Integral test 1

Motivation: Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$

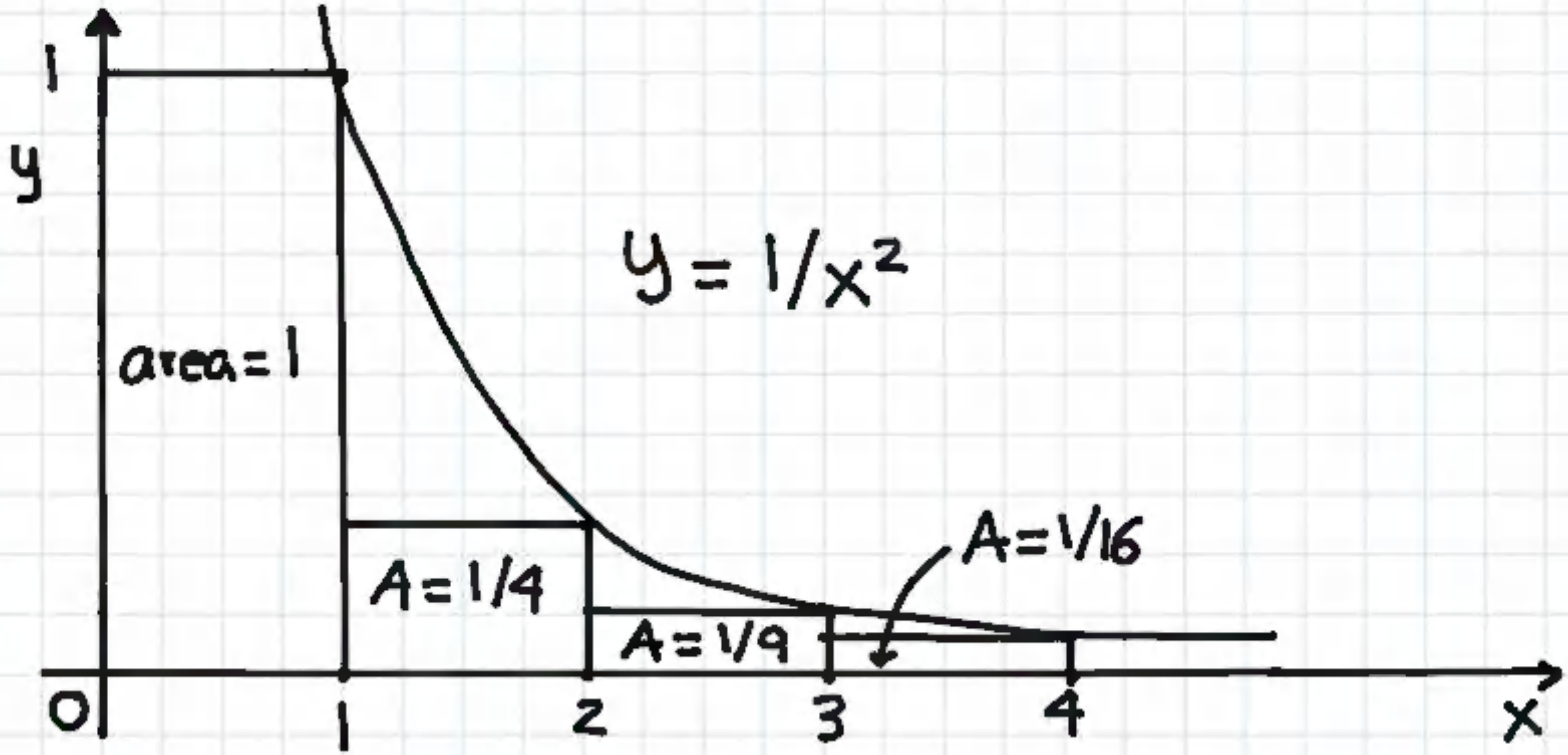
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

Does the above series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge ?

Consider $f(x) = 1/x^2$

let's find $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx = \lim_{b\to\infty} -x^{-1}\Big|_1^b$

$$= \lim_{b\to\infty} \frac{-1}{b} - -1 = \lim_{b\to\infty} \frac{-1}{b} + 1 = 0 + 1 = 1$$



$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots +$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + 1 = 2$$

Since the areas of each of these rectangles is smaller than the corresponding area under the curve $y = 1/x^2$:

$$S = \sum_{n=1}^{\infty} 1/n^2 < 1 + \int_1^{\infty} 1/x^2 \, dx = 1 + 1 = 2$$

$\therefore \sum_{n=1}^{\infty} 1/n^2$ converges since the area under the curve $y = 1/x^2$ on $[1, \infty)$ forms an upper bound for the series $\sum_{n=1}^{\infty} 1/n^2$ and since the area under the curve is finite, the series $\sum_{n=1}^{\infty} 1/n^2$ also converges.

The Integral Test

Suppose $f(x)$ is a continuous, positive valued and decreasing function on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

C] If $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges

D] If $\int_1^{\infty} f(x)\,dx$ Diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ Diverges

Remarks:

1] Convergence or Divergence of a series is not affected by adding/subtracting a finite number of terms to a series, so we dont need to start the series or the integral at $n=1$, $\sum_{n=N}^{\infty} a_n$ or $\int_N^{\infty} f(x)\,dx$

2] It is not necessary that $f(x)$ always be decreasing but rather that $f(x)$ is eventually decreasing for x larger than N.

3] $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$ The integral test does not give the sum of series.

Ex] Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or Divergence

$f(x) = \frac{1}{x^2+1}$ is continuous, positive and decreasing on $[1, \infty)$ so we can apply the Integral Test:

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{p\to\infty} \int_1^{p} \frac{1}{x^2+1}\,dx = \lim_{p\to\infty} \tan^{-1}x \Big|_1^p$$

$$= \lim_{p\to\infty} \left(\tan^{-1}p - \tan^{-1}(1) \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Since $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges, by the Integral Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

Ex] Determine if the following series converge or Diverge by applying the Integral test.

a) $\sum_{n=3}^{\infty} \frac{\ln n}{n}$

b) $\sum_{n=1}^{\infty} n \cdot e^{-n}$

a) Solution: $f(x) = \frac{\ln x}{x}$ is positive and continuous for $x \geq 3$ since $\ln x > 0$ for $x > 1$ and $\ln x$ is cont. for $x > 0$, $\therefore$ $f(x) = \frac{\ln x}{x}$ is positive and continuous for $x \geq 3$. But it is not obvious that $f(x) = \frac{\ln x}{x}$ is decreasing, so let's prove it!

$f(x) = \frac{\ln x}{x}$; $f'(x) = \frac{x \cdot 1/x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$

$f'(x) < 0 \Rightarrow 1 - \ln x < 0$ since $x^2 > 0$ for all $x > 0$

$1 - \ln x < 0 \Rightarrow 1 < \ln x \Rightarrow e^1 < e^{\ln x} \Rightarrow x > e$, that is $f'(x) < 0$ or $f(x)$ is decreasing when $x > e$

and to summarize we have proven that $f(x) = \frac{\ln x}{x}$ is positive valued, continuous and decreasing for $x \geq 3$.

So let's apply the Integral Test to find out if the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ converges or Diverges ?

Integral Test

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{p\to\infty}\int_3^p \frac{\ln x}{x}\,dx = \lim_{p\to\infty}\frac{(\ln x)^2}{2}\Big|_3^p$$

U-Subst.

$u = \ln x \quad du = 1/x\,dx$

$$= \lim_{p\to\infty}\frac{(\ln P)^2}{2} - \frac{(\ln 3)^2}{2} = \infty \quad \text{Diverges}$$

Since the improper integral $\int_3^\infty \frac{\ln x}{x}\,dx$ Diverges

the series $\sum_{n=3}^{\infty}\frac{\ln n}{n}$ also Diverges by the Integral Test.

b) $\sum_{n=1}^{\infty} ne^{-n}$

$f(x) = xe^{-x}$ is positive, continuous for $x \geq 1$ but it is not obvious that $f(x) = xe^{-x}$ is decreasing, so let's prove it by taking the derivative.

$f'(x) = 1e^{-x} + xe^{-x}(-1) = e^{-x}(1-x)$

Since e^{-x} is positive for all x, $f'(x) < 0$ when $1 - x < 0 \Rightarrow x > 1$, so $f(x)$ is decreasing on $(1, \infty)$ so let's apply the Integral Test to find out if the series $\sum_{n=1}^{\infty} ne^{-n}$ converges or Diverges?

Integral Test

Apply $\int$ by Parts

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$\int_1^\infty xe^{-x}\,dx$$

$$\int_1^\infty xe^{-x}\,dx = \lim_{p\to\infty}\int_1^p xe^{-x}\,dx$$

$$= \lim_{p\to\infty}\left(-xe^{-x}\Big|_1^p - \int_1^p -e^{-x}\,dx\right)$$

$$= \lim_{p\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^p$$

$$= \lim_{p\to\infty}\left[\frac{-p}{e^p} - \frac{1}{e^p} - \left(\frac{-1}{e} - \frac{1}{e}\right)\right]$$

$$= \lim_{p \to \infty} \left[\frac{-p}{e^p} - \frac{1}{p} + \frac{2}{e} \right]$$

since $e^p >> p \Rightarrow \lim_{p \to \infty} -p/e^p = 0$

and $\frac{-1}{p} \to 0$ as $p \to \infty$

$\therefore \lim_{p \to \infty} \left[\frac{-p}{e^p} - \frac{1}{p} + \frac{2}{e} \right] = \frac{2}{e}$ Converges

$\therefore \int_1^{\infty} x \cdot e^{-x}\, dx = 2/e$ converges

Since the improper integral $\int_1^{\infty} xe^{-x} dx$ converges

the series $\sum_{n=1}^{\infty} ne^{-n}$ also converges by the

Integral Test.

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Integral Test 3

Ex] Find the values of P for which the series $\sum_{n=1}^{\infty} \frac{1}{n^P}$ is convergent ?

Solution: If $P<0$ then $\lim_{n\to\infty} \frac{1}{n^P} = \lim_{n\to\infty} n^{-P} = \infty$

then $\sum_{n=1}^{\infty} \frac{1}{n^P}$ diverges by the Divergence Test.

If $P=0$ then $\lim_{n\to\infty} \frac{1}{n^P} = \lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} \frac{1}{1} = 1$

then $\sum_{n=1}^{\infty} \frac{1}{n^P}$ diverges by the Divergence Test.

If $P=1$ then we have the series $\sum_{n=1}^{\infty} \frac{1}{n}$

which is the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we have proven earlier to be Divergent.

Now consider $P>0$ and $P \neq 1$; $\sum_{n=1}^{\infty} 1/n^P$

The corresponding function is $f(x) = \frac{1}{x^P}$, which is clearly positive, continuous and decreasing.

$$\int_1^{\infty} \frac{1}{x^P}\,dx = \int_1^{\infty} x^{-P}\,dx = \left.\frac{x^{-P+1}}{-P+1}\right|_1^{\infty}$$

$$= \frac{1}{1-P}\left(\lim_{x\to\infty} x^{1-P} - 1\right)$$

Since $\lim_{x\to\infty} x^{1-P} = 0$ if the exponent $1-P<0$

$\Rightarrow -P < -1 \Rightarrow P > 1$

For $P>1 \Rightarrow \int_1^\infty \frac{1}{x^P}dx = \frac{1}{1-P}\left(\lim_{x\to\infty} x^{1-P} - 1\right) = \frac{1}{1-P}(0-1)$

$\therefore \int_1^\infty \frac{1}{x^P}dx = \frac{1}{P-1}$ converge for $P>1$

Since $\int_1^\infty \frac{1}{x^P}dx$ converges for $P>1$, the series

$\sum_{n=1}^\infty \frac{1}{n^P}$ converges for $P>1$ by the Integral Test.

Now consider $0<P<1 \Rightarrow \int_1^\infty \frac{1}{x^P}dx = \frac{1}{1-P}\left(\lim_{x\to\infty} x^{1-P} - 1\right)$

For $0<P<1$ $\lim_{x\to\infty} x^{1-P} = \infty$ and hence

$\int_1^\infty \frac{1}{x^P}dx$ Diverges and by the Integral Test the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ also Diverges.

Summary p series convergence

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and Divergent if $p \le 1$

Ex| $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent because it is a p_series with $p = 3/2 > 1$

Ex| $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is Divergent because it is a p- series with $p = 1/2 < 1$

Ex] Does the following series converge or Diverge?

A] $\sum_{n=1}^{\infty}\left(\frac{1}{n^{1.1}}+\frac{1}{n^{0.9}}\right)$

B] $\sum_{n=1}^{\infty}\frac{1}{n+3}$

A] Solution: $\sum_{n=1}^{\infty}\left(\frac{1}{n^{1.1}}+\frac{1}{n^{0.9}}\right)=\sum_{n=1}^{\infty}\frac{1}{n^{1.1}}+\sum_{n=1}^{\infty}\frac{1}{n^{0.9}}$

$\sum_{n=1}^{\infty}\frac{1}{n^{1.1}}$ is a Convergent P-Series with $P=1.1>1$

however $\sum_{n=1}^{\infty}\frac{1}{n^{0.9}}$ is a Divergent P-Series with

$P=0.9<1$, $\therefore$ The entire series $\sum_{n=1}^{\infty}\left(\frac{1}{n^{1.1}}+\frac{1}{n^{0.9}}\right)$

Diverges. Key concept: Since $\sum_{n=1}^{\infty} \frac{1}{n^{0.9}}$ Diverges the entire series $\sum_{n=1}^{\infty} \left(\frac{1}{n^{1.1}} + \frac{1}{n^{0.9}}\right)$ Diverges.

B] $\sum_{n=1}^{\infty} \frac{1}{n+3}$; Note we cannot apply the P-Series Test directly.

So let's write $\sum_{n=1}^{\infty} \frac{1}{n+3}$ as $\sum_{n=4}^{\infty} \frac{1}{n}$

Notice both series are equivalent since we subtracted 3 from n+3 and added 3 to the index n=1 to compensate.

Since $\sum_{n=4}^{\infty} \frac{1}{n}$ is a divergent P-series with P=1 it follows that $\sum_{n=1}^{\infty} \frac{1}{n+3}$ also Diverges

Note: $\sum_{n=1}^{\infty} \frac{1}{n+3} = \sum_{n=4}^{\infty} \frac{1}{n} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$

is a Divergent P-series ($P=1$) without the first 3 terms. But we know that adding/subtracting a finite number of terms does not affect the convergence or Divergence of a Series.

Integral Test 4

Ex] Find the values of P for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^P} \text{ is convergent.}$$

For $P>0$ and $x \geq 2$ $f(x)=\dfrac{1}{x(\ln x)^P}$ is positive,

Continuous and decreasing on $[2,\infty)$ so we can apply the Integral Test

$$\int_2^{\infty} \frac{1}{x(\ln x)^P}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^P}\,du$$

U-Substitution

$u=\ln x \quad du=\frac{1}{x}\,dx$

$x=2 \Rightarrow u=\ln 2$

$x=\infty \Rightarrow u=\infty$

$$\int_{\ln 2}^{\infty} \frac{1}{u^P}\,du = \frac{u^{-P+1}}{1-P}\Bigg|_{\ln 2}^{\infty} = \frac{1}{1-P}\left(\lim_{u\to\infty} u^{1-P} - (\ln 2)^{1-P}\right)$$

$$\lim_{u\to\infty} u^{1-P} = 0 \text{ if } 1-P<0 \Rightarrow -P<-1 \Rightarrow P>1$$

$$\int_{\ln 2}^{\infty} \frac{1}{u^P}\,du = \frac{1}{1-P}\left(0-(\ln 2)^{1-P}\right) = \frac{(\ln 2)^{1-P}}{P-1}$$

Since $\displaystyle\int_{2}^{\infty} \frac{1}{x(\ln x)^P}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^P}\,du$ converges for $P>1$

then by the Integral Test the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^P}$ also converges for $P>1$.

Ex] Determine if the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges or Diverges?

Solution: $f(x) = \frac{e^{1/x}}{x^2}$ is positive and continuous

but it is not clear that $f(x) = \frac{e^{1/x}}{x^2}$ is decreasing,

so let's prove it by taking the derivative.

$$f'(x) = \frac{\cancel{x^2} e^{1/x} (-1/\cancel{x^2}) - e^{1/x} . 2x}{x^4} = \frac{-e^{1/x}(1+2x)}{x^4}$$

Clearly $f'(x) < 0$ on $[1, \infty)$ so let's apply the Integral Test.

Integral Test

U-Subst.
$u = 1/x$
$du = -1/x^2\, dx$

$$\int_1^\infty \frac{e^{1/x}}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{e^{1/x}}{x^2}\,dx$$

$$= \lim_{b\to\infty} -e^{1/x}\Big|_1^b = \lim_{b\to\infty} \left[-e^{1/b} - -e^{1/1}\right]$$

$$= \lim_{b\to\infty} -e^{1/b} + e = -e^0 + e = -1 + e = e - 1$$

Since the improper integral $\int_1^\infty \frac{e^{1/x}}{x^2}\,dx$ converges the series $\sum_{n=1}^\infty \frac{e^{1/n}}{n^2}$ also converges by the Integral Test.

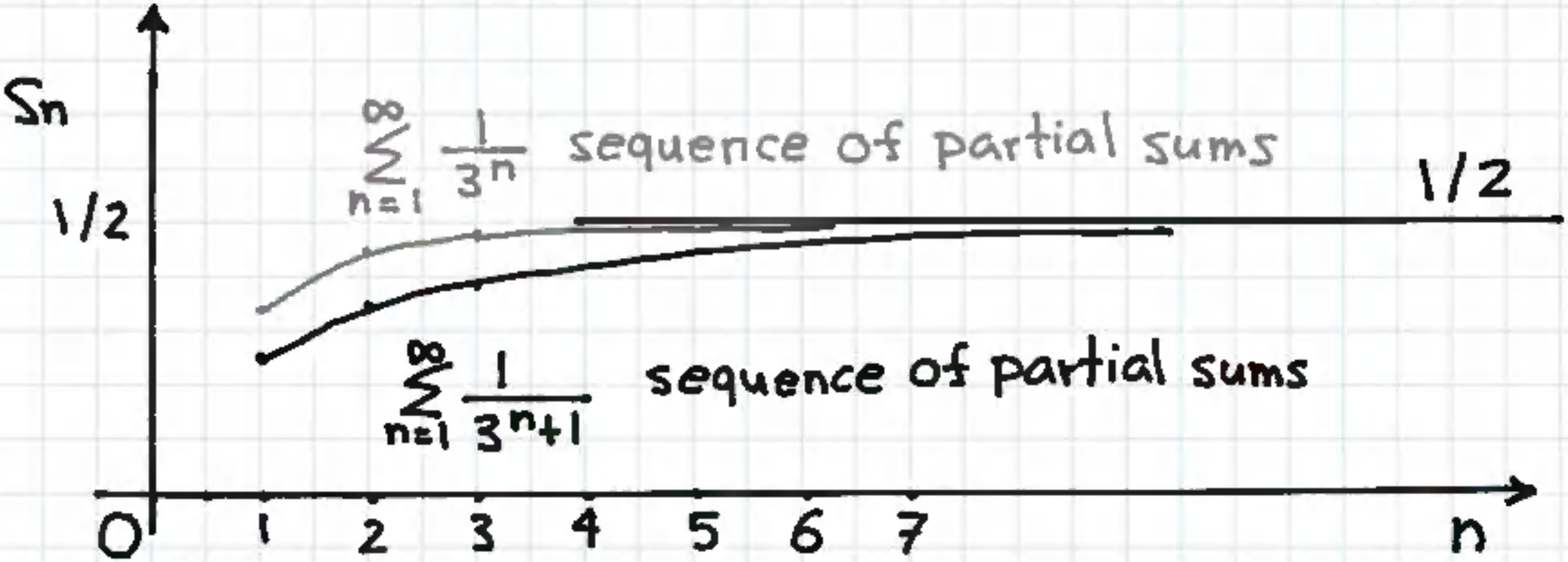
Basic Comparison Test 1

Motivation: Basic idea is to compare a given series $\sum a_k$ to another series $\sum b_k$ for which we know its convergence or divergence. (Geometric series, P series)

Ex| $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ is very similiar to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a convergent geometric series with $a=1/3$ and $r=1/3 < 1$, so intuitively we feel that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ also converges.

Notice that $0 < \frac{1}{3^n+1} < \frac{1}{3^n}$ for $n \geq 1$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3^n+1} < \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{a}{1-r} = \frac{1/3}{1-1/3} = \frac{1}{2}$$



Since our given series $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ has smaller terms than the geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ which converges with a sum $S = 1/2$, therefore the partial sums of $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ form a bounded increasing sequence, which converges to a finite sum which is less than the sum of the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{3^n+1}$ is a Convergent series

<u>Summary</u>: $$\sum_{n=1}^{\infty} \frac{1}{3^n+1} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{3^n}}$$

Convergent Geometric Series with $r = 1/3 < 1$

Basic Comparison Test

Assume that $\sum a_n$ and $\sum b_n$ are series with positive terms that is $a_n > 0$ and $b_n > 0$

1. If $\sum b_n$ converges and $0 < a_n \leq b_n$, then $\sum a_n$ also converges.
2. If $\sum b_n$ diverges and $a_n \geq b_n > 0$, then $\sum a_n$ also Diverges.

Ex| Determine whether the series converges or Diverges?

$$\sum_{n=0}^{\infty} \frac{1+\sin n}{5^n}$$

$$\sum_{n=0}^{\infty} \frac{1+\sin n}{5^n}$$ we know that $\sin n \le 1$ for all n

$$\frac{1+\sin n}{5^n} \le \frac{1+1}{5^n} = \frac{2}{5^n}$$ Since $\sum_{n=0}^{\infty} \frac{2}{5^n}$ is a geometric

series with $a = 2/5^0 = 2$ and $|r| = 1/5 < 1$, and since

$\sum_{n=0}^{\infty} 2 \cdot \frac{1}{5^n}$ is a convergent geometric series, the

given series $\sum_{n=0}^{\infty} \frac{1+\sin n}{5^n}$ converges by the Basic

Comparison Test.

Summary: $$\sum_{n=0}^{\infty} \frac{1+\sin n}{5^n} < \underbrace{\sum_{n=0}^{\infty} \frac{2}{5^n}}_{\text{Convergent Geometric series with } r = 1/5 < 1}$$

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Basic Comparison Test 2

Determine whether the following series converges or diverges?

A] $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^3}$

B] $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

C] $\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n}$

A] Since $\lim_{n\to\infty} \tan^{-1} n = \pi/2$ and $\tan^{-1} n$ is monotonically increasing, we know that $\tan^{-1} n < \pi/2$ for all $n \geq 1$.

Since $\dfrac{\tan^{-1} n}{1+n^3} \leq \dfrac{\pi/2}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{\pi/2}{n^3} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent P-series with $P=3$ the given series $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^3}$ is convergent by the Basic Comparison Test.

Summary: $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^3} < \boxed{\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}}$

Convergent P-series

$P=3$

B] $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

Consider $f(x) = e^{1/x} \Rightarrow f'(x) = e^{1/x} \cdot \frac{-1}{x^2} < 0$ for all x and since $\lim_{x \to \infty} e^{1/x} = 1$ and $f(x)$ is monotonically decreasing, then $e^{1/x} > 1$ for all $x \geq 1$

Since $\frac{e^{1/n}}{n} > \frac{1}{n}$ for all $n \geq 1$ and we know that $\sum_{n=1}^{\infty} 1/n$ is a Divergent P-series with $P=1$, the given series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ Diverges by the Basic Comparison Test.

Summary: $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n} > \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Divergent P- series with } P=1}$

Recall: $\sum_{n=1}^{\infty} \frac{1}{n^P}$ converges if $P>1$ and Diverges if $P \leq 1$

C] $\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n} = \sum_{n=1}^{\infty} \frac{2\ln n}{n} = 2 \sum_{n=1}^{\infty} \frac{\ln n}{n}$

We know that $\ln x > 1$ when $x > e$, so $\ln n > 1$ when $n \geq 3$, and since adding/subtracting a finite number of terms from a series does not affect the convergence/Divergence of the series.

We can consider $\sum_{n=3}^{\infty} \frac{2\ln n}{n} = 2 \sum_{n=3}^{\infty} \frac{\ln n}{n}$

We know that $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$ and

$\frac{2\ln n}{n} > \frac{2}{n}$ for $n \geq 3$ and since $2\sum_{n=3}^{\infty} \frac{1}{n}$

is a Divergent P-series with $P=1$, the given series

$\sum_{n=3}^{\infty} \frac{2\ln n}{n}$ Diverges by the Basic Comparison

Test and $\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n}$ is Divergent.

Summary

$$\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n} = \sum_{n=1}^{\infty} \frac{2\ln n}{n}$$

Recall: $\ln x^r = r \ln x$

for $n \geq 3$ $\ln n > 1$ so we considered $\sum_{n=3}^{\infty} \frac{2\ln n}{n}$

$$\sum_{n=3}^{\infty} \frac{2\ln n}{n} > \boxed{2\sum_{n=3}^{\infty} \frac{1}{n}}$$

Divergent P-series with $P=1$

$\therefore \sum_{n=1}^{\infty} \frac{\ln(n^2)}{n}$ Diverges by the Basic Comparison Test

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Basic Comparison Test 3

Determine whether the following series converges or diverges?

A] $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

B] $\sum_{n=1}^{\infty} \frac{3^n+n}{5^n}$

C] $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$

A] $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$; we know that $\ln n < \sqrt{n}$ for $n \geq 1$ since $\ln x < kx^p$ for $p > 0$ and $k > 0$ and large x

A] $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$; since $\frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{1.5}}$ and

$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent P-series with $P=1.5>1$,

the given series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent by the

Basic Comparison Test.

Summary: $\sum_{n=1}^{\infty} \frac{\ln n}{n^2} < \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \boxed{\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}}$

Convergent P-series
$P=1.5>1$

B] $\sum_{n=1}^{\infty} \frac{3^n+n}{5^n}$; we know that $n<3^n$ for $n \geq 1$

B] $\sum_{n=1}^{\infty} \frac{3^n + n}{5^n}$; looking at dominant term in the numerator 3^n and denominator 5^n, it is logical that this series $\sum_{n=1}^{\infty} \frac{3^n+n}{5^n}$ behaves like $\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ which is a convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ with $|r| = 3/5 < 1$, so intuitively we know that the given series $\sum_{n=1}^{\infty} \frac{3^n+n}{5^n}$ converges.

Let's formally verify our intuition by applying the Basic Comparison Test.

B] $\sum_{n=1}^{\infty} \frac{3^n+n}{5^n}$; since $n < 3^n$ for all $n \geq 1$

$$3^n + n < 3^n + 3^n \Rightarrow \frac{3^n+n}{5^n} < \frac{3^n+3^n}{5^n} = \frac{2(3^n)}{5^n}$$

Since $\sum_{n=1}^{\infty} \frac{2(3^n)}{5^n} = \sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^n$ is a convergent geometric series with $|r| = 3/5 < 1$, the given series $\sum_{n=1}^{\infty} \frac{3^n+n}{5^n}$ converges by the Basic Comparison Test:

<u>Summary</u>: $\sum_{n=1}^{\infty} \frac{3^n+n}{5^n} < \boxed{\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^n}$ $|r| = 3/5 < 1$

Convergent Geometric Series

c] $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$; we know that $(-1)^n = 1$ for n even

and $(-1)^n = -1$ for n odd, so $1 \leq 2+(-1)^n \leq 3$ for

all $n \geq 1$; so $2+(-1)^n \geq 1 \Rightarrow \frac{2+(-1)^n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$; Since

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a Divergent P-Series with $P = 1/2 < 1$

the given series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$ Diverges by the Basic

Comparison Test.

<u>Summary:</u> $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}} \geq \boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}}$

$P = 1/2 < 1$ Divergent

Divergent P-series

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Basic Comparison Test 4

Ex] Prove that if $a_n \geq 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ also converges.

Solution: Since $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

Since $\lim_{n\to\infty} a_n = 0$, then for sufficiently large $n \geq N$

$a_n < 1$ and since $0 \leq a_n < 1$ then $a_n^2 \leq a_n$ since a fraction less than 1 squared is less than the original fraction. Example $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$

Since $a_n^2 \leq a_n$ for $n \geqslant N$ and $\sum_{n=N}^{\infty} a_n$ converges by assumption, then $\sum_{n=N}^{\infty} a_n^2$ also converges by the Basic Comparison Test and since adding/subtracting a finite number of terms from a series does not affect the convergence/divergence of the series we can conclude that $\sum_{n=1}^{\infty} a_n^2$ Converges.

Summary : $a_n^2 \leq a_n$ for $n \geqslant N \Rightarrow \sum_{n=N}^{\infty} a_n^2 \leq \underbrace{\sum_{n=N}^{\infty} a_n}_{\text{Convergent series}}$

$\sum_{n=N}^{\infty} a_n^2$ Converges by Comparison Test.

Ex| Does the following series converge or Diverge?

$$\sum_{k=2}^{\infty} \left(\frac{2k}{4k-2}\right)^k$$

Intuition: $\left(\frac{2k}{4k-2}\right)^k \cong \left(\frac{2k}{4k}\right)^k = \left(\frac{1}{2}\right)^k$ for large k

Since $\sum_{k=2}^{\infty} \left(\frac{2k}{4k-2}\right)^k$ behaves like $\sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k$

which is a convergent geometric series $\sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k$

with $|r| = \frac{1}{2} < 1$, so intuitively we know

that the given series $\sum_{k=2}^{\infty} \left(\frac{2k}{4k-2}\right)^n$ converges.

Let's formally verify our intuition by applying the Basic Comparison Test.

$$\sum_{k=2}^{\infty}\left(\frac{2k}{4k-2}\right)^k \; ; \quad \text{since } k \geq 2 \Rightarrow -k \leq -2$$

$$\Rightarrow -k+4k \leq -2+4k \quad \Rightarrow \quad 4k-2 \geq 3k \Rightarrow \frac{1}{4k-2} \leq \frac{1}{3k}$$

$$\Rightarrow \frac{2k}{4k-2} \leq \frac{2k}{3k} \Rightarrow \left(\frac{2k}{4k-2}\right)^k \leq \left(\frac{2k}{3k}\right)^k = \left(\frac{2}{3}\right)^k$$

Since $\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k$ is a convergent geometric series

with $|r| = \frac{2}{3} < 1$, the given series $\sum_{k=2}^{\infty}\left(\frac{2k}{4k-2}\right)^k$

also converges by the Basic Comparison Test.

Summary: $\sum_{k=2}^{\infty}\left(\frac{2k}{4k-2}\right)^k < \boxed{\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k}$

Convergent Geometric Series with $|r| = 2/3 < 1$

Limit Comparison Test 1

Limit Comparison Test: Assume that $\sum a_n$ and $\sum b_n$ are series with positive terms, that is $a_n \geq 0$ and $b_n \geq 0$ for all n. If $\lim_{n \to \infty} \frac{a_n}{b_n} = K$ where K is a finite number $K > 0$, then either both series converge or both series diverge.

Note: $\lim_{n \to \infty} \frac{a_n}{b_n} = K \Rightarrow a_n \cong K b_n$ for large n

which means the two series $\sum a_n$ and $\sum b_n$ behave similiarly, either both series converge or both series Diverge.

Ex| Test the series $\sum_{n=1}^{\infty} \frac{1}{3^n - n}$ for convergence or Divergence.

Solution: looking at the dominant term in the denominator $3^n > n$ we expect for large n, $\sum_{n=1}^{\infty} \frac{1}{3^n - n}$ to behave like the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ with $|r| = 1/3 < 1$, so intuitively we expect that the given series $\sum_{n=1}^{\infty} \frac{1}{3^n - n}$ should also converge.

Let's formally verify our intuition by applying the Limit Comparison Test.

Given $\sum_{n=1}^{\infty} \frac{1}{3^n - n}$ let's use $a_n = \frac{1}{3^n - n}$ and $b_n = \frac{1}{3^n}$

So we apply the limit comparison test with

$b_n = \frac{1}{3^n}$; $\lim_{n\to\infty} \frac{\frac{1}{3^n - n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n - n}$

$$\lim_{n\to\infty} \frac{3^n}{3^n - n} = \lim_{n\to\infty} \frac{3^n/3^n}{\frac{3^n}{3^n} - \frac{n}{3^n}} = \lim_{n\to\infty} \frac{1}{1 - \frac{n}{3^n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1-\frac{n}{3^n}}$$

so we need to find $\lim_{n\to\infty} \frac{n}{3^n}$; since numerator n approaches ∞ and denominator 3^n also approaches ∞

$\lim_{n\to\infty} \frac{n}{3^n}$ has the $\frac{\infty}{\infty}$ pattern as $n \to \infty$, so we apply L'Hopital's rule to evaluate the limit.

$$\lim_{n\to\infty} \frac{n}{3^n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{3^n \ln 3} = 0$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1-\frac{n}{3^n}} = 1 > 0$$

Since this limit exists and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series with $|r| = 1/3 < 1$, the given series $\sum_{n=1}^{\infty} \frac{1}{3^n - n}$ also converges by the Limit Comparison Test.

Key Idea: Compare the given series to another simpler series (P-series or Geometric Series) whose convergence or divergence is known.

Limit Comparison Test 2

Ex] Test the following series for Convergence or Divergence.

A] $$\sum_{k=1}^{\infty} \frac{2k^4 - k^2 + 1}{k^6 - k + 2}$$

B] $$\sum_{n=1}^{\infty} \frac{2n^2 + 4n}{\sqrt{n^6 + n^2}}$$

Hint: Compare the given series to another simpler series (P-series or Geometric Series) whose convergence or divergence is known.

A] $\displaystyle\sum_{k=1}^{\infty} \frac{2k^4-k^2+1}{k^6-k+2}$

As $k \to \infty$, the dominant term in the numerator is $2k^4$ and the dominant term in the denominator is k^6.

So as $k \to \infty$ $\quad \dfrac{2k^4-k^2+1}{k^6-k+2} \approx \dfrac{2k^4}{k^6} = \dfrac{2}{k^2}$

So we should apply the Limit Comparison Test with $b_k = \dfrac{2}{k^2}$ and $a_k = \dfrac{2k^4-k^2+1}{k^6-k+2}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k^4-k^2+1}{k^6-k+2} \Big/ \frac{2}{k^2}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k^4 - k^2 + 1}{k^6 - k + 2} \cdot \frac{k^2}{2}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k^6 - k^4 + k^2}{2k^6 - 2k + 4}$$

$$= \lim_{k\to\infty} \frac{2k^6/k^6 - k^4/k^6 + k^2/k^6}{2k^6/k^6 - 2k/k^6 + 4/k^6}$$

$$= \lim_{k\to\infty} \frac{2 - 1/k^2 + 1/k^4}{2 - 2/k^5 + 4/k^6} = \frac{2}{2} = 1 > 0$$

Since $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k}$ exists and $\sum\limits_{k=1}^{\infty} \dfrac{2}{k^2}$ is a convergent

convergent P- series with $P=2>1$, the given series $\sum_{k=1}^{\infty} \frac{2k^4-k^2+1}{k^6-k+2}$ also converges by the Limit Comparison Test

---

B] $\sum_{n=1}^{\infty} \frac{2n^2+4n}{\sqrt{n^6+n^2}}$

For large n, the dominant term in the numerator is $2n^2$ and the dominant term in the denominator is $\sqrt{n^6} = n^3$ so as $n \to \infty$, $\frac{2n^2+4n}{\sqrt{n^6+n^2}} \approx \frac{2n^2}{n^3}$

as $n \to \infty$, $\dfrac{2n^2+4n}{\sqrt{n^6+n^2}} \approx \dfrac{2n^2}{n^3} = \dfrac{2}{n}$

So we should apply the Limit Comparison Test with

$a_n = \dfrac{2n^2+4n}{\sqrt{n^6+n^2}}$ and $b_n = \dfrac{2}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+4n}{\sqrt{n^6+n^2}} \Big/ \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2n^2+4n}{\sqrt{n^6+n^2}} \cdot \frac{n}{2} = \lim_{n\to\infty} \frac{2n^3+4n^2}{2\sqrt{n^6+n^2}}$$

Strategy: Divide Num. and Denom. by $\sqrt{n^6} = n^3$

Note: $\sqrt{n^6} = n^3$

$$= \lim_{n\to\infty} \frac{2n^3/n^3 + 4n^2/n^3}{2\sqrt{n^6/n^6 + n^2/n^6}}$$

$$= \lim_{n\to\infty} \frac{2 + 4/n}{2\sqrt{1 + 1/n^4}} = \frac{2}{2} = 1 > 0$$

since $\lim_{n\to\infty} 4/n = 0$ and $\lim_{n\to\infty} 1/n^4 = 0$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and exists and $\sum_{n=1}^{\infty} \frac{2}{n}$ is a Divergent P-series with $P = 1$, the given series $\sum_{n=1}^{\infty} \frac{2n^2+4n}{\sqrt{n^6+n^2}}$ also Diverges by the Limit Comparison Test.

Limit Comparison Test 3

Ex] Test the following series for convergence or Divergence

A] $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k+1}}{\sqrt[2]{k^3+4}}$

B] $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3+2n^2}{2^n(n^3+1)}$

Hint: Compare the given series to another simpler series (P-series or Geometric Series) whose convergence or divergence is known.

A] $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k+1}}{\sqrt[2]{k^3+4}}$

For large k, the dominant part of the numerator is $\sqrt[3]{k}$ and the dominant part of the denominator is $\sqrt[2]{k^3}$ so as $k \to \infty$, $\dfrac{\sqrt[3]{k+1}}{\sqrt[2]{k^3+4}} \approx \dfrac{k^{1/3}}{k^{3/2}} = \dfrac{1}{k^{7/6}}$

So we should apply the Limit Comparison Test with $a_k = \dfrac{\sqrt[3]{k+1}}{\sqrt[2]{k^3+4}}$ and $b_k = \dfrac{1}{k^{7/6}}$

$$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{\sqrt[3]{k+1}}{\sqrt[2]{k^3+4}}\Big/\frac{1}{k^{7/6}}=\lim_{k\to\infty}\frac{k^{7/6}\sqrt[3]{k+1}}{\sqrt[2]{k^3+4}}$$

$$=\lim_{k\to\infty}\frac{k^{7/6}\sqrt[3]{k(1+1/k)}}{\sqrt[2]{k^3(1+4/k^3)}}=\lim_{k\to\infty}\frac{k^{7/6}k^{1/3}\sqrt[3]{1+1/k}}{k^{3/2}\sqrt[2]{1+4/k^3}}$$

$$=\lim_{k\to\infty}\frac{k^{9/6}}{k^{3/2}}\frac{\sqrt[3]{1+1/k}}{\sqrt[2]{1+4/k^3}}=\lim_{k\to\infty}\frac{\cancel{k^{3/2}}\sqrt[3]{1+1/k}}{\cancel{k^{3/2}}\sqrt[2]{1+4/k^3}}=1>0$$

Since $\lim\limits_{k\to\infty}\frac{a_k}{b_k}=1$ and exists and $\sum\limits_{k=1}^{\infty}\frac{1}{k^{7/6}}$ is a convergent P-series with $P=7/6>1$, the given series

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k+1}}{\sqrt[2]{k^3+4}}$$ also converges by the Limit Comparison Test.

B] $$\sum_{n=1}^{\infty} \frac{4n^3+2n^2}{2^n(n^3+1)}$$

As $n \to \infty$, the dominant part of the numerator is $4n^3$ and the dominant part of the denominator is $2^n \cdot n^3$ so as $n \to \infty$, $\frac{4n^3+2n^2}{2^n \cdot n^3+2^n} \approx \frac{4\cancel{n^3}}{2^n \cdot \cancel{n^3}} = \frac{4}{2^n}$

So we should apply the Limit Comparison Test with $a_n = \dfrac{4n^3 + 2n^2}{2^n \cdot n^3 + 2^n}$ and $b^n = \dfrac{4}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4n^3+2n^2}{2^n \cdot n^3 + 2^n} \Big/ \frac{4}{2^n} = \lim_{n\to\infty} \frac{2^n(4n^3+2n^2)}{2^n(n^3+1)\cdot 4}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}(4n^3+2n^2)}{\cancel{2^n}(n^3+1)\cdot 4} = \lim_{n\to\infty} \frac{4n^3+2n^2}{4n^3+4}$$

$$= \lim_{n\to\infty} \frac{4n^3/n^3 + 2n^2/n^3}{4n^3/n^3 + 4/n^3} = \lim_{n\to\infty} \frac{4 + 2/n}{4 + 4/n^3} = 1 > 0$$

Since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$ exists and $\sum\limits_{n=1}^{\infty} 4\left(\dfrac{1}{2}\right)^n$ is a convergent geometric series with $|r| = \dfrac{1}{2} < 1$

the given series $\sum_{n=1}^{\infty} \frac{4n^3+2n^2}{2^n(n^3+1)}$ also converges

by the Limit Comparison Test.

Limit Comparison Test 4

Ex] Test the following series for convergence or Divergence ?

A] $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3e^{-n}}{\sqrt{4n^6-n}}$

B] $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

A] $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3e^{-n}}{\sqrt{4n^6-n}}$

As $n \to \infty$, the dominant part of the numerator is $2n$ as $3e^{-n} = \frac{3}{e^n} \Rightarrow 0$ as $n \to \infty$ and the dominant part of the denominator $\sqrt{4n^6 - n}$ is $\sqrt{4n^6}$ as $n << 4n^6$

So as $n \to \infty$, $\frac{2n+3e^{-n}}{\sqrt{4n^6-n}} \approx \frac{2n}{\sqrt{4n^6}} = \frac{2n}{2n^3} = \frac{1}{n^2}$

So we should apply the *Limit Comparison Test* with $a_n = \frac{2n+3e^{-n}}{\sqrt{4n^6-n}}$ and $\boxed{b_n = \frac{1}{n^2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+3e^{-n}}{\sqrt{4n^6-n}} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{(2n+3e^{-n})n^2}{\sqrt{4n^6-n}}$$

$$= \lim_{n\to\infty} \frac{(2n^3+3n^2e^{-n})}{\sqrt{4n^6-n}}$$

$$= \lim_{n\to\infty} \frac{(2n^3+3n^2e^{-n})}{\sqrt{4n^6-n}} = \lim_{n\to\infty} \frac{(2n^3/n^3 + 3n^2e^{-n}/n^3)}{\sqrt{4n^6/n^6 - n/n^6}}$$

Dividing by n^6 inside the radical $\sqrt{\ }$ is equivalent to dividing by n^3 outside the radical since $\sqrt{n^6} = n^3$ for $n > 0$

$$= \lim_{n\to\infty} \frac{2 + 3/ne^n}{\sqrt{4 - 1/n^5}} = \frac{2+0}{\sqrt{4-0}} = \frac{2}{2} = 1 > 0$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ exists and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent P-series with $P = 2 > 1$, the given series $\sum_{n=1}^{\infty} \frac{2n+3e^{-n}}{\sqrt{4n^6-n}}$ also converges by the Limit Comparison Test.

B] $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Since we have $\frac{1}{n}$ inside the sine function, this suggests that we should apply Limit Comparison Test with $b_n = 1/n$ and $a_n = \sin(1/n)$

let $u = \frac{1}{n}$, as $n \to \infty \Rightarrow u \to 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{u\to 0} \frac{\sin(u)}{u}$$

$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{u \to 0} \frac{\sin u}{u}$; since we have the 0/0 pattern we will apply L'Hopital's Rule.

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{u \to 0} \frac{\sin u}{u} \overset{H}{=} \lim_{u \to 0} \frac{\cos u}{1} = 1 > 0$$

Since $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ and exists and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a Divergent P-series with P=1, the given series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ also diverges by the Limit Comparison Test.

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Alternating Series Test I

Let's consider a series whose terms alternate in sign, $+-+-+-\cdots$

EX] $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+-\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

EX] $-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\frac{1}{25}+\frac{1}{36}-+\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n^2}$

Notice the n-th term of an alternating series has the pattern : $a_n=(-1)^{n+1}b_n$ or $a_n=(-1)^n b_n$

where $b_n>0$, $|a_n|=|(-1)^n b_n|=b_n$

Basic Idea: If the terms of an alternating series decrease in magnitude towards zero, then the series converges and has a finite sum. $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Alternating Series Test

The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges provided:

1] $0 < a_{n+1} \leq a_n$; terms of the series decrease in magnitude

2] $\lim_{n \to \infty} a_n = 0$

If these 2 conditions are satisfied, then the Alternating series converges and has a finite sum.

Caution: For positive term series $\sum_{n=1}^{\infty} a_n$; $\lim_{n\to\infty} a_n = 0$ does not imply convergence, but for alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ with decreasing terms, $\lim_{n\to\infty} a_n = 0$ does imply convergence.

Ex] $\sum_{n=1}^{\infty} \frac{1}{n}$ is the Divergent P-series with $P=1$

Ex] $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent by the Alternating Series Test.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges by the Alternating Series Test

Since $a_{n+1} < a_n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$; terms decrease in magnitude and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$; therefore $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent by the Alternating Series Test.

Ex] Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ for convergence or divergence?

1] $a_{n+1} < a_n \Rightarrow$ It is not clear that the terms given by $a_n = \frac{n}{n^2+1}$ are decreasing, so lets consider $f(x) = \frac{x}{x^2+1}$

By the Quotient Rule: $f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$

$f'(x) = \frac{1-x^2}{(x^2+1)^2}$; Since $f'(x) < 0 \Rightarrow 1 - x^2 < 0 \Rightarrow x^2 > 1$

$\Rightarrow x > 1$ or $x < -1$, so for $x > 1$, $f(x)$ is decreasing on $(1, \infty)$, hence $a_{n+1} < a_n$ for $n > 1$

2] $\lim_{n\to\infty} a_n = 0$; $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n/n^2}{n^2/n^2 + 1/n^2}$

$$\lim_{n\to\infty} a_n = \frac{1/n}{1+1/n^2} = \frac{0}{1+0} = 0$$

∴ The given series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ is convergent

by the Alternating Series Test.

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Alternating Series Test 2

Ex| Test the following series for convergence or divergence?

A| $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$

B| $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+5}$

C| $\sum_{n=1}^{\infty} (-1)^n \frac{\log e^n}{\cos(n\pi)\, n^4}$

A] $\sum_{n=2}^{\infty} (-1)^n \frac{\text{Ln}\, n}{n}$

1] $a_{n+1} < a_n \Rightarrow$ It is not clear that the terms given by $a_n = \frac{\text{Ln}\, n}{n}$ are decreasing, so let's consider

$f(x) = \frac{\text{Ln}\, x}{x}$; By the Quotient Rule:

$$f'(x) = \frac{x \cdot 1/x - \text{Ln}\, x \cdot 1}{x^2} = \frac{1 - \text{Ln}\, x}{x^2}$$

$f'(x) < 0 \Rightarrow 1 - \text{Ln}\, x < 0 \Rightarrow \text{Ln}\, x > 1 \Rightarrow e^{\text{Ln}\, x} > e^1$

$\Rightarrow x > e \Rightarrow$ the terms $a_n = \text{Ln}\, n/n$ are decreasing for $n \geq 3$, hence $a_{n+1} < a_n$ for $n \geq 3$

2] $\lim_{n\to\infty} a_n = 0$; $\lim_{n\to\infty} \frac{\ln n}{n}$ $\frac{\infty}{\infty}$ pattern

Since $\lim_{n\to\infty} \frac{\ln n}{n}$ has the $\frac{\infty}{\infty}$ pattern we need to

apply L'Hopital's rule :

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} \overset{H}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$$

$\therefore$ The given series $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ is convergent

by the Alternating Series Test.

B] $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+5}$

Although this series is alternating but:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+5} = \lim_{n\to\infty} \frac{n/n}{2n/n + 5/n}$$

$$= \lim_{n\to\infty} \frac{1}{2+5/n} = \frac{1}{2}$$

$$\lim_{n\to\infty} (-1)^n \frac{n}{2n+5} = \begin{cases} \frac{1}{2} & \text{for } n \text{ even } (2,4,6,\dots) \\ \frac{-1}{2} & \text{for } n \text{ odd } (1,3,5,\dots) \end{cases}$$

Since $\lim_{n\to\infty} (-1)^n \frac{n}{2n+5} \neq 0$, this series Diverges by the Divergence Test.

c] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \log e^n}{\cos(n\pi)\, n^4}$

let's simplify $(-1)^n a_n$ first:

$\log e^n = \ln e^n = n$

$\{\cos n\pi\}_{n=1}^{\infty} = \{\cos\pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \ldots\}$

$= \{-1, 1, -1, 1, \ldots\}$

$\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \ldots\}$

$\underline{\underline{(-1)^n = \cos(n\pi)}}$

c] $\displaystyle\sum_{n=1}^{\infty} \frac{\cancel{(-1)^n} \log e^n}{\cancel{\cos(n\pi)}\, n^4} = \sum_{n=1}^{\infty} \frac{n}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

After simplification we realize that the given series is not an Alternating Series, infact

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \log e^n}{\cos(n\pi)\, n^4} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a P-series with $P = 3 > 1$

So the given series is convergent because it is a P-series with $P = 3 > 1$.

Alternating Series 3 Estimating Sums

If a series $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=1}^{\infty} (-1)^n a_n$ converges to a finite sum S, then the remainder is $R_n = S - S_n$ where S_n is the partial sum of the first n terms. $|R_n| = |S - S_n|$ is the absolute error in approximating the sum of the series $S = \sum_{n=1}^{\infty} (-1)^n a_n$ with $S_n = \sum_{k=1}^{n} (-1)^k a_k$, and for a converging alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$, the size of the error $|R_n| = |S - S_n|$ is less than a_{n+1}, (The first omitted term of the series.)

Alternating Series Remainder Theorem

If $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is the sum of an alternating series that satisfies:

1] $0 \leq a_{n+1} < a_n$ and 2] $\lim_{n \to \infty} a_n = 0$

then $|R_n| = |S - S_n| \leq a_{n+1}$

Ex] How many terms of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ have to be added up so that the error is less than 10^{-6} ?

Strategy: 1] Show that given series converges.

2] Apply AST Remainder Theorem.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ is a convergent alternating series

since: 1] $a_{n+1} < a_n \Rightarrow \frac{1}{(n+1)^2} < \frac{1}{n^2}$

2] $\lim_{n \to \infty} a_n = 0 \Rightarrow \lim_{n \to \infty} \frac{1}{n^2} = 0$

let's express the given series as the sum of the first n terms plus the remainder.

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \underbrace{1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + \frac{(-1)^{n+1}}{n^2}}_{S_n = n\text{th partial sum}} + \underbrace{\frac{(-1)^{n+2}}{(n+1)^2} + \cdots}_{|R_n| = |S - S_n|}$$

$|R_n| = |S - S_n|$ is less than this term

$|S - S_n| \leq |a_{n+1}|$ By the Remainder Theorem

$$a_{n+1} = \left| \frac{(-1)^{n+2}}{(n+1)^2} \right| = \frac{1}{(n+1)^2}$$

We need five decimal place accuracy

$$|S - S_n| \leq a_{n+1} < 0.000001$$

$$a_{n+1} = \frac{1}{(n+1)^2} < 10^{-6} \Rightarrow (n+1)^2 > 10^6$$

$$\Rightarrow \sqrt{(n+1)^2} > \sqrt{10^6} \Rightarrow |n+1| > 10^3 \Rightarrow n+1 > 1000$$

$$\Rightarrow n > 999 \Rightarrow n = 1000$$

therefore we need to add up $n = 1000$ terms of this series to approximate the sum of this series with error less than 0.000001 or 10^{-6}.

<u>Summary:</u>

$$S_{1000} = \sum_{n=1}^{1000} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + \frac{(-1)^{1001}}{(1000)^2}$$

$$|S - S_{1000}| < 10^{-6}$$

Alternating Series Test 4

Ex] For the following convergent alternating series, Find the smallest number of terms that have to be added up so that the error is less than 10^{-6} ?

A] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{10^n \cdot n}$

B] $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n)!}$

A] $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{10^n \cdot n}$

let's check conditions for the AST:

1] $a_{n+1} < a_n \Rightarrow \frac{1}{10^{n+1}(n+1)} < \frac{1}{10^n \cdot n}$ YES ✓

2] $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{10^n \cdot n} = 0$ YES ✓

We need 5 decimal place accuracy

$|R_n| = |S - S_n| \leq a_{n+1} < 0.000001$

$a_{n+1} = \frac{1}{10^{n+1}(n+1)} < 10^{-6}$

$$a_{n+1} = \frac{1}{10^{n+1}(n+1)} < \frac{1}{10^6}$$

let's try $n=4 \Rightarrow \dfrac{1}{10^5(5)} > \dfrac{1}{10^6}$ NO!

let's try $n=5 \Rightarrow \dfrac{1}{10^6(6)} < \dfrac{1}{10^6}$ YES ✓

∴ We need to add up the first 5 terms of this series to approximate the sum of this series with error less than 0.000001 or 10^{-6}.

$$S_5 = \sum_{n=1}^{5} \frac{(-1)^n}{10^n\, n} = \frac{-1}{10} + \frac{1}{10^2 \cdot 2} - \frac{1}{10^3 \cdot 3} + \frac{1}{10^4 \cdot 4} - \frac{1}{10^5 \cdot 5}$$

B] $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n)!}$

Let's check the conditions for the AST:

$a_{n+1} < a_n \Rightarrow \frac{1}{(2(n+1))!} < \frac{1}{(2n)!}$ YES ✓

$\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{(2n)!} = 0$ YES ✓

We need 5 decimal place accuracy

$|R_n| = |S - S_n| \leq a_{n+1} < 0.000001$

$a_{n+1} = \frac{1}{(2(n+1))!} = \frac{1}{(2n+2)!} < 10^{-6}$

let's expand the series $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n)!}$

$$S = \frac{-1}{0!} + \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \frac{1}{8!} + \frac{1}{10!} - \frac{1}{12!}$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5 \quad n=6$

$$a_{n+1} = \frac{1}{(2n+2)!} < 10^{-6}$$ Remainder Theorem

Let's try $n=3 \Rightarrow a_4 = \frac{1}{8!} = 0.0000248 > 10^{-6}$

Let's try $n=4 \Rightarrow a_5 = \frac{1}{10!} = 0.00000028 < 10^{-6}$

$$|R_4| = |S - S_4| \leq a_5 < 10^{-6}$$

$\therefore$ $n=4$ works but because $\sum_{n=0}^{\infty}(-1)^{n+1}a_n$ starts at $n=0$, we need to add 1 to n to get the number of terms needed. $\Rightarrow n+1=4+1=5$ terms needed.

$$S_4=\frac{-1}{0!}+\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}-\frac{1}{8!}$$

Therefore we need to add up the first 5 terms of this series to approximate the sum of this alternating series with error less than 10^{-6}.

$$|S-S_4|<10^{-6}$$

Absolute Convergence I

Basic Idea: Consider a series $\sum_{n=1}^{\infty} a_n$, let's look at $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$ whose terms are the absolute values $|a_n|$ of the terms a_n.

Definition: A series $\sum_{n=1}^{\infty} a_n$ is called Absolutely Convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Ex) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - \cdots$ is absolutely convergent since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is a convergent P-series with $P = 2 > 1$ and so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ Converges

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + - \cdots$ is convergent by AST since $\sum_{n=1}^{\infty} (-1)^n a_n$ is alternating, the terms $a_n = \frac{1}{\sqrt{n}}$ decrease in magnitude towards zero since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, But the series is not Absolutely Convergent since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$ is a

Divergent P-series with $P = \frac{1}{2} < 1$

Definition: If $\sum_{n=1}^{\infty} |a_n|$ Diverges and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n$ is conditionally Convergent.

Note: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is Conditionally Convergent

Since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a Divergent P-series with $P = 1/2 < 1$, but $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is a convergent series by the AST.

(Alternating Series Test)

Ex| Determine if the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \frac{\sin 4}{4^2}$$

This series has both positive and negative terms since $\sin 1 > 0$, $\sin 2 > 0$, $\sin 3 > 0$, $\sin 4 < 0$, ... but is not alternating so we can't apply the AST, so we should look at $\sum_{n=1}^{\infty} |a_n|$

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$ but $|\sin n| \leq 1$ for all n

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent P-series with $P = 2 > 1$

$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ also converges by the Comparison Test, therefore $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is Absolutely Convergent and therefore is a convergent series.

Review of key concepts

1] If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ is Absolutely Convergent.

2] Absolute Convergence $\Rightarrow$ Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

3] $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if

$\sum_{n=1}^{\infty} |a_n|$ Diverges and $\sum_{n=1}^{\infty} a_n$ Converges.

4] If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges. This is called the Divergence Test.

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Absolute Convergence Test 2

Ex] Test the following series for convergence or Divergence ?

A] $\sum_{n=1}^{\infty} \frac{\sin n \sqrt{n^2+1}}{n^3 - n + 2}$

B] $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

A] $\sum_{n=1}^{\infty} \frac{\sin n \sqrt{n^2+1}}{n^3 - n + 2}$; This series has both Positive and negative terms since $\sin 1 > 0$, $\sin 2 > 0$,

$\sin 3 > 0$, $\sin 4 < 0$, ... but it is not alternating so we can't apply the AST, so we should look at $\sum_{n=1}^{\infty} |a_n|$

$$\sum_{n=1}^{\infty} \left| \frac{\sin n \sqrt{n^2+1}}{n^3-n+2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n| \sqrt{n^2+1}}{n^3-n+2}$$

but $|\sin n| \leq 1$ for all n.

$$\sum_{n=1}^{\infty} \frac{|\sin n| \sqrt{n^2+1}}{n^3-n+2} \leq \sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^3-n+2}$$

Let's check $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^3-n+2}$

Since $\frac{\sqrt{n^2+1}}{n^3-n+2} \approx \frac{n}{n^3} = \frac{1}{n^2}$ for Large n

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^3-n+2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Let's apply Limit Comparison Test with $b_n = \frac{1}{n^2}$

and $a_n = \frac{\sqrt{n^2+1}}{n^3-n+2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n^3-n+2} \bigg/ \frac{1}{n^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n^3-n+2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2+1} \cdot n^2}{n^3-n+2}$$

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}} \cdot \frac{n^2}{n^2}}{\frac{n^3}{n^3}-\frac{n}{n^3}+\frac{2}{n^3}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}}{1-\frac{1}{n^2}+\frac{2}{n^3}}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n^2}}}{1-\frac{1}{n^2}+\frac{2}{n^3}} = 1 > 0 \text{ , finite limit}$$

Since $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$ exists and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent P-series with $P = 2 > 1$, the series $\sum_{n=1}^{\infty}\frac{\sqrt{n^2+1}}{n^3-n+2}$ also converges by the Limit Comparison Test.

And since $\sum_{n=1}^{\infty}\left|\frac{\sin n\sqrt{n^2+1}}{n^3-n+2}\right| \le \sum_{n=1}^{\infty}\frac{\sqrt{n^2+1}}{n^3-n+2}$

$\therefore$ By Basic Comparison Test $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin n\sqrt{n^2+1}}{n^3-n+2}\right|$

also converges, so the original series

$\displaystyle\sum_{n=1}^{\infty}\frac{\sin n\sqrt{n^2+1}}{n^3-n+2}$ converges Absolutely and is a convergent series.

<u>Summary</u>: We applied 3 convergence tests to solve this question.

1. Absolute Convergence
2. Limit Comparison Test
3. Basic Comparison Test

B] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

Recall: $\ln x < kx^P$ For $k>0$, $P>0$ and x Large.

Since $\ln(n+1) < n+1$

$$\frac{1}{\ln(n+1)} > \frac{1}{n+1}$$

Let's do an Absolute Test

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\ln(n+1)} \right| = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} > \sum_{n=1}^{\infty} \frac{1}{n+1}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ is a Divergent Series as it can be compared to the Divergent P series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ with $P=1$. (Limit Comparison Test)

$\sum_{n=1}^{\infty} \frac{1}{Ln(n+1)}$ Also Diverges by the Basic Comparison Test, so the given series $\sum_{n=1}^{\infty} \frac{(-1)^n}{Ln(n+1)}$ does not converge absolutely.

So let's apply the Alternating Series Test (AST)

$\sum_{n=1}^{\infty} \frac{(-1)^n}{Ln(n+1)}$; Let's check conditions for AST

1] $a_{n+1} < a_n \Rightarrow \frac{1}{Ln(n+2)} < \frac{1}{Ln(n+1)}$ YES ✓

2] $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{Ln(n+1)} = 0$ YES ✓

So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ converges by the AST, but

Since the given series does not converge absolutely, this series is Conditionally Convergent.

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Ratio Test 1

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series and $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

1] If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is Absolutely Convergent by the Ratio Test.

2] If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is Divergent by the Ratio Test.

3] If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is Inconclusive. $\sum_{n=1}^{\infty} a_n$ may be Convergent or Divergent.

Ratio Test (Outline of the proof)

Let's assume that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \Rightarrow |a_{n+1}| \approx L|a_n|$

for Large n; Similiarly $|a_{n+2}| \approx L|a_{n+1}| = L^2|a_n|$

and $|a_{n+3}| \approx L|a_{n+2}| = L^3|a_n|$ and so on...

Therefore the infinite tail of the series $\sum_{n=1}^{\infty} a_n$

behaves like $|a_n| + |a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \cdots$

$= |a_n| + L|a_n| + L^2|a_n| + L^3|a_n| + \cdots$

$= |a_n|(1 + L + L^2 + L^3 + L^4 + \cdots)$

$= |a_n|(1+L+L^2+L^3+L^4+\cdots)$ which is a Geometric series with common ratio L; We know that a Geometric Series converges when the common Ratio $0 \leq L < 1$ and Diverges when $L > 1$ which is the conclution of the Ratio Test.

Summary: Ratio Test is asking Does the tail end of the series $\sum_{n=1}^{\infty} a_n$ behave like a Geometric Series and if the common ratio L of successive terms $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series converges absolutely.

Ex] Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n}$ for Convergence?

Let's apply the Ratio test with $a_n = (-1)^n \frac{n^2}{2^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^2}{2^{n+1}} \Big/ \frac{(-1)^n n^2}{2^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| \cdot \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}}$$

$$L = \lim_{n\to\infty} |-1| \cdot \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$$

Since $L = 1/2 < 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n}$ Converges Absolutely by the Ratio Test.

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Ratio Test 2

Ex] Test the following series for Convergence or Divergence ?

A] $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

B] $\sum_{n=1}^{\infty} \frac{n(-2)^n}{3^{2n+1}}$

C] $\sum_{n=1}^{\infty} \frac{(-1)^n (1.1)^n}{n^{10}}$

A] $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Let's apply the Ratio Test with $a_n = \dfrac{n!}{n^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \Big/ \frac{n!}{n^n}$$

$$L = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\cancel{(n+1)}(\cancel{n!})(n^n)}{(n+1)^n \cancel{(n+1)} (\cancel{n!})}$$

$$L = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$L = \frac{1}{e} < 1$$

Recall: $\displaystyle\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

Since $L = 1/e < 1$, the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Converges by the Ratio Test.

B) $\sum_{n=1}^{\infty} \frac{n(-2)^n}{3^{2n+1}} = \sum_{n=1}^{\infty} \frac{n(-1)^n 2^n}{3^{2n}\cdot 3} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n n}{3\cdot 9^n}$

let's apply Ratio Test with $a_n = \frac{(-1)^n \cdot 2^n \cdot n}{3\cdot 9^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 2^{n+1} (n+1)}{3\cdot 9^{n+1}} \Big/ \frac{(-1)^n 2^n n}{3\cdot 9^n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| \cdot \left(\frac{n+1}{n} \right) \cdot \frac{9^n}{9^{n+1}} \cdot \frac{2^{n+1}}{2^n} \Bigg|$$

$$L = \lim_{n\to\infty} \left|-1\right| \left(1+\frac{1}{n}\right)\cdot\frac{1}{9}\cdot 2 = \frac{2}{9} < 1$$

Since $L = 2/9 < 1$, the series Converges Absolutely by the Ratio Test.

C] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (1.1)^n}{n^{10}}$

Let's apply the ratio test with $a_n = \dfrac{(-1)^n (1.1)^n}{n^{10}}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (1.1)^{n+1}}{(n+1)^{10}} \Big/ \frac{(-1)^n (1.1)^n}{n^{10}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| \cdot \frac{(1.1)^{n+1}}{(1.1)^n} \cdot \frac{n^{10}}{(n+1)^{10}}$$

$$L = \lim_{n\to\infty} \left| -1 \right| (1.1) \left(\frac{n}{n+1} \right)^{10} = 1.1 > 1$$

Since $L = 1.1 > 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (1.1)^n}{n^{10}}$

Diverges by the Ratio Test.

Ratio Test 3

Ex] Test the following series for Convergence or Divergence ?

A] $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

B] $\sum_{n=1}^{\infty} \frac{n!}{1000^n}$

C] $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 4^n}{n!}$

A] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Let's apply the Ratio Test with $a_n = \dfrac{(-1)^n (n!)^2}{(2n)!}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\left((n+1)!\right)^2}{(2(n+1))!} \Big/ \frac{(-1)^n (n!)^2}{(2n)!} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| \frac{(n+1)!\,(n+1)!}{(2n+2)!} \; \frac{(2n)!}{n!\; n!}$$

$$L = \lim_{n\to\infty} \left| -1 \right| \frac{(n+1)\,n!\,(n+1)\,n! \quad (2n)!}{(2n+2)(2n+1)\,(2n)!\; n!\; n!}$$

$$L = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\,(n+1)\cancel{n!}\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{n!}\,\cancel{n!}}$$

$$L = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$$

Since $L = 1/4 < 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Converges Absolutely by the Ratio Test.

B] $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{1000^n}$

Let's apply the Ratio Test with $a_n = \dfrac{n!}{1000^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{1000^{n+1}} \bigg/ \frac{n!}{1000^n}$$

$$L = \lim_{n\to\infty} \frac{(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{n!} = \lim_{n\to\infty} \frac{(n+1)\,\cancel{n!}\,(1000)^n}{1000^{n+1}\,\cancel{n!}}$$

$$L = \lim_{n\to\infty} \frac{(n+1)}{1000} = \infty > 1$$

Since $L = \infty > 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{1000^n}$ Diverges

By the Ratio Test.

C] $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 4^n}{n!}$

Let's apply the Ratio Test with $a_n = \frac{(-1)^n n^2 4^n}{n!}$

$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^2 4^{n+1}}{(n+1)!} \Big/ \frac{(-1)^n n^2 4^n}{n!} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| \frac{(n+1)^2}{n^2} \; \frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!}$$

$$L = \lim_{n\to\infty} |-1| \left(\frac{n+1}{n}\right)^2 . 4 \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty} \frac{4(1+1/n)^2}{(n+1)}$$

$$L = 0 < 1$$

Since $L = 0 < 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 4^n}{n!}$

Converges Absolutely by the Ratio Test.

Ratio Test 4

Ex] Test the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n n!}{5.7.9.11\ldots(2n+3)}$ for

Convergence or Divergence?

Let's apply the Ratio Test with $a_n = \frac{(-1)^n 3^n n!}{5.7.9.11\ldots(2n+3)}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 3^{n+1} (n+1)!}{5.7.9.11\ldots(2n+3)(2n+5)} \Big/ \frac{(-1)^n 3^n n!}{5.7.9.11\ldots(2n+3)} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| \frac{3^{n+1} (n+1) n!}{5.7.9.11\ldots(2n+3)(2n+5)} \cdot \frac{5.7.9.11\ldots(2n+3)}{3^n n!}$$

$$L = \lim_{n\to\infty} |-1| \frac{3^{n+1}}{3^n} \frac{(n+1)\cancel{n!}}{\cancel{n!}} \frac{\cancel{5}.\cancel{7}.\cancel{9}.\cancel{11}\ldots\cancel{(2n+3)}}{\cancel{5}.\cancel{7}.\cancel{9}.\cancel{11}\ldots\cancel{(2n+3)}(2n+5)}$$

$$L = \lim_{n\to\infty} \frac{3(n+1)}{2n+5} = \lim_{n\to\infty} \frac{3n+3}{2n+5} = \frac{3}{2} > 1$$

Since $L = 3/2 > 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n n!}{5.7.9.11\ldots(2n+3)}$

Diverges by the Ratio Test.

Ex] The terms of a series $\sum_{n=1}^{\infty} a_n$ are given by the recursive relation:

$a_1 = 2$; $a_{n+1} = \dfrac{2n+1}{4n-2}\, a_n$

Determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges?

Let's apply the recursive relation $a_{n+1} = \dfrac{2n+1}{4n-2}\, a_n$ to form the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2n+1}{4n-2} \right| = \lim_{n\to\infty} \frac{2n+1}{4n-2} = \frac{1}{2} < 1$$

Since $L = \lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \dfrac{1}{2} < 1$, the series $\sum_{n=1}^{\infty} a_n$

Converges Absolutely by the Ratio Test.

Ex] Test the given series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^{n!}}$ for Convergence or Divergence?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^{n!}}$$

Let's apply the Ratio Test with $a_n = \dfrac{(-1)^{n+1}}{e^{n!}}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{e^{(n+1)!}} \Big/ \frac{(-1)^{n+1}}{e^{n!}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \right| \frac{e^{n!}}{e^{(n+1)!}} = \lim_{n\to\infty} |-1| \frac{1}{e^{(n+1)!-n!}}$$

$$L = \lim_{n\to\infty} \frac{1}{e^{(n+1)n!-n!}} = \lim_{n\to\infty} \frac{1}{e^{n!(n+1-1)}}$$

$$L = \lim_{n\to\infty} \frac{1}{e^{n!\,n}} = 0 < 1$$; Since $L = 0 < 1$;

the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^{n!}}$ Converges Absolutely by the Ratio Test.