded below by $-2/3$ and bounded above by $2/3$ so $\{a_n\}$ is bounded.

The $(-1)^n$ factor oscillates between -1 and 1, since $(-1)^{odd} = -1$ and $(-1)^{even} = 1$ while the factor $\dfrac{2n}{3n+5}$ approaches $\dfrac{2}{3}$ as $n \to \infty$ since

$$\lim_{n\to\infty} \frac{2n/n}{\frac{3n}{n}+\frac{5}{n}} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}$$

Now let's look at how the terms of $\{a_n\}$ increase or decrease.

For n even $a_n = (-1)^n \dfrac{2n}{3n+5} = \dfrac{2n}{3n+5}$

Consider $f(x) = \dfrac{2x}{3x+5}$

$$f'(x) = \frac{(3x+5)(2) - 2x(3)}{(3x+5)^2} = \frac{\cancel{6x} + 10 - \cancel{6x}}{(3x+5)^2}$$

$$f'(x) = \frac{10}{(3x+5)^2} > 0 \text{ for all } x \geq 1$$

$\therefore$ $f(n) = \dfrac{2n}{3n+5}$ is increasing and approaching $2/3$

Similiarly for n Odd $\Rightarrow a_n = (-1)^n (2n)/(3n+5)$

$$a_n = \frac{-2n}{3n+5}$$

We can show by taking derivative that

$$f'(x) = \frac{-10}{(3x+5)^2} < 0 \quad \text{for all } x \geq 1$$

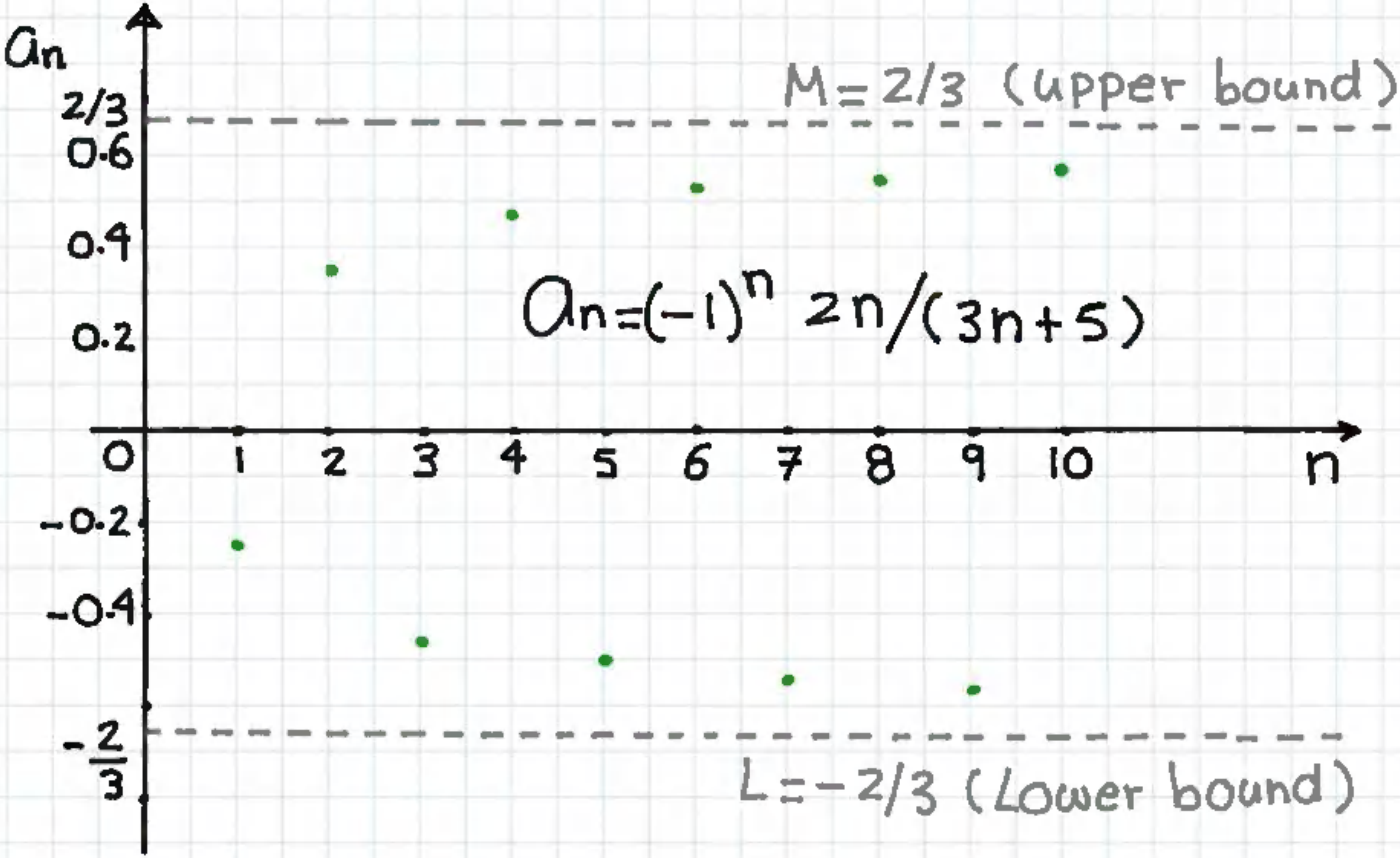
$\therefore f(n) = \dfrac{-2n}{3n+5}$ is decreasing and approaching $-2/3$

$\therefore$ The even terms of $a_n = (-1)^n (2n)/(3n+5)$ form an increasing sequence approaching $2/3$ while the odd terms of the sequence $\{a_n\}$ form a decreasing sequence approaching $-2/3$

$\therefore$ The sequence $\{a_n\}$ diverges. It is interesting to note this is a bounded sequence ($\{a_n\}$ is bounded above by 2/3 and bounded below by -2/3) that is not Monotonic, in fact the factor $(-1)^n$ causes the oscillatory behavior.

Summary

The sequence $a_n = (-1)^n \dfrac{2n}{3n+5}$ Diverges as the odd terms approach $-2/3$ while the even terms approach $2/3$ as $n \to \infty$.


a_n
$M=2/3$ (upper bound)
2/3
0.6
0.4
0.2
$a_n=(-1)^n\ 2n/(3n+5)$
0
1
2
3
4
5
6
7
8
9
10
n
-0.2
-0.4
$-\frac{2}{3}$
$L=-2/3$ (Lower bound)

Infinite Sequences 9

Ex| Find the limit of the following sequences if the limit exists.

a) $a_n = \dfrac{(1+(-1)^n)\sqrt{n}}{2^n}$ for $n \geq 1$

b) $b_n = \ln(\sin(1/n)) + \ln n$ for $n \geq 1$

a) $a_n = \dfrac{(1+(-1)^n)\sqrt{n}}{2^n}$, let's compute a few terms

$a_1 = (1+-1)\sqrt{1}/2 = 0$, $a_2 = 2\sqrt{2}/4 = \sqrt{2}/2$

$a_3 = 0$, $a_4 = 2\sqrt{4}/16 = 1/4$, $a_5 = 0, \ldots$

It is clear that when n is an odd integer

$a_n = (1+(-1)^n)\sqrt{n}/2^n = 0$ since $1+(-1)^n = 0$ for odd n and when n is an even integer

$a_n = (1+(-1)^n)\sqrt{n}/2^n = 2\sqrt{n}/2^n$, so it is logical to apply squeeze law to find the limit of a_n

$$0 \le \frac{(1+(-1)^n)\sqrt{n}}{2^n} \le \frac{2\sqrt{n}}{2^n}$$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{(1+(-1)^n)\sqrt{n}}{2^n} \le \lim_{n\to\infty} \frac{2\sqrt{n}}{2^n}$$

Let's consider $\lim_{n\to\infty} \frac{2\sqrt{n}}{2^n}$; since both $2\sqrt{n}$ and 2^n approach infinity as $n\to\infty$, we have the ∞/∞ pattern, so we will apply L'Hopital's rule to compute the limit.

$$\lim_{n\to\infty} \frac{2\sqrt{n}}{2^n} \overset{H}{=} \lim_{n\to\infty} 2\cdot\frac{1}{2\sqrt{n}} \Big/ 2^n \ln 2$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\, 2^n \ln 2} = 0$$

$\therefore$ By L'Hopital's Rule $\lim_{n\to\infty} \frac{2\sqrt{n}}{2^n} = 0$

Now we can compute the limit of a_n as $n \to \infty$ by applying the squeeze law.

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{(1+(-1)^n)\sqrt{n}}{2^n} \le \lim_{n\to\infty} \frac{2\sqrt{n}}{2^n}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

By Squeeze Law

$$\therefore\ a_n = \frac{(1+(-1)^n)\sqrt{n}}{2^n} \text{ converges to } 0$$

b) $b_n = \ln(\sin(1/n)) + \ln n$ for $n \geq 1$

$b_n = \ln(n \sin(1/n))$ $\qquad \ln x + \ln y = \ln(xy)$

$$b_n = \ln\left(\frac{\sin(1/n)}{(1/n)}\right)$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{\sin(1/n)}{1/n}\right) = \ln\left(\lim_{n\to\infty} \frac{\sin(1/n)}{1/n}\right)$$

Notice that we have $0/0$ pattern inside the log function as $\sin(\lim_{n\to\infty} \frac{1}{n}) = \sin 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n}$ is also 0, so we should apply L'Hopital's rule.

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \overset{H}{=} \lim_{n\to\infty} \frac{\cos(1/n)\cancel{(-1/n^2)}}{\cancel{-1/n^2}}$$

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \overset{H}{=} \lim_{n\to\infty} \cos(1/n) = \cos 0 = 1$$

$$\therefore \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \ln\left(\frac{\sin(1/n)}{1/n}\right) = \ln 1 = 0$$

$\therefore$ $b_n = \ln\left(\dfrac{\sin(1/n)}{1/n}\right)$ converges to 0

Infinite Sequences 10

Ex] Show that the sequence $\{a_n\}$ defined recursively as $a_1 = 1$, $a_{n+1} = \sqrt{2 + a_n}$, is increasing and bounded above by 2 and find $\lim_{n\to\infty} a_n$.

Lets compute a few terms of the sequence

$$a_1 = 1, \quad a_2 = \sqrt{2 + a_1} = \sqrt{2+1} = \sqrt{3} \cong 1.73$$

$$a_3 = \sqrt{2 + a_2} = \sqrt{2 + \sqrt{3}} \cong 1.93$$

$$a_4 = \sqrt{2 + a_3} = \sqrt{2 + \sqrt{2 + \sqrt{3}}} \cong 1.98$$

The first 4 terms of the sequence suggest that this sequence is increasing and is bounded above by 2.

Let's formally verify our intuition by applying Mathematical induction.

Let's prove that $a_{n+1} > a_n$ for all $n \geq 1$ by induction.

$a_1 = 1$, $a_2 = \sqrt{3}$ $\Rightarrow a_2 > a_1$ since $\sqrt{3} > 1$, holds for $n=1$

Let's assume $a_{n+1} > a_n$ is true for $n=k$, that is

$a_{k+1} > a_k \Rightarrow a_{k+1} = \sqrt{2+a_k} < \sqrt{2+a_{k+1}} = a_{k+2}$

since $a_k < a_{k+1}$ but now we have proven that

$a_{k+1} < a_{k+2}$ or $a_{(k+1)+1} > a_{k+1}$, $\therefore$ we have

shown that $a_{n+1} > a_n$ is true for $n = k+1$

and since $a_{n+1} > a_n$ is also true for $n=k$,

hence by mathematical induction $a_{n+1} > a_n$ for all n.
Now we need to verify that the sequence $\{a_n\}$ is bounded above by $M=2$ that is $a_n < 2$ for all n. We have already proven that $\{a_n\}$ is increasing so $\{a_n\}$ must have a lower bound : $a_n \geq a_1 = 1$ for all n. That is $\{a_n\}$ has a lower bound $L=1$. Let's apply induction to prove that $\{a_n\}$ has an upper bound $M=2$, that is $a_n < 2$ for all n.

Recall $a_{n+1} = \sqrt{2+a_n}$ and $a_1 = 1$ for $n \geq 1$

Let's start with $a_1 < 2$ since $1 < 2$, hence

$a_n < 2$ holds for $n=1$. Assume that $a_n < 2$ holds for $n=k$ then $a_k < 2$ then:

$a_{k+1} = \sqrt{2+a_k} < \sqrt{2+2} = 2$ So $a_{k+1} < 2$

So $a_n < 2$ holds for $n = k+1$

We have shown that $a_n < 2$ is true for $n=k+1$ and since $a_n < 2$ is also true for $n=k$, hence by mathematical induction $a_n < 2$ for all n.

$\therefore M=2$ is an upper bound for the sequence $\{a_n\}$

We have proven that the sequence $\{a_n\}$ is monotonically increasing and is bounded above by 2, therefore the sequence $\{a_n\}$ must converge to a limit P, but what is the limit P?

We know $P = \lim_{n\to\infty} a_n$ exists, Let's use the recurrence relation to find P.

$a_{n+1} = \sqrt{2+a_n}$, $a_1 = 1$

Key concept: as $a_n \to P \Rightarrow a_{n+1} \to P$

as $n \to \infty \Rightarrow n+1 \to \infty$

$$\lim_{n\to\infty} a_n = P \Rightarrow \lim_{n\to\infty} a_{n+1} = P$$

$$\therefore \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+\lim_{n\to\infty} a_n}$$

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2+\lim_{n\to\infty} a_n} \Rightarrow P=\sqrt{2+P}$$

$$P=\sqrt{2+P} \Rightarrow P^2 = 2+P \Rightarrow P^2-P-2=0$$

$(P-2)(P+1)=0 \Rightarrow P=2\ ;\ P=-1$ and since $a_n \geq 1$ since $L=1$ is the lower bound for $\{a_n\}$ we reject $P=-1$ and we accept $P=2$.

$\therefore \{a_n\}$ converges to 2 as we predicted.

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