Integration by parts theory explained as reverse of the product rule

Integration by Parts I

$$\int u\,dv = uv - \int v\,du$$

Reverse of Product Rule

Proof:

$$\frac{d}{dx}\left[u(x)\,v(x)\right] = u'(x)\,v(x) + u(x)\,v'(x)$$

Product Rule

$$\int \frac{d}{dx}\left[u(x)\,v(x)\right]dx = \int u'(x)\,v(x)\,dx + \int u(x)\,v'(x)\,dx$$

Integrate Both sides

$$u(x)\,v(x) = \int u'(x)\,v(x)\,dx + \int u(x)\,v'(x)\,dx$$

$$\int \overbrace{u(x)}^{u}\;\overbrace{v'(x)\,dx}^{dv} = \overbrace{u(x)\,v(x)}^{uv} - \int \overbrace{v(x)}^{v}\;\overbrace{u'(x)\,dx}^{du}$$

$$\int u\,dv = uv - \int v\,du$$ I.B.P Formula

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$ Definite Integral

$$\int_a^b u\,dv = u(b)v(b) - u(a)v(a) - \int_a^b v\,du$$ Definite integral

Strategy

1) Choose u easy to differentiate and choose dv easy to integrate. If Both u and dv are easy to differentiate and integrate then choose $u =$ Polynomial ex. $u = x, u = x^2$

This strategy will work most of the time.

2) A similiar strategy is LIATE

Choose u to be the function that comes first in this list

L : Logarithmic function ex: $u = \ln x$

I : Inverse Trig. function ex: $u = \tan^{-1} x$

A : Algebraic Function ex: $u = \sqrt{x}$, $u = x^2$

T : Trig. Function ex: $u = \sin x$, $u = \cos x$

E : Exponential Function ex: $u = e^x$, $u = 2^x$

Find ∫x e^(2x) dx by applying the LIATE principle (Integration by Parts)

$\boxed{\text{Ex}}$ $\int \underbrace{x}_{u}\underbrace{e^{2x}\,dx}_{dv}$

LIATE

$\downarrow$ x algebraic

$\searrow$ e^{2x} exponential

$$\boxed{\begin{array}{ll} u = x & du = dx \\ dv = e^{2x}\,dx & v = \dfrac{e^{2x}}{2} \end{array}}$$

Recall: $\int e^{kx}\,dx = \dfrac{e^{kx}}{k} + C$

Substitution

$$\underbrace{\frac{xe^{2x}}{2}}_{u\ v} - \int \underbrace{\frac{e^{2x}}{2}}_{v} \cdot \underbrace{dx}_{du} = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\,\frac{e^{2x}}{2} + C$$

$$\int x e^{2x}\,dx = \boxed{\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}$$

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Integration by Parts 2

Ex Evaluate $\int_1^e \ln x \, dx$

Solution: Let's rewrite integrand so we have a product of two functions

$$\int_1^e \ln x \, dx = \int_1^e \ln x \cdot 1 \, dx$$

Recall: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

$$\int_1^e \overbrace{\ln x}^{u} \cdot \overbrace{1 \, dx}^{dv}$$

$$\int_1^e \ln x \cdot 1\, dx$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = 1\, dx \\ du = \frac{1}{x}\, dx & v = x \end{array}}$$

$$\int_1^e \overbrace{\ln x}^{u} \cdot \overbrace{1\, dx}^{dv} = \overbrace{\ln x}^{u} \cdot \overbrace{x}^{v} \Big|_1^e - \int_1^e \overbrace{x}^{v} \cdot \overbrace{\frac{1}{x}\, dx}^{du}$$

$$= x \ln x \Big|_1^e - \int_1^e 1\, dx$$

$$= x \ln x \Big|_1^e - x \Big|_1^e = x \ln x - x \Big|_1^e$$

Apply F.T.C

$$= e \overset{=1}{\ln e} - e - \{ 1 \overset{=0}{\ln 1} - 1 \} = e - e - \{ 0 - 1 \}$$

$$= 0 + 1 = 1$$

Strategy Review

$$\int_1^e \underbrace{\ln x}_{u} \cdot \underbrace{1\,dx}_{dV}$$

Recall

Choose u to be function that comes first on list

LIATE

$L \rightarrow$ logs $\ln x$

$A \rightarrow$ Algebraic 1

Choose $u = \ln x$ $\quad dv = 1\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = x$

$\boxed{\text{Ex}}$ Evaluate $\int_{\pi/6}^{\pi/4} x \cdot \sec^2 x \, dx$

Solution: $\int_{\pi/6}^{\pi/4} \underbrace{x}_{u} \cdot \underbrace{\sec^2 x \, dx}_{dv}$

LIATE

$\downarrow$ Algebraic: x

$\rightarrow$ Trig.: $\sec^2 x$

Choose u to be function that comes first on list

$$\boxed{\begin{array}{ll} u = x & dv = \sec^2 x \, dx \\ du = dx & v = \tan x \end{array}}$$

$$\int_{\pi/6}^{\pi/4} x \cdot \sec^2 x \, dx$$

$$\boxed{\begin{array}{ll} u = x & dv = \sec^2 x \, dx \\ du = dx & v = \tan x \end{array}}$$

$$\int_{\pi/6}^{\pi/4} \overbrace{x}^{u} \; \overbrace{\sec^2 x \, dx}^{dv} = \overbrace{x}^{u} \cdot \overbrace{\tan x}^{v} \Big|_{\pi/6}^{\pi/4} - \int_{\pi/6}^{\pi/4} \overbrace{\tan x}^{v} \; \overbrace{dx}^{du}$$

***

$$= x \tan x - \boxed{\int \frac{\sin x}{\cos x} \, dx}$$

Apply subst. Rule

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$\int -\frac{du}{u} = -\ln|u|$$

$$= \boxed{-\ln|\cos x|} \; ***$$

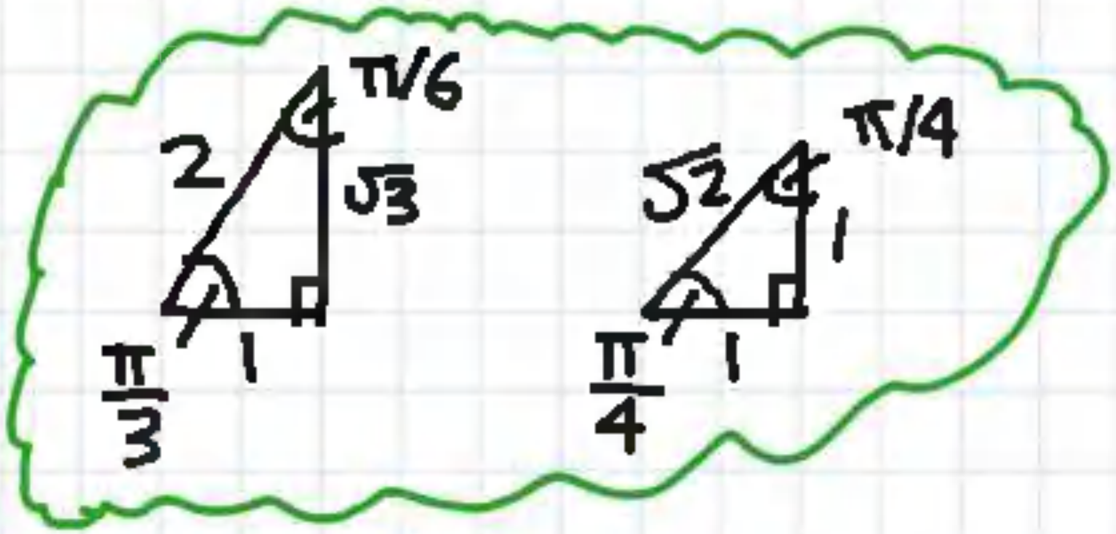
$$\int_{\pi/6}^{\pi/4} x\sec^2 x\,dx = \Big[x\tan x + \ln|\cos x|\Big]\Big|_{\pi/6}^{\pi/4}$$

Apply F.T.C

$$= \frac{\pi}{4}\overset{=1}{\tan\pi/4} + \ln\left|\overset{=1/\sqrt{2}}{\cos\frac{\pi}{4}}\right| - \left[\frac{\pi}{6}\overset{=1/\sqrt{3}}{\tan\frac{\pi}{6}} + \ln\left|\overset{=\frac{\sqrt{3}}{2}}{\cos\frac{\pi}{6}}\right|\right]$$

Note: $\ln\left|\frac{1}{\sqrt{2}}\right| = \ln(1/\sqrt{2})$

Since $\frac{1}{\sqrt{2}} > 0$



$$\frac{\pi}{4}\cdot 1 + \ln(1/\sqrt{2}) - \frac{\pi}{6}\cdot\frac{1}{\sqrt{3}} - \ln\left(\frac{\sqrt{3}}{2}\right)$$

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Integration by Parts 3

Ex] Evaluate $\int_0^{\pi/2} x \sin x \, dx$

Solution:

LIATE

Algebraic: x

Trig.: $\sin x$

LIATE

Choose u to be function that comes first!

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$u = x \qquad du = dx \qquad dv = \sin x\,dx \qquad v = -\cos x$

$$\int_0^{\pi/2} x \cdot \sin x \, dx$$

$$\boxed{\begin{array}{ll} u = x & dv = \sin x \, dx \\ du = dx & v = -\cos x \end{array}}$$

$$\int_0^{\pi/2} \underbrace{x}_{u} \cdot \underbrace{\sin x \, dx}_{dv} = \underbrace{x}_{u}\,(\underbrace{-\cos x}_{v})\Big|_0^{\pi/2} - \int_0^{\pi/2} \underbrace{-\cos x}_{v} \, \underbrace{dx}_{du}$$

$$= -x\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx$$

$$= -x\cos x\Big|_0^{\pi/2} + \sin x\Big|_0^{\pi/2}$$

$$\int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx = -x\cos x + \sin x \Big|_0^{\pi/2}$$

$$= -\frac{\pi}{2}\underbrace{\cos \pi/2}_{=0} + \underbrace{\sin\frac{\pi}{2}}_{=1} - \left\{ -0\underbrace{\cos 0}_{=1} + \underbrace{\sin 0}_{=0} \right\}$$

$$= 0 + 1 - \{0 + 0\} = 1$$

$$\int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx = 1$$

Evaluate ∫ √x Ln(x) dx integration by parts solved example

Ex| Evaluate $\int \sqrt{x}\, Lnx\, dx$

Solution :

$$\boxed{\begin{array}{ll} u = Lnx & dv = x^{1/2}\, dx \\ du = \frac{1}{x}\, dx & V = \frac{2}{3} x^{3/2} \end{array}}$$

$$\int u\, dv = uv - \int v\, du$$

$$\int x^{1/2} Lnx\, dx = Lnx . \frac{2}{3} x^{3/2} - \int \frac{2}{3} x^{3/2} . \frac{1}{x}\, dx$$

$$= \frac{2}{3} Lnx . x^{3/2} - \frac{2}{3} \int x^{1/2}\, dx$$

$$= \frac{2}{3} Lnx . x^{3/2} - \frac{2}{3} \frac{2}{3} x^{3/2} + C$$

$$= \frac{2}{3} \ln x \cdot x^{3/2} - \frac{4}{9} x^{3/2} + C$$

$$\int \sqrt{x} \ln x \, dx = \frac{2}{3} \ln x \cdot x^{3/2} - \frac{4}{9} x^{3/2} + C$$

Strategy Review $\int x^{1/2} \ln x \, dx$

L I A T E

Logs $\ln x$

Algebraic $x^{1/2}$

Choose $u = \ln x$

Choose u to be function that comes first.

$u = \ln x \quad du = \frac{1}{x} dx \quad dv = x^{1/2} dx \quad v = \frac{2}{3} x^{3/2}$

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Integration by Parts 4

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

I.B.P Formula

Ex) Evaluate $\int_1^e (\ln x)^2\,dx$

LIATE

logs: $(\ln x)^2$

Algebraic: 1

$$\int_1^e \underbrace{(\ln x)^2}_{u}\ \underbrace{1\,dx}_{dv}$$

$$\int_1^e (\ln x)^2 \cdot 1 \, dx$$

$$\boxed{\begin{array}{ll} u = (\ln x)^2 & dv = dx \\ du = 2(\ln x)\cdot\frac{1}{x}dx & v = x \end{array}}$$

$$= (\ln x)^2 \cdot x \Big|_1^e - \int_1^e \cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} \, dx$$

Apply I.B.P again

$$= x\cdot(\ln x)^2 \Big|_1^e - 2\boxed{\int_1^e \ln x \, dx}^{***}$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}dx & v = x \end{array}}$$

$$\left[x\ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx \right]$$

$$= \boxed{x\ln x - x} \; ***$$

$$\int_1^e (\ln x)^2\,dx = x(\ln x)^2\Big|_1^e - 2\left[x\ln x - x\right]\Big|_1^e$$

$$= x(\ln x)^2 - 2x\ln x + 2x\Big|_1^e$$

Simplify

$$= e(\overset{1}{\ln e})^2 - 2e\overset{1}{\ln e} + 2e - \left\{1(\overset{0}{\ln 1})^2 - 2(1)\overset{0}{\ln 1} + 2\right\}$$

$$= e(1)^2 - \cancel{2e} + \cancel{2e} - \left\{0 - 0 + 2\right\}$$

$$= \boxed{e - 2}$$

$$\int_1^e (\ln x)^2\,dx = e - 2$$

Recall

$\ln e = \log_e e = 1$

$\ln 1 = 0$

Evaluate ∫ x^2 sin(πx) dx integration by parts twice solved example

Ex Evaluate $\int x^2 \sin(\pi x)\,dx$

Solution:

L I A T E

Algebraic

x^2

Trig.

$\sin(\pi x)$

LIATE

Choose u to be the function that comes first

$$\int \underbrace{x^2}_{u} \underbrace{\sin(\pi x)\,dx}_{dv}$$

$u = x^2$ $\quad dv = \sin(\pi x)\,dx$

$du = 2x\,dx$ $\quad v = -\dfrac{\cos(\pi x)}{\pi}$

$$\int x^2 \sin(\pi x)\,dx$$

$$u = x^2 \qquad dv = \sin(\pi x)\,dx$$

$$du = 2x\,dx \qquad v = \frac{-\cos(\pi x)}{\pi}$$

$$= x^2\left(\frac{-\cos(\pi x)}{\pi}\right) - \int \frac{-\cos(\pi x)}{\pi}\cdot 2x\,dx$$

$$= -\frac{1}{\pi}x^2\cos(\pi x) + \frac{2}{\pi}\boxed{\int x\cos(\pi x)\,dx}$$

$$\int x\cos(\pi x)\,dx$$

$$u = x \qquad dv = \cos(\pi x)\,dx$$

$$du = dx \qquad v = \frac{\sin(\pi x)}{\pi}$$

$$\frac{x\cdot\sin(\pi x)}{\pi} - \int \frac{\sin(\pi x)}{\pi}\,dx$$

$$= -\frac{1}{\pi} x^2 \cos(\pi x) + \frac{2}{\pi} \boxed{\int x \cdot \cos(\pi x)\, dx} \ ***$$

$$= -\frac{1}{\pi} x^2 \cos(\pi x) + \frac{2}{\pi} \left[\frac{x \sin(\pi x)}{\pi} - \frac{1}{\pi} \int \sin(\pi x)\, dx \right]$$

$$= -\frac{1}{\pi} x^2 \cos(\pi x) + \frac{2}{\pi} \left[\frac{x \cdot \sin(\pi x)}{\pi} - \frac{1}{\pi} \left[\frac{-\cos(\pi x)}{\pi} \right] \right]$$

$$= -\frac{1}{\pi} x^2 \cos(\pi x) + \frac{2x \sin(\pi x)}{\pi^2} + \frac{2}{\pi^3} \cos(\pi x) + C$$

$$\int x^2 \sin(\pi x)\, dx = \text{ABOVE}$$

Notes

$$\int \sin(\pi x)\,dx = -\frac{\cos(\pi x)}{\pi} + C$$

Apply Subst. Rule

$u = \pi x \qquad du = \pi\, dx$

$\frac{1}{\pi}\, du = dx$

Evaluate ∫ sin^(-1)X dx from x=0 to x=1/2 integration by parts solved example

Integration by Parts 5

Ex] Evaluate $\int_0^{1/2} \sin^{-1}x \, dx$

L I A T E

I: Inverse Trig. $\sin^{-1}x$

A: Algebraic 1

$$\int_0^{1/2} \underbrace{\sin^{-1}x}_{u} \; \underbrace{1\,dx}_{dv}$$

Recall
Choose u to be function that comes first on list

$u = \sin^{-1}x \qquad dv = 1\,dx$

$du = \dfrac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$

$$\int_0^{1/2} \sin^{-1}x \, dx$$

$u = \sin^{-1}x \qquad dv = 1\,dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$

$$= (\sin^{-1}x)\,x \Big|_0^{1/2} - \boxed{\int_0^{1/2} x \cdot \frac{1}{\sqrt{1-x^2}}\,dx}$$

Apply Subst. method

$t = 1 - x^2$

$dt = -2x\,dx$

$-\frac{1}{2}\,dt = x\,dx$

$$\int_0^{1/2} (1-x^2)^{-1/2}\, x\,dx$$

$$\int t^{-1/2}\left(-\frac{1}{2}\right) dt$$

$$-\frac{1}{\cancel{2}}\, t^{1/2} \cdot \cancel{2} = \boxed{-\sqrt{1-x^2}\Big|_0^{1/2}}$$

$$\int_0^{1/2} \sin^{-1}x \, dx = x\sin^{-1}x \Big|_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx$$

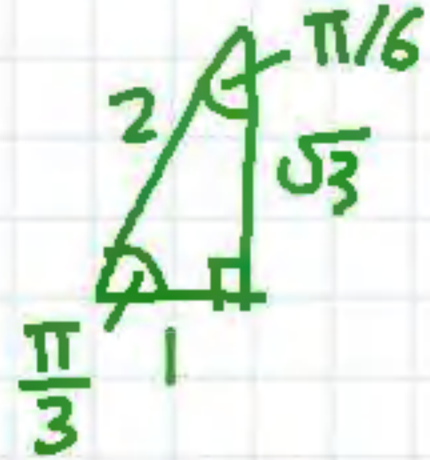
$$= x\sin^{-1}x \Big|_0^{1/2} - -\sqrt{1-x^2} \Big|_0^{1/2}$$

$$= x\sin^{-1}x + \sqrt{1-x^2} \Big|_0^{1/2}$$

$$= \frac{1}{2}\overset{\pi/6}{\sin^{-1}(1/2)} + \sqrt{1-1/4} - \left\{ \overset{0}{0\sin^{-1}0} + \sqrt{1-0^2} \right\}$$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \sqrt{3/4} - \left\{ 0 + 1 \right\}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$



$$\int_0^{1/2} \sin^{-1}x \, dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

Evaluate ∫ arctan(x) dx integration by parts solved example

Ex] Evaluate $\int \tan^{-1}x \, dx$

$$\int \underbrace{\tan^{-1}x}_{u} \cdot \underbrace{1\,dx}_{dv}$$

$$\boxed{\begin{array}{ll} u = \tan^{-1}x & dv = 1\,dx \\ du = \frac{1}{1+x^2}\,dx & v = x \end{array}}$$

$$= (\tan^{-1}x) \cdot x - \int x \cdot \frac{1}{1+x^2}\,dx$$

Apply Subst. method

$t = 1+x^2 \quad dt = 2x\,dx$

$\frac{1}{2}dt = x\,dx$

$$\int \frac{1}{t} \cdot \frac{1}{2}\,dt$$

$$\frac{1}{2}\ln|t| = \boxed{\frac{1}{2}\ln(1+x^2)}$$

$$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

Evaluate ∫ x^3 e^(x^2) dx by applying U-subst. and Integration by parts solved

Integration by Parts 6

Hint: first apply change of variables $t = x^k$, then apply Integration by Parts

Ex| Evaluate $\int x^3 e^{x^2} dx$

Let $t = x^2$ $\quad dt = 2x\,dx$

$$\frac{1}{2}dt = x\,dx$$

$$\int e^{x^2} \cdot x^2 \cdot x\,dx = \int e^t \cdot t \cdot \frac{1}{2}dt = \frac{1}{2}\int t \cdot e^t dt$$

Now Apply Integration by parts

$$\frac{1}{2}\int t e^t dt$$

$u = t$	$dv = e^t dt$
$du = dt$	$v = e^t$

$$= \frac{1}{2}\int \underbrace{t}_{u} \cdot \underbrace{e^t\,dt}_{dv}$$

$u = t$	$dv = e^t\,dt$
$du = dt$	$v = e^t$

$$= \frac{1}{2}\left[\overset{u}{t} \cdot \overset{v}{e^t} - \int \overset{v}{e^t}\, \overset{du}{dt} \right]$$

$$= \frac{1}{2}\left[te^t - e^t \right] + C$$

Subst. $t = x^2$

$$= \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2} \right] + C$$

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2} \right] + C$$

Ex| Evaluate $\int_0^{\frac{\pi^2}{16}} \cos(\sqrt{x})\,dx$

Strategy:

1| Apply $t = x^k$ subst. method

2| Integrate by Parts

$$\int_0^{\pi^2/16} \cos(\sqrt{x})\,dx$$

$t = x^{1/2}$ $\quad dt = \frac{1}{2\sqrt{x}}\,dx$

$$dx = 2\sqrt{x}\,dt = 2t\,dt$$

Change limits : $\quad x = 0 \quad t = \sqrt{x} \quad t = 0$

$x = \pi^2/16 \quad t = \sqrt{x} \quad t = \pi/4$

$$\int_0^{\frac{\pi}{4}} \cos t \,.\, 2t\,dt = 2\int_0^{\frac{\pi}{4}} t \,.\, \cos t\,dt$$

$$\int_0^{\pi^2/16} \cos(\sqrt{x})\,dx = 2\int_0^{\pi/4} t.\cos t\,dt$$

Now Apply Integration by Parts

$$= 2\int_0^{\pi/4} \underbrace{t}_{u}.\underbrace{\cos t\,dt}_{dv}$$

$$\boxed{\begin{array}{ll} u = t & dv = \cos t\,dt \\ du = dt & v = \sin t \end{array}}$$

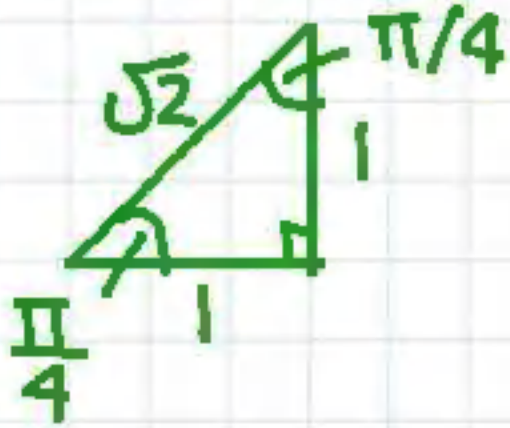
$$= 2\left[\underbrace{t}_{u}.\underbrace{\sin t}_{v}\Big|_0^{\pi/4} - \int_0^{\pi/4} \underbrace{\sin t}_{v}\,\underbrace{dt}_{du}\right]$$

$$= 2\left[t.\sin t\Big|_0^{\pi/4} - -\cos t\Big|_0^{\pi/4}\right]$$

$$= 2\left[t\sin t \Big|_0^{\pi/4} + \cos t \Big|_0^{\pi/4} \right]$$

$$= 2\left[\left[t\sin t + \cos t \right]\Big|_0^{\pi/4} \right]$$

$$= 2\left[\frac{\pi}{4} \underbrace{\sin\frac{\pi}{4}}_{=1/\sqrt{2}} + \underbrace{\cos\frac{\pi}{4}}_{=1/\sqrt{2}} \right] - 2\left[0\cdot \underbrace{\sin 0}_{=0} + \underbrace{\cos 0}_{=1} \right]$$



$$= 2\left[\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] - 2\left[0 + 1 \right]$$

$$= 2\left[\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} \right] - 2 = \frac{2\pi}{4\sqrt{2}} + \frac{2}{\sqrt{2}} - 2$$

$$\int_0^{\pi^2/16} \cos(\sqrt{x})\,dx = \frac{\pi}{2\sqrt{2}} + \sqrt{2} - 2$$

Some notes

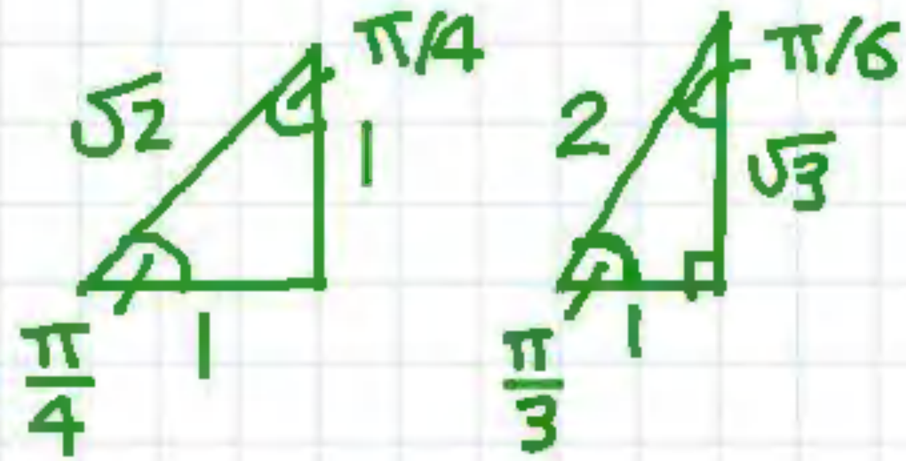
$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2}\,\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = \sqrt{2}$$

$\sin 0 = 0$ $\quad$ $\sin \frac{\pi}{2} = 1$

$\cos 0 = 1$ $\quad$ $\cos \frac{\pi}{2} = 0$

$\cos \pi/4 = 1/\sqrt{2}$ $\quad$ $\sin \frac{\pi}{4} = 1/\sqrt{2}$

Special Triangles



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Evaluate ∫ e^(2x) sin(πx) dx integration by parts loopy example solved

Integration by Parts 7

Ex| Evaluate $\int e^{2x} \sin(\pi x)\,dx$

Solution: Since the integrand is a product of two different functions and U substitution will not solve the integral, we will apply Integration by Parts. $\int u\,dv = uv - \int v\,du$

$$\int \underbrace{e^{2x}}_{u} \underbrace{\sin(\pi x)\,dx}_{dv}$$

$u = e^{2x}$ $\qquad dv = \sin(\pi x)\,dx$

$du = e^{2x} \cdot 2\,dx$ $\qquad V = \dfrac{-\cos(\pi x)}{\pi}$

$$\int \underbrace{e^{2x}}_{u} \underbrace{\sin(\pi x)\, dx}_{dv}$$

$$\boxed{\begin{array}{ll} u = e^{2x} & dv = \sin(\pi x)\, dx \\ du = 2e^{2x} dx & v = \dfrac{-\cos(\pi x)}{\pi} \end{array}}$$

$$= e^{2x} \cdot \frac{-\cos(\pi x)}{\pi} - \int \frac{-\cos(\pi x)}{\pi} \cdot 2e^{2x}\, dx$$

$$= -\frac{1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi} \boxed{\int e^{2x} \cos(\pi x)\, dx}$$

Note: we make "same" choices for u and dv
u = expon.
dv = trig.

$$\boxed{\begin{array}{ll} u = e^{2x} & dv = \cos(\pi x)\, dx \\ du = 2e^{2x} dx & v = \dfrac{\sin(\pi x)}{\pi} \end{array}}$$

$$\int e^{2x}\sin(\pi x)\,dx = -\frac{1}{\pi}e^{2x}\cos(\pi x) + \frac{2}{\pi}\boxed{\int e^{2x}\cos(\pi x)\,dx} \ *$$

Note: we make "same" choices for u and dv

u = expon. dv = trig.

$$\boxed{\begin{array}{ll} u = e^{2x} & dv = \cos(\pi x)\,dx \\ du = e^{2x}\cdot 2\,dx & v = \dfrac{\sin(\pi x)}{\pi} \end{array}}$$

$$e^{2x}\cdot\frac{\sin(\pi x)}{\pi} - \int \frac{\sin(\pi x)}{\pi}\cdot 2e^{2x}\,dx$$

$$= \boxed{\frac{e^{2x}\cdot\sin(\pi x)}{\pi} - \frac{2}{\pi}\int e^{2x}\sin(\pi x)\,dx} \ *$$

Notice how the integral $\int e^{2x}\sin(\pi x)\,dx$ reappeared on the right side of the equation This is the Loopy nature of this integral.

$$I = \int e^{2x} \cdot \sin(\pi x)\, dx$$

$$I = -\frac{1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi} \boxed{\int e^{2x} \cos(\pi x)\, dx}$$

$$\int e^{2x} \cos(\pi x)\, dx = \frac{e^{2x} \sin(\pi x)}{\pi} - \frac{2}{\pi} \underbrace{\boxed{\int e^{2x} \sin(\pi x)\, dx}}_{I}$$

$$I = -\frac{1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi}\left[\frac{e^{2x} \sin(\pi x)}{\pi} - \frac{2}{\pi} I\right]$$

$$I = -\frac{1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi^2} e^{2x} \sin(\pi x) - \frac{4}{\pi^2} I$$

$$I + \frac{4}{\pi^2} I = -\frac{1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi^2} e^{2x} \sin(\pi x) + C$$

$$I + \frac{4}{\pi^2} I = \frac{-1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi^2} e^{2x} \sin(\pi x)$$

$$I\left(1 + \frac{4}{\pi^2}\right) = \frac{-1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi^2} e^{2x} \sin(\pi x)$$

$$I = \frac{\frac{-1}{\pi} e^{2x} \cos(\pi x) + \frac{2}{\pi^2} e^{2x} \sin(\pi x)}{(1 + 4/\pi^2)} + C$$

$$I = \int e^{2x} \sin(\pi x)\, dx = \text{ABOVE}$$

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Find the reduction formula for ∫ (sinx)^n dx integration by parts solved example

Integration by Parts 8

Ex| Prove the Reduction Formula $n \geq 2$

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Strategy: Apply Integration by Parts + trig. identity

$$\int u \, dv = uv - \int v \, du$$

$$I = \int \sin^n x \, dx$$

$$I = \int \underbrace{\sin^{n-1} x}_{u} \; \underbrace{\sin x \, dx}_{dv}$$

$$\boxed{\begin{aligned} u &= \sin^{n-1} x \\ du &= (n-1) \sin^{n-2} x . \cos x \, dx \\ dv &= \sin x \, dx \\ v &= -\cos x \end{aligned}}$$

$$I = \int \underbrace{\sin^{n-1}x}_{u} \, \underbrace{\sin x \, dx}_{dv}$$

$u = \sin^{n-1}x$

$du = (n-1)\sin^{n-2}x \cos x \, dx$

$\int dv = \int \sin x \, dx$

$v = -\cos x$

$$I = \sin^{n-1}x \cdot (-\cos x) - \int (-\cos x) \cdot (n-1)\sin^{n-2}x \cos x \, dx$$

$$I = -\cos x \sin^{n-1}x + (n-1)\int \cos^2 x \sin^{n-2}x \, dx$$

$\cos^2 x = 1 - \sin^2 x$

$$I = -\cos x \sin^{n-1}x + (n-1)\int (1 - \sin^2 x)\sin^{n-2}x \, dx$$

$\parallel I$

$$I = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \, dx - (n-1)\boxed{\int \sin^n x \, dx}$$

$$I + (n-1)I = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \, dx$$

$$nI = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \, dx$$

$$\frac{nI}{n} = \frac{-\cos x \sin^{n-1} x}{n} + \frac{(n-1)\int \sin^{n-2} x \, dx}{n}$$

$$I = -\frac{1}{n} \cos x \sin^{n-1} x + \left(\frac{n-1}{n}\right) \int \sin^{n-2} x \, dx$$

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \left(\frac{n-1}{n}\right) \int \sin^{n-2} x \, dx$$

We can use above Reduction formula to solve

$\int \sin^4 x \, dx$ $\int \sin^8 x \, dx$ $\int \sin^7 x \, dx$

$n = 4$ $n = 8$ $n = 7$

Apply Reduction formula for $\int \sin^n x \, dx$ to solve $\int \sin^6 x \, dx$

$$\int \sin^n x \, dx = \frac{-1}{n} \cos x \sin^{n-1} x + \left(\frac{n-1}{n}\right) \int \sin^{n-2} x \, dx$$

$$n=6 \Rightarrow \int \sin^6 x \, dx = -\frac{1}{6} \cos x \sin^5 x + \frac{5}{6} \boxed{\int \sin^4 x \, dx}$$

now apply formula for $n=4$

$$n=4 \Rightarrow \int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} \boxed{\int \sin^2 x \, dx}$$

$n=2$

$$n=2 \Rightarrow \int \sin^2 x \, dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} \int \sin^0 x \, dx$$

$\sin^0 x = (\sin x)^0 = 1$

$$= -\frac{1}{2} \cos x \sin x + \frac{1}{2} x$$

Putting it all together!

$$\int \sin^6 x\, dx =$$

$$= -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\left[-\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left[-\frac{1}{2}\cos x \sin x + \frac{x}{2}\right]\right] + C$$

$$= -\frac{1}{6}\cos x \sin^5 x - \frac{5}{24}\cos x \sin^3 x - \frac{15}{48}\cos x \sin x + \frac{15}{48}x + C$$

Review: we have proved the reduction formula:

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

and we applied above formula for $\int \sin^6 x\, dx$

Evaluate ∫ x e^(x^2) cos (x^2) dx integration by parts solved example (hard)

Integration by Parts 9

Hard Example

$\boxed{\text{Ex}}$ Evaluate $\int x \cdot e^{x^2} \cos(x^2)\, dx$

Apply Subst. method
$t = x^2$
$dt = 2x\,dx$
$\frac{1}{2} dt = x\,dx$

$$\boxed{\begin{array}{ll} u = e^{x^2} & dv = x\cos(x^2)\,dx \\ du = e^{x^2} \cdot 2x\,dx & V = \dfrac{\sin(x^2)}{2} \end{array}}$$

$$\int u\,dv = uv - \int v\,du$$

$$= e^{x^2} \cdot \frac{\sin(x^2)}{2} - \int \frac{\sin(x^2)}{\cancel{2}} \cdot \cancel{2} x e^{x^2} dx$$

$$= \frac{1}{2} e^{x^2} \sin(x^2) - \int x e^{x^2} \sin(x^2)\, dx$$

$$I = \int x e^{x^2}\cos(x^2)\,dx = \frac{1}{2}e^{x^2}\sin(x^2) - \boxed{\int x e^{x^2}\sin(x^2)\,dx}\ *$$

Apply Integ. By Parts again!

$$\int x e^{x^2}\sin(x^2)\,dx$$

$$\boxed{\begin{array}{ll} u = e^{x^2} & dv = x\sin(x^2)\,dx \\ du = e^{x^2}\cdot 2x\,dx & v = \dfrac{-\cos(x^2)}{2} \end{array}}$$

$$= e^{x^2}\cdot\frac{-\cos(x^2)}{2} - \int \frac{-\cos(x^2)}{\cancel{2}}\cdot e^{x^2}\cdot\cancel{2}x\,dx$$

$$= \boxed{-\frac{1}{2}e^{x^2}\cos(x^2) + \int x e^{x^2}\cos(x^2)\,dx}\ *$$

Notice the reappearance of $\int x e^{x^2}\cos(x^2)\,dx$

This is the Loopy nature of this type of Integral.

$$I = \int x e^{x^2} \cos(x^2)\,dx = \frac{1}{2} e^{x^2} \sin(x^2) - \boxed{\int x e^{x^2} \sin(x^2)\,dx}$$

$$\int x e^{x^2} \sin(x^2)\,dx = -\frac{1}{2} e^{x^2} \cos(x^2) + \underbrace{\boxed{\int x e^{x^2} \cos(x^2)\,dx}}_{= I}$$

$$I = \frac{1}{2} e^{x^2} \sin(x^2) - \left[-\frac{1}{2} e^{x^2} \cos(x^2) + I \right]$$

$$I = \frac{1}{2} e^{x^2} \sin(x^2) + \frac{1}{2} e^{x^2} \cos(x^2) - I$$

$$\frac{2I}{2} = \frac{\frac{1}{2} e^{x^2} \sin(x^2) + \frac{1}{2} e^{x^2} \cos(x^2)}{2}$$

$$I = \frac{1}{4} e^{x^2} \sin(x^2) + \frac{1}{4} e^{x^2} \cos(x^2) + C$$

$$\int x e^{x^2} \cos(x^2)\,dx = \text{ABOVE}$$

Strategy Review

$$I = \int x e^{x^2} \cos(x^2)\,dx$$

Apply Integration by Parts $u = e^{x^2}$

$dv = x\cos(x^2)\,dx$

We label the integral as I so that when this integral reappears we can algebraically solve for it.

$$\underbrace{\int x e^{x^2}\cos(x^2)\,dx}_{I} = \frac{1}{2}e^{x^2}\sin(x^2) - \left[-\frac{1}{2}e^{x^2}\cos(x^2) + \underbrace{\int x e^{x^2}\cos(x^2)\,dx}_{I}\right]$$

$$I = \frac{1}{2}e^{x^2}\sin(x^2) + \frac{1}{2}e^{x^2}\cos(x^2) - I$$

Solve for I

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Evaluate ∫ xe^x/(x+1)^2 dx , Integration by parts hard example solved

Integration by Parts 10

Ex] $\int \frac{xe^x}{(x+1)^2}\,dx$ Hard Integral !

Strategy: let's think of all possible choices for u

1] $u = \frac{x}{(x+1)^2}$ $dv = e^x\,dx$

2] $u = xe^x$ $dv = (x+1)^{-2}\,dx$

3] $u = e^x$ $dv = \frac{x}{(x+1)^2}\,dx$

When solving difficult integration by parts integrals don't hesitate to try different options for u and once u is chosen, dv is the remaining part of the integral.

$$\int \frac{xe^x}{(x+1)^2}\,dx$$

$$\boxed{\begin{aligned} u &= xe^x \\ du &= (1e^x + xe^x)\,dx \\ dv &= (x+1)^{-2}\,dx \\ v &= \frac{(x+1)^{-1}}{-1} \end{aligned}}$$

$\int (x+1)^{-2}\,dx$

let $t = x+1$ $\quad dt = dx$

$\int t^{-2}\,dt = \frac{t^{-1}}{-1} + C$

$\int (x+1)^{-2}\,dx = -(x+1)^{-1} + C$

$$\int \frac{xe^x}{(x+1)^2}\,dx$$

$u = xe^x \qquad dv = (x+1)^{-2}\,dx$

$du = (e^x + xe^x)\,dx \qquad v = -(x+1)^{-1}$

$$\int u\,dv = uv - \int v\,du$$

$$= -xe^x(x+1)^{-1} - \int -(x+1)^{-1}(e^x + xe^x)\,dx$$

$$= \frac{-xe^x}{(x+1)} + \int \frac{(e^x + xe^x)}{(x+1)}\,dx$$

$$= \frac{-xe^x}{(x+1)} + \int \frac{e^x\cancel{(1+x)}}{\cancel{(x+1)}}\,dx$$

$$= \frac{-xe^x}{(x+1)} + \int e^x\,dx = \frac{-xe^x}{x+1} + e^x + C$$

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Integration by Parts II

Ex Evaluate $\int_1^3 x \cdot f''(x)\,dx$

Given: $f(1)=5 \quad f'(1)=2 \quad f(3)=6 \quad f'(3)=4$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\int_1^3 x \cdot f''(x)\,dx$$

$u = x$	$dv = f''(x)\,dx$
$du = dx$	$v = f'(x)$

$$\int_1^3 x f''(x)\,dx = x f'(x)\Big|_1^3 - \int_1^3 f'(x)\,dx$$

$$\int_1^3 x \cdot f''(x)\,dx = x \cdot f'(x)\Big|_1^3 - \int_1^3 f'(x)\,dx$$

$$= x f'(x)\Big|_1^3 - f(x)\Big|_1^3$$

$$= x f'(x) - f(x)\Big|_1^3$$

Given

$f(1)=5 \quad f'(1)=2$

$f(3)=6 \quad f'(3)=4$

$$= 3f'(3) - f(3) - [1f'(1) - f(1)]$$

$$= 3f'(3) - f(3) - f'(1) + f(1)$$

$$= 3(4) - 6 - 2 + 5 = 12 - 6 - 2 + 5 = 9$$

$$\int_1^3 x f''(x)\,dx = 9$$