Infinite Series

Infinite Sequence: $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$

Infinite Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

We can define an infinite series as a sequence of Partial Sums.

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_k + a_{k+1} + \cdots$$

$S_1 = a_1$; $S_2 = a_1 + a_2$; $S_3 = a_1 + a_2 + a_3$; … ; $S_n = a_1 + a_2 + \cdots + a_n$

Now consider the sequence of Partial Sums:

$\{S_1, S_2, S_3, \ldots, S_n\}$, If this sequence $\{S_n\}$ converges to S then the series $\sum_{k=1}^{\infty} a_k$ converges to S

where $S = \sum_{k=1}^{\infty} a_k$; S is the sum of the series, if the sequence of partial sums $\{S_1, S_2, S_3, \ldots, S_n\}$ or $\{S_n\}$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ Diverges (No Sum)

<u>Summary</u>: If the sequence $\{S_n\}$ converges and $\lim_{n\to\infty} S_n = S$ exists then the series $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = S$; S is the sum of the series otherwise if $\lim_{n\to\infty} S_n$ does not exist $\sum_{k=1}^{\infty} a_k$ is said to Diverge.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = S$$ Series Converges to Sum S.

Ex| Does the series $\sum_{k=1}^{\infty} k$ converge or Diverge?

Let's consider $S_n = \sum_{k=1}^{n} k = 1+2+3+\cdots+n$

Let's look at the Sequence of partial Sums:

$\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, S_3, \ldots, S_n, \ldots\}$

$S_1 = 1$; $S_2 = 1+2 = 3$; $S_3 = 1+2+3 = 6$; ... $S_n = 1+2+3+\cdots+n$

We have a closed formula for $S_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{\infty} k = \lim_{n\to\infty} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{n(n+1)}{2} \Rightarrow \infty$$

$\therefore$ the series $\sum_{k=1}^{\infty} k$ Diverges to ∞

Ex] Prove that the series $\sum_{K=1}^{\infty} \left(\frac{1}{3}\right)^K$ converges and find its sum S.

$$\sum_{K=1}^{\infty} \left(\frac{1}{3}\right)^K = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots + \frac{1}{3^K} + \cdots$$

let's consider $S_n = \sum_{K=1}^{n} (1/3)^K$

$S_1 = \frac{1}{3}$; $S_2 = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$; $S_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{13}{27}$; $S_4 = \frac{40}{81}$; ...

The pattern for the nth partial sum seems to be:

$S_n = \frac{1}{2}(3^n - 1)/3^n$ For example: $S_3 = \frac{1}{2}(3^3 - 1)/3^3 = \frac{13}{27}$

We can verify this by Mathematical induction:

$S_n = \frac{1}{2}(3^n - 1)/3^n$

$S_{n+1} = S_n + a_{n+1} = \dfrac{3^n - 1}{2\cdot 3^n} + \dfrac{1}{3^{n+1}} = \dfrac{3(3^n-1)+2}{2\cdot 3^{n+1}}$

$S_{n+1} = \dfrac{3^{n+1} - 1}{2\cdot 3^{n+1}}$ $\therefore$ By induction Formula for S_n holds for S_{n+1} and hence is true for all $n \geq 1$

$\therefore$ The sum of the given series is :

$S = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} (1/3)^k = \lim\limits_{n\to\infty} \dfrac{(3^n - 1)}{2\cdot 3^n}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{3^n}}{2\cdot\cancel{3^n}} - \dfrac{1}{2\cdot 3^n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{2} - \dfrac{1}{2\cdot 3^n}\right) = \dfrac{1}{2}$

$\therefore \sum_{k=1}^{\infty} (1/3)^k$ converges to the sum $S = 1/2$

$$\sum_{k=1}^{\infty} (1/3)^k = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots + \frac{1}{3^k} + \cdots = 1/2$$

This infinite series converges to the sum $S = 1/2$

Infinite Series 2 Geometric Series

An infinite geometric series has the pattern:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^{n-1} + \dots \qquad a \neq 0$$

Consider the nth partial sum $S_n = \sum_{k=0}^{n-1} ar^k$

$$S_n = \sum_{k=0}^{n-1} ar^k = a + ar + ar^2 + \dots + ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n$$ multiply by r

$$S_n - rS_n = a + ar + ar^2 + \dots + ar^{n-1}$$
$$\underline{\quad - (ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n)}$$

$$S_n - rS_n = a - ar^n$$ subtract

$S_n - rS_n = a - ar^n$ Assume $r \neq 1$ and $a \neq 0$

$S_n(1-r) = a - ar^n$ Factor out S_n

$$S_n = \frac{a(1-r^n)}{1-r}$$

<u>Consider $|r|<1$ or $-1<r<1$</u> $\Rightarrow \lim\limits_{n\to\infty} r^n = 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} (1-r^n)$$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

<u>Consider $r=1$</u>

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} = a + a + a + \dots + a = na$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na \to \pm\infty$$

Case $r=-1$

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a - a + a + \cdots + a(-1)^{n-1}$$

$S_1 = a$; $S_2 = 0$; $S_3 = a$; $S_4 = 0$; ...

$\lim\limits_{n\to\infty} S_n$ Does not exist since S_n oscillates between 0 and a. $\therefore \sum\limits_{k=0}^{\infty} a(-1)^k$ Diverges

Case $|r|>1$ or $r>1$, $r<-1$

$S_n = \sum\limits_{k=0}^{n-1} ar^k$ Diverges since $\lim\limits_{n\to\infty} r^n$ does not exist

$\therefore \sum\limits_{k=0}^{\infty} ar^k$ Diverges

Summary Geometric Series

Let $a \neq 0$ and r be real numbers then the series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^{n-1} + \dots$$

converges for $|r| < 1$ or $-1 < r < 1$ and its sum is:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for } |r| < 1$$

If $|r| \geq 1$, the geometric series Diverges

For finite n: $\displaystyle\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$

Provided $r \neq 1$

Ex] Find the sum of the following geometric series if it exists.

a) $1 - \frac{4}{3} + \frac{16}{9} - \frac{64}{27} + - \cdots \left(\frac{-4}{3}\right)^{k-1} + \cdots$

b) $1000 + 500 + 250 + \cdots + 1000\left(\frac{1}{2}\right)^{k-1} + \cdots$

a) Solution: $\sum_{k=1}^{\infty} \left(\frac{-4}{3}\right)^{k-1}$ Diverges since $r = -\frac{4}{3}$ and $|r| > 1$

b) $\sum_{k=1}^{\infty} 1000\left(\frac{1}{2}\right)^{k-1}$ Converges since $r = \frac{1}{2} < 1$ or $|r| < 1$

$$\sum_{k=1}^{\infty} 1000\,(1/2)^{k-1} \qquad\qquad \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r|<1$$

$a = 1000$ is the first term and $r = 1/2$ is the common ratio and since $r = \frac{1}{2} < 1$

$$S = \frac{a}{1-r} = \frac{1000}{1-1/2} = \frac{1000}{1/2} = 2000$$

$$\therefore 1000 + 500 + 250 + \cdots + 1000\left(\frac{1}{2}\right)^{k-1} + \cdots = 2000$$

$\sum_{k=1}^{\infty} 1000\,(1/2)^{k-1}$ converges to $S = 2000$

Infinite Series 3

Ex] Find the sum of the following geometric series if the series converges.

a) $\sum_{k=2}^{\infty} 3^k \, 2^{2-2k}$

solution: let's rewrite in the form $\sum_{k=1}^{\infty} ar^{k-1}$

$$\sum_{k=2}^{\infty} \frac{3^k}{2^{2k-2}} = \sum_{k=2}^{\infty} \frac{3^k \cdot 4}{4^k} = \sum_{k=2}^{\infty} 4\left(\frac{3}{4}\right)^k$$

but we need index k to start at k=1

$$\sum_{k=1}^{\infty} 4\left(\frac{3}{4}\right)^{k+1}$$

subtract 1 from index k
add 1 to exponent

$$\sum_{k=1}^{\infty} 4\left(\frac{3}{4}\right)^{k+1}$$ We need $\sum_{k=1}^{\infty} ar^{k-1}$

$$\sum_{k=1}^{\infty} 4\left(\frac{3}{4}\right)^{2}\left(\frac{3}{4}\right)^{k-1}$$ Factor out $(3/4)^2$

$$\sum_{k=1}^{\infty} \frac{9}{4}\left(\frac{3}{4}\right)^{k-1}$$ Simplify

This is a geometric series with $a=\frac{9}{4}$ and $r=3/4$ and since $r=3/4<1$, this geometric series converges with $S=\frac{a}{1-r} \Rightarrow S=\frac{9/4}{(1-3/4)}=\frac{9/4}{1/4}=9$

$\therefore \sum_{k=2}^{\infty} 3^k 2^{2-2k}$ converges with $S=9$

Ex] Express the number $5.2\overline{13}$ as a ratio of integers by applying geometric series.

$$5.2131313\ldots = 5.2 + \frac{13}{10^3} + \frac{13}{10^5} + \frac{13}{10^7} + \cdots$$

After the first term 5.2 we have a geometric series starting with $a = 13/10^3$ and $r = \frac{1}{10^2}$

To find common ratio r: $r = \frac{13/10^5}{13/10^3} = \frac{1}{10^2}$

$$5.2\overline{13} = 5.2 + \frac{13/10^3}{1 - 1/10^2}$$

$S = \frac{a}{1-r}$; $|r| < 1$

$$5.2\overline{13} = 5.2 + \frac{13/1000}{99/100} = 5.2 + \frac{13}{1000} \cdot \frac{100}{99}$$

$$5.2\overline{13} = 5.2 + \frac{13}{990} = \frac{52}{10} + \frac{13}{990} = \frac{99(52)+13}{990}$$

$$5.2\overline{13} = \frac{5161}{990}$$

Summary: We have applied Geometric Series to express a repeating decimal expansion as a fraction.

$$5.21313131\ldots = 5.2 + \underbrace{\frac{13}{10^3} + \frac{13}{10^5} + \frac{13}{10^7} + \cdots}_{\text{geometric series with } a = 13/10^3 \text{ and } r = 1/10^2} = \frac{5161}{990}$$

Infinite Series 4 (Telescoping Series)

Evaluate the following Series

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$

b) $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k}\right)$

Strategy: Find a formula for the partial Sum S_n and take $\lim\limits_{n\to\infty} S_n$ to evaluate the series.

a) $\displaystyle S_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots + \frac{1}{n\times(n+2)}$

Let's apply partial fractions so we can obtain a closed form for the partial sum S_n.

$$\frac{1}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2} = \frac{A(k+2)+Bk}{k(k+2)} = \frac{(A+B)k+2A}{k(k+2)}$$

multiply both sides by $k(k+2)$ to clear out the fractions

$$1 = (A+B)k + 2A$$

$A+B=0$ and $2A=1 \Rightarrow A = 1/2$, $B = -A = -1/2$

$$\therefore \frac{1}{k(k+2)} = \frac{1/2}{k} + \frac{-1/2}{k+2}$$

$$\therefore S_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \sum_{k=1}^{n} \frac{1/2}{k} + \frac{-1/2}{k+2}$$

$$S_n = \sum_{k=1}^{n} \frac{1/2}{k} - \frac{1/2}{k+2}$$

$$S_n = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+2}$$

Factor out 1/2

$$S_n = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right]$$

$$S_n = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cdots + \frac{1}{n-2} - \cancel{\frac{1}{n}} + \frac{1}{n-1} - \frac{1}{n+1} + \cancel{\frac{1}{n}} - \frac{1}{n+2}\right]$$

The first and third term remain from the left end and last term $\frac{-1}{n+2}$ and third last term $\frac{-1}{n+1}$ remain.

$$S_n = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

Notice the telescoping nature of this infinite series with almost all of the interior terms of the sum cancelling out we are left with a simple formula for Sn.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

∴ The given series converges with sum = 3/4

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{3}{4}$$

b) $S_n = \sum_{k=1}^{n} \ln\left(\frac{k+2}{k}\right) = \sum_{k=1}^{n} \left(\ln(k+2) - \ln k\right)$

$S_n = (\cancel{\ln 3} - \ln 1 + \cancel{\ln 4} - \ln 2 + \ln 5 - \cancel{\ln 3} + \ln 6 - \cancel{\ln 4} + \cdots +$
$+ \cancel{\ln(n-1)} - \ln(n-3) + \cancel{\ln(n)} - \ln(n-2) + \ln(n+1) - \cancel{\ln(n-1)}$
$+ \ln(n+2) - \cancel{\ln n})$

$S_n = -\ln 1 - \ln 2 + \ln(n+1) + \ln(n+2)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \underbrace{-\ln 1}_{=0} - \ln 2 + \ln(n+1) + \ln(n+2)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln 2 + \ln[(n+1)(n+2)] \Rightarrow \infty$

$\therefore \sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k}\right)$ Diverges to ∞

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Infinite Sequences 5

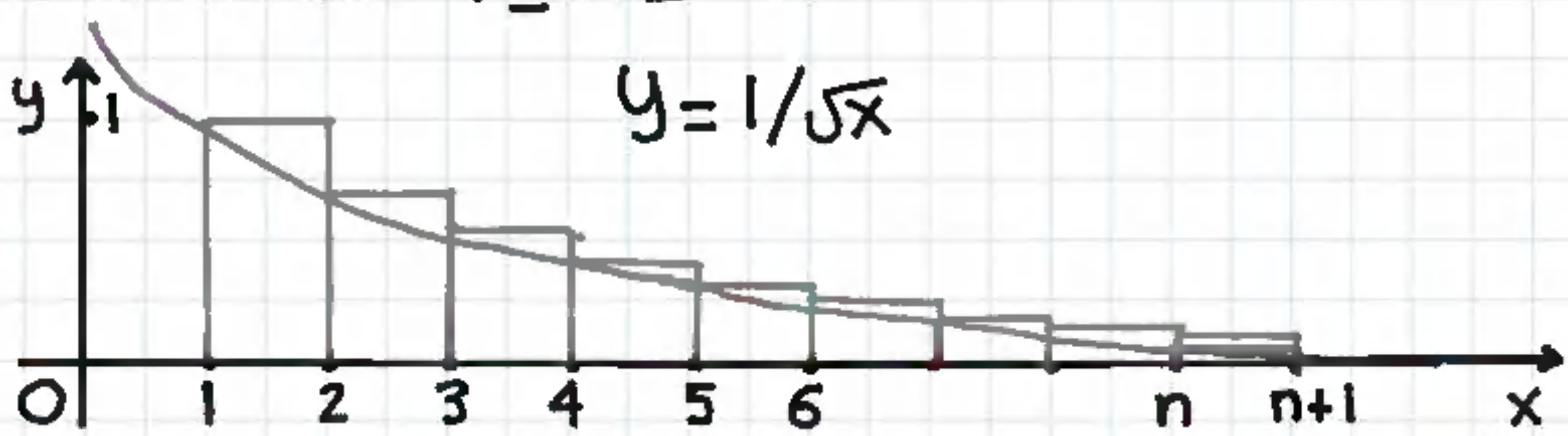
Ex] Show that the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges to ∞

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

The n-th term of the series is $f(n) = a_n = 1/\sqrt{n}$

Let's consider the related function $f(x) = 1/\sqrt{x}$

on the interval $1 \le x \le n+1$



$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

= sum of the areas of rectangles $A_1 + A_2 + \cdots + A_n$

where $A_1 = f(1)(1) = \frac{1}{\sqrt{1}} \times 1 = 1$; $A_2 = f(2)(1) = \frac{1}{\sqrt{2}}$

$A_3 = f(3)(1) = \frac{1}{\sqrt{3}} \times 1 = \frac{1}{\sqrt{3}}$; ... ; $A_n = f(n)(1) = \frac{1}{\sqrt{n}}$

Notice: $A_1 + A_2 + A_3 + \cdots + A_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = S_n$

Since the Areas of each of these rectangles is larger than the area under the curve $y = 1/\sqrt{x}$

$$S_n \geq \int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = \int_1^{n+1} x^{-1/2}\, dx = \left. \frac{x^{1/2}}{1/2} \right|_1^{n+1}$$

$$S_n \geq \int_1^{n+1} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\,\Big|_1^{n+1} = 2\sqrt{n+1} - 2$$

Since $\lim\limits_{n\to\infty} (2\sqrt{n+1} - 2) = \infty$

$\therefore \lim\limits_{n\to\infty} S_n = \infty$ and $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ Diverges to ∞

Theorem: If the infinite series $\sum\limits_{n=1}^{\infty} a_n$ is convergent then $\lim\limits_{n\to\infty} a_n = 0$

Proof: Let $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

$$S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1}$$

then $S_n - S_{n-1} = a_n$ and since $\sum_{n=1}^{\infty} a_n$ converges then the sequence of partial sums $\{S_n\}$ also converges, Assume $\lim_{n\to\infty} S_n = S$ and since $n-1 \to \infty$ as $n \to \infty$; $\lim_{n\to\infty} S_{n-1} = S$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

$\therefore$ we have proven that if $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$

Note: The converse of this theorem is not generally true as we proved earlier that

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ Diverges to ∞ while $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

<u>Divergence Test</u> : If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist then the series $\sum_{n=1}^{\infty} a_n$ is Divergent

Ex| Show that the series $\sum_{n=1}^{\infty} Ln\left(\frac{2n}{n+1}\right)$ Diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} Ln\left(\frac{2n}{n+1}\right) = Ln\left(\lim_{n\to\infty} \frac{2n}{n+1}\right) = Ln 2 \neq 0$$

Since $\lim_{n\to\infty} a_n = Ln 2 \neq 0$, This series Diverges by the Divergence Test.

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Infinite Series 6

Theory: If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series with the given sums A and B then:

a) $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to the sum $A+B$

b) $\sum_{n=1}^{\infty} (a_n - b_n)$ converges to the sum $A-B$

c) If C is a real number, $\sum_{n=1}^{\infty} Ca_n$ converges to the sum CA since $\sum_{n=1}^{\infty} Ca_n = C\sum_{n=1}^{\infty} a_n = CA$

Ex] Find the sum of the following series if the series converges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+2^n)}{4^n}$ b) $\displaystyle\sum_{n=1}^{\infty} \left[\left(\frac{e}{\pi}\right)^n + \left(\frac{\pi}{e}\right)^n\right]$

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n} + \left(\frac{2}{4}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

We have a sum of two convergent geometric series with both $r = 1/4 < 1$ and $r = 1/2 < 1$

So we can apply $S = \dfrac{a}{1-r}$ to each geometric series and add up the sums.

$$\sum_{n=1}^{\infty} \frac{(1+2^n)}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1/4}{(1-1/4)} + \frac{1/2}{(1-1/2)}$$

For $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$; $a = \frac{1}{4}$ plug $n=1$ to get first term a

$r = 1/4$ common ratio

$$\sum_{n=1}^{\infty} \frac{(1+2^n)}{4^n} = \frac{1/4}{3/4} + \frac{1/2}{1/2} = \frac{1}{3} + 1 = \frac{4}{3}$$

∴ The given series which is the sum of two convergent geometric series converges with sum $= 4/3$.

b) $\displaystyle\sum_{n=1}^{\infty}\left[\left(\frac{e}{\pi}\right)^n+\left(\frac{\pi}{e}\right)^n\right]$

let's split up the sum and evaluate each series separately.

$$\sum_{n=1}^{\infty}\left[\left(\frac{e}{\pi}\right)^n+\left(\frac{\pi}{e}\right)^n\right]=\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n+\sum_{n=1}^{\infty}\left(\frac{\pi}{e}\right)^n$$

Both series have the pattern $\displaystyle\sum_{n=1}^{\infty} r^n$ and are geometric series, the first series has $r=e/\pi$ and since $e<\pi \Rightarrow r<1$ and converges, however the second series has $r=\pi/e>1$ and Diverges.

$\therefore$ The entire series $\sum_{n=1}^{\infty}\left(\left(\frac{e}{\pi}\right)^n + \left(\frac{\pi}{e}\right)^n\right)$ Diverges

Theory: If the series $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ is a Divergent series, then $\sum (a_n + b_n)$ Diverges.

Summary: Both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ must converge to a sum in order for the entire series $\sum_{n=1}^{\infty} (a_n + b_n)$ to converge.

Infinite series 7

Ex| Show that the harmonic series $\sum_{n=1}^{\infty} 1/n$ Diverges

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

lets consider sequence of partial sums according to the powers of 2^n ; $S_1, S_2, S_4, S_8, S_{16}, S_{32}, \dots$

$S_1 = 1$; $S_2 = 1 + \frac{1}{2} = \frac{3}{2}$; $S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$S_8 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$S_8 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

Applying the same maneuver we can show that

$S_{16} > 1 + \frac{4}{2}$ and $S_{32} > 1 + \frac{5}{2}$

Now we see a pattern developing!

$$S_8 > 1 + \frac{3}{2} \Rightarrow S_{2^3} > 1 + \frac{3}{2}$$

$$S_{16} > 1 + \frac{4}{2} \Rightarrow S_{2^4} > 1 + \frac{4}{2}$$

the general pattern is $S_{2^n} > 1 + \frac{n}{2}$

$$\lim_{n\to\infty} S_{2^n} > \lim_{n\to\infty} \left(1 + \frac{n}{2}\right) = \infty$$

We have proven that the partial sum $S_{2^n} \to \infty$ as $n \to \infty$, therefore $\lim_{n\to\infty} S_n = \infty$ and therefore the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges to ∞.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is an example of a Divergent series whose terms a_n approach 0 as $n \to \infty$.

Ex] Evaluate the series $\sum_{n=2}^{\infty}\left(\frac{3}{n}+\frac{1}{3^n}\right)$

Let's split up the sum into the sum of two separate series and evaluate each sum separately.

$$\sum_{n=2}^{\infty}\left(\frac{3}{n}+\frac{1}{3^n}\right)=\sum_{n=2}^{\infty}\frac{3}{n}+\sum_{n=2}^{\infty}\frac{1}{3^n}$$

The first series $\sum_{n=2}^{\infty}\frac{3}{n}=3\sum_{n=2}^{\infty}\frac{1}{n}$ is a harmonic series that was proven to diverge to ∞ in the last example, the second series $\sum_{n=2}^{\infty} 1/3^n$ is a geometric series with $r=1/3<1$ and converges. However the entire series $\sum_{n=2}^{\infty} (3/n + 1/3^n)$ Diverges

since $\sum_{n=2}^{\infty} \frac{3}{n}$ is a Divergent Harmonic Series.

Key Concept: For the entire series $\sum_{n=1}^{\infty} (a_n + b_n)$ to be convergent both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have to be convergent and if one of the series say $\sum_{n=1}^{\infty} a_n$ Diverges then the entire series $\sum_{n=1}^{\infty} (a_n + b_n)$ also Diverges.

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Infinite series 8

For the following series find the value of x for which the series converges and find the sum.

a) $\sum_{n=2}^{\infty} (\tan x)^n$

b) $\sum_{n=1}^{\infty} (\ln x)^n$

c) $\sum_{n=0}^{\infty} \frac{(x+2)^n}{4^n}$

a) Solution: $\sum_{n=2}^{\infty} (\tan x)^n$ is a geometric series

with $a = (\tan x)^2$ as the first term and $r = \tan x$

and $r = \tan x$ as the common ratio, we need $|r| < 1$ so that the given geometric series converges.

$|r| < 1 \Rightarrow |\tan x| < 1 \Rightarrow -1 < \tan x < 1$

From the graph of $y = \tan x$ we obtain:

$-\frac{\pi}{4} < x < \frac{\pi}{4}$. Since $\tan x$ is periodic with period π

$-\frac{\pi}{4} + n\pi < x < \frac{\pi}{4} + n\pi$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\sum_{n=2}^{\infty} (\tan x)^n = \frac{\tan^2 x}{1 - \tan x}$$

For these values of x the geometric series converges to a sum.

$a = \tan^2 x$ and $r = \tan x$ $\quad S = \frac{a}{1-r}$

b) $\sum_{n=1}^{\infty} (\ln x)^n$

Solution: $\sum_{n=1}^{\infty} (\ln x)^n$ is a geometric series with the first term $a = \ln x$ and $r = \ln x$ as the common ratio.

We need $|r| < 1$ so that the given geometric series converges. $|r| < 1 \Rightarrow |\ln x| < 1 \Rightarrow -1 < \ln x < 1$

$-1 < \ln x < 1 \Rightarrow e^{-1} < e^{\ln x} < e^{1} \Rightarrow e^{-1} < x < e$

$\therefore$ For these values of x the geometric series has sum $= \dfrac{a}{1-r} \Rightarrow \sum_{n=1}^{\infty} (\ln x)^n = \dfrac{\ln x}{1 - \ln x}$

where $a = \ln x$ and $r = \ln x$

c) $\sum_{n=0}^{\infty} \frac{(x+2)^n}{4^n}$

Solution: $\sum_{n=0}^{\infty} \left(\frac{x+2}{4}\right)^n$ is a geometric series with

$a = \left(\frac{x+2}{4}\right)^0 = 1$ as the first term and $r = \frac{x+2}{4}$

as the common ratio. We need $|r| < 1$ so that the given geometric series converges.

$|r| < 1 \Rightarrow \left|\frac{x+2}{4}\right| < 1 \Rightarrow |x+2| < 4 \Rightarrow -4 < x+2 < 4$

$-6 < x < 2$. Therefore for these values of x the geometric series has sum $= \frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{4^n} = \frac{1}{1-\left(\frac{x+2}{4}\right)}$$

where $a=1$ and $r=\dfrac{x+2}{4}$

Infinite Series 9

Ex| Evaluate the following series :

a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^2}$

b) $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{n}{n+1}\right)$

c) $\sum_{n=1}^{\infty} \cos(n\pi) \frac{n}{2n+1}$

Solution: a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{2n}}{n^2}$ $\quad \frac{\infty}{\infty}$ Pattern

Let's apply L'Hopital's Rule

a) $\lim_{n\to\infty} \frac{e^{2n}}{n^2} \overset{H}{=} \lim_{n\to\infty} \frac{e^{2n}\cdot \cancel{2}}{\cancel{2}n} \overset{H}{=} \lim_{n\to\infty} \frac{e^{2n}\cdot 2}{1} = \infty \neq 0$

Since $\lim_{n\to\infty} a_n = \infty \neq 0$, this series Diverges

by the Divergence test.

b) $\lim_{n\to\infty} \tan^{-1}\left(\frac{n}{n+1}\right) = \tan^{-1}\left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \tan^{-1}(1)$

$= \pi/4 \neq 0$

Since $\lim_{n\to\infty} a_n = \pi/4 \neq 0$, this series Diverges

by the Divergence Test.

c) Let's investigate $a_n = \cos(n\pi)\dfrac{n}{2n+1}$

$$\cos(n\pi) = \begin{cases} 1 & \text{For } n \text{ even} \quad \cos(2\pi) = \cos(4\pi) = \cdots = 1 \\ -1 & \text{For } n \text{ odd} \quad \cos(\pi) = \cos(3\pi) = \cdots = -1 \end{cases}$$

$\therefore \cos(n\pi) = (-1)^n$

Let's rewrite $a_n = \cos(n\pi)\dfrac{n}{2n+1}$ as $a_n = (-1)^n \dfrac{n}{2n+1}$

$a_n = (-1)^n \dfrac{n}{2n+1} = \dfrac{-n}{2n+1}$ for n ODD

$a_n = (-1)^n \dfrac{n}{2n+1} = \dfrac{n}{2n+1}$ for n EVEN

$$a_n = (-1)^n \frac{n}{2n+1} = \begin{cases} \dfrac{-n}{2n+1} & \text{for } n \text{ odd} \\[2ex] \dfrac{n}{2n+1} & \text{for } n \text{ even} \end{cases}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-n}{2n+1} = -\frac{1}{2} \quad \text{for } n \text{ odd}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \quad \text{for } n \text{ even}$$

$\therefore \lim_{n\to\infty} a_n$ Oscillates between -1 and 1

Depending on n odd or n even

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n\pi)\frac{n}{2n+1}$ does not exist.

This series Diverges by the Divergence Test.

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Infinite Series 10

Ex] Find the value of C such that $\sum_{n=0}^{\infty} e^{nc} = 20$

Solution: Let's expand the series term by term.

$$\sum_{n=0}^{\infty} e^{nc} = 1 + e^{c} + e^{2c} + e^{3c} + \cdots + (e^{c})^{n} + \cdots = 20$$

Clearly this is a geometric series with the first term $a=1$ and common ratio $r=e^{c}$. For this infinite geometric series to converge to a finite sum we need $|r|<1 \Rightarrow |e^{c}|<1 \Rightarrow e^{c}<1$

$\ln e^{c} < \ln 1 \Rightarrow C<0$, Now let's find C.

$$\sum_{n=0}^{\infty} e^{nc} = 20 \quad \text{with } a=1 \text{ and } r=e^{c}$$

$$S = \frac{a}{1-r} \Rightarrow 20 = \frac{1}{1-e^{c}} \Rightarrow 20-20e^{c}=1$$

$$20e^{c} = 19 \Rightarrow e^{c} = \frac{19}{20} \Rightarrow \operatorname{Ln} e^{c} = \operatorname{Ln}\left(\frac{19}{20}\right)$$

$$\Rightarrow C = \operatorname{Ln}\left(\frac{19}{20}\right)$$

$\therefore$ If $C = \operatorname{Ln}\left(\frac{19}{20}\right)$ the geometric series

$\sum_{n=0}^{\infty} e^{nc}$ converges and has Sum $= 20$

$$\sum_{n=0}^{\infty} e^{n\operatorname{Ln}(19/20)} = 20$$

Ex| Given the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$S_n = \frac{2n-1}{2n+1}$ Find a_n and $\sum_{n=1}^{\infty} a_n$

Solution: For this type of question we are given the partial sum S_n and we have to work "backwards" to recover the general term a_n.

$$S_n - S_{n-1} = a_n$$

$$S_n - S_{n-1} = \frac{2n-1}{2n+1} - \frac{2(n-1)-1}{2(n-1)+1} = \frac{2n-1}{2n+1} - \frac{2n-3}{2n-1}$$

$$a_n = \frac{(2n-1)(2n-1)-(2n-3)(2n+1)}{(2n+1)(2n-1)} = \frac{4n^2-4n+1-(4n^2-4n-3)}{(2n+1)(2n-1)}$$

$$S_n - S_{n-1} = \frac{4}{(2n+1)(2n-1)} = \frac{4}{4n^2-1} = a_n$$

Now find the sum of the series $\sum_{n=1}^{\infty} a_n = S$

Take $\lim_{n\to\infty} S_n = S$

$$S_n = \frac{2n-1}{2n+1} \qquad a_n = \frac{4}{4n^2-1}$$

Partial Sum S_n and general term a_n

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2n-1}{2n+1} = 1 = S$$

$$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{4}{4n^2-1} = 1$$

Ex] Show that for all real values of x:

$$\sin x - \frac{1}{3}\sin^2 x + \frac{1}{9}\sin^3 x - \frac{1}{27}\sin^4 x + \cdots = \frac{3\sin x}{3+\sin x}$$

Solution: Clearly this is the expanded form of a geometric series with the first term $a = \sin x$ and the common ratio $r = \frac{-\sin x}{3}$, now for this geometric series to have a finite sum S we need

$$|r| < 1 \Rightarrow \left|\frac{-\sin x}{3}\right| < 1 \Rightarrow \frac{|\sin x|}{3} < 1$$

Since $|\sin x| \leq 1$ for all $x \Rightarrow \frac{|\sin x|}{3} \leq \frac{1}{3} < 1$

$\therefore |r| = \left|\frac{-\sin x}{3}\right| < 1$ for all x.

$\therefore\ |r| = \left|\dfrac{-\sin x}{3}\right| < 1$ and the infinite geometric series converges to a finite sum $S = \dfrac{a}{1-r}$

with $a = \sin x$ and $r = \dfrac{-\sin x}{3}$

$$\therefore\ S = \frac{a}{1-r} = \frac{\sin x}{1 - -\frac{\sin x}{3}} = \frac{3\sin x}{3+\sin x}$$

$$\therefore\ \sin x - \frac{1}{3}\sin^2 x + \frac{1}{9}\sin^3 x - \frac{1}{27}\sin^4 x + \cdots = \frac{3\sin x}{3+\sin x}$$

Infinite Series || Geometric Series Application

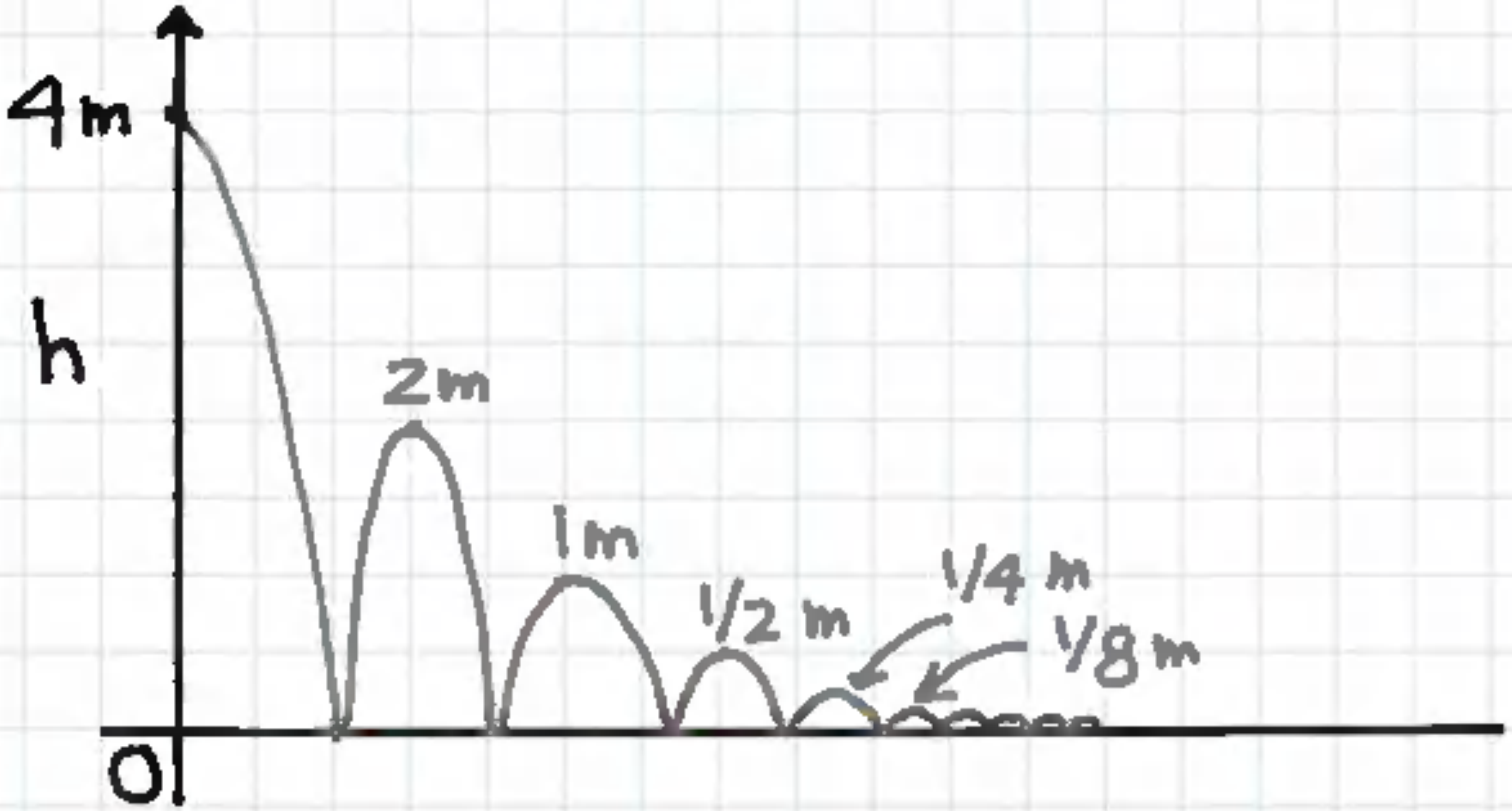
Ex| A basketball has the property that each time it falls from a height of h metres onto a hard level surface it rebounds to 50% of the height dropped. Assume that the ball is dropped from an initial height of 4 metres.

a) Assume that the ball continues to bounce up and down indefinitely, find the total distance the ball travels.

b) Find the total time that the ball travels up and

b) up and down. (Apply the fact that the ball falls a distance $\frac{1}{2}gt^2$ meters in t seconds.)

a) Solution: As the ball is dropped from a height of 4 metres it rebounds to a height of $\frac{1}{2}(4) = 2$ metres the ball drops down 2 metres and rebounds again to a height of $\frac{1}{2}(2) = 1$ metre, the ball then drops down 1 metre and rebounds again to a height of $\frac{1}{2}(1) = \frac{1}{2}$ metre. This process of up and down bouncing continues indefinitely. Let's draw a diagram.



So the total distance the ball travels is :

$$D = 4 + \left(\frac{1}{2}(4) + \frac{1}{2}(4)\right) + \left(\frac{1}{2}(2) + \frac{1}{2}(2)\right) + \left(\frac{1}{2}(1) + \frac{1}{2}(1)\right) + \left(\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)\right) + \cdots$$

$$D = 4 + 4 + 2 + 1 + \frac{1}{2} + \cdots$$

It is clear that after the first term 4, we have a geometric series starting with 4 and with a common ratio of $\frac{1}{2}$ so total distance travelled is:

$$D = 4 + \underbrace{4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}_{\text{Infinite Geometric Series with } a=4 \text{ and } r=1/2}$$

$$D = 4 + \frac{4}{1-1/2} = 4 + 8 = 12\,m \qquad \text{Apply } S = \frac{a}{1-r}$$

b) Initially ball has height of 4 metres, to find the time it takes for the ball to drop 4 metres, we

b) we apply $h=\frac{1}{2}gt^2$ to find the drop time.

$$h=\frac{1}{2}gt^2 \Rightarrow t=\sqrt{\frac{2h}{g}} \Rightarrow t=\sqrt{\frac{2(4)}{9.8}}$$

where $h=4\,m$ is initial height and $g=9.8\,m/s^2$

So the total time the ball travels up and down is:

$$T=\sqrt{\frac{2(4)}{9.8}}+2\sqrt{\frac{2(2)}{9.8}}+2\sqrt{\frac{2(1)}{9.8}}+2\sqrt{\frac{2(1/2)}{9.8}}+\cdots$$

$$T=\sqrt{\frac{2(4)}{9.8}}+2\sqrt{\frac{2}{9.8}}\;\underbrace{\left(\sqrt{2}+\sqrt{1}+\sqrt{1/2}+\sqrt{1/4}+\cdots+\right)}$$

Infinite Geometric series with $a=\sqrt{2}$ and $r=1/\sqrt{2}$

$$T=\sqrt{\frac{2(4)}{9.8}}+2\sqrt{\frac{2}{9.8}}\Big(\underbrace{\sqrt{2}+\sqrt{1}+\sqrt{1/2}+\sqrt{1/4}+\sqrt{1/8}+\dots}_{\text{Geometric}\quad a=\sqrt{2}\quad r=1/\sqrt{2}}\Big)$$

$$T=\sqrt{\frac{8}{9.8}}+2\sqrt{\frac{2}{9.8}}\left(\frac{\sqrt{2}}{1-1/\sqrt{2}}\right)$$

$$T=\sqrt{\frac{8}{9.8}}+2\sqrt{\frac{2}{9.8}}\left(\frac{2}{\sqrt{2}-1}\right)$$

$T=0.904+0.904(4.828)\cong 5.27$ seconds

$\therefore$ Total time the ball travels up and down is approximately 5.27 seconds.

Infinite Series 12 Geometric Series Application

Suppose in a hypothetical isolated community, the local government spends 1 billion dollars to stimulate the economy. Assume that the 1 billion is distributed equally to all citizens and that each citizen spends 80% of his or her received share of the 1 billion dollar and in turn the secondary recepient spends 80% of the money they received, and every recepient of spent money spends 80% of the money they have received and this cycle of spending goes on and on for n months, a) Find an expression for the total spending that has been generated after n months, that is find an expression for S_n.

b) Find the total spending that results from the government stimulus spending of 1 billion dollars. Hint: Apply $\lim_{n \to \infty} S_n$

a) Solution: Find S_n

Money spent by the government = 1 billion Dollars

Money spent by all the people who receive \$1 billion

$= 0.80(1 \text{ billion}) = 0.80$ billion dollars

Money spent by all the people who receive \$0.80 billion

$= 0.80(\$0.80 \text{ billion}) = (0.80)^2 . 1 = \$0.64$ billion

Money spent by all the people who receive \$0.64 billion

$= 0.80(0.8)^2 = (0.8)^3 . 1 = 0.512$ billion dollars

As the cycle of receiving and spending goes on and on

Total money spent by the people in the n th month

$= 0.8^{n-1} . 1$ billion

So the total spending that has been generated after n months is:

$$S_n = 1 + 0.8 + 0.8^2 + 0.8^3 + \dots + 0.8^{n-1}$$

S_n is a finite geometric series with n terms, with $a = 1$ and a common ratio of $r = 0.8 < 1$

$\therefore$ Apply Formula: $S_n = \dfrac{a(1-r^n)}{1-r}$

$$S_n = 1 + 0.8 + (0.8)^2 + (0.8)^3 + \dots + (0.8)^{n-1} = \frac{1(1-(0.8)^n)}{1-0.8}$$

$$S_n = \frac{1(1-(0.8)^n)}{0.2} = 5(1-(0.8)^n)$$ After n months

b) Find the total spending as $n \to \infty$

From part a) we found $S_n = 5(1-(0.8)^n)$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 5(1-(0.8)^n) = \lim_{n\to\infty} 5 - 5\lim_{n\to\infty} (0.8)^n$$

$\lim_{n\to\infty} S_n = 5 - 0$ since $\lim_{n\to\infty} (0.8)^n = 0$

$\lim_{n\to\infty} S_n = 5$ billion dollars

$\therefore$ 5 billion dollars of total spending has resulted from 1 billion stimulus spending by the government. Economists call this the multiplier effect of government spending.

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