Numerical Integration 1

Why do we need Numerical Integration?

Case I To evaluate $\int_a^b f(x)\,dx$ by applying the Fundamental Theorem of Calculus we need to find an antiderivative $F(x)$ of $f(x)$, and sometimes this is an impossible task. Here are some examples:

$$\int_0^1 e^{x^2}\,dx\ ,\quad \int_1^2 \cos(x^2)\,dx\ ,\quad \int_2^3 \frac{1}{1+\ln x}\,dx$$

Case II This case results from functional values recorded at equally spaced intervals.

Case II Example Velocity measurments of a runner

t (sec)	0	1	2	3	4	5	6	7	8
$v(t)$ (m/sec)	0	4.5	6	7.9	8.2	9.3	10.2	10.9	10.5

Estimate the distance covered $\int_0^8 v(t)\,dt$

In both of these cases we need to find approximate values of the definite integral of the form $\int_a^b f(x)\,dx$

Let's Review Riemann sums we have covered before.

Left hand, Right hand and midpoint Riemann Sums

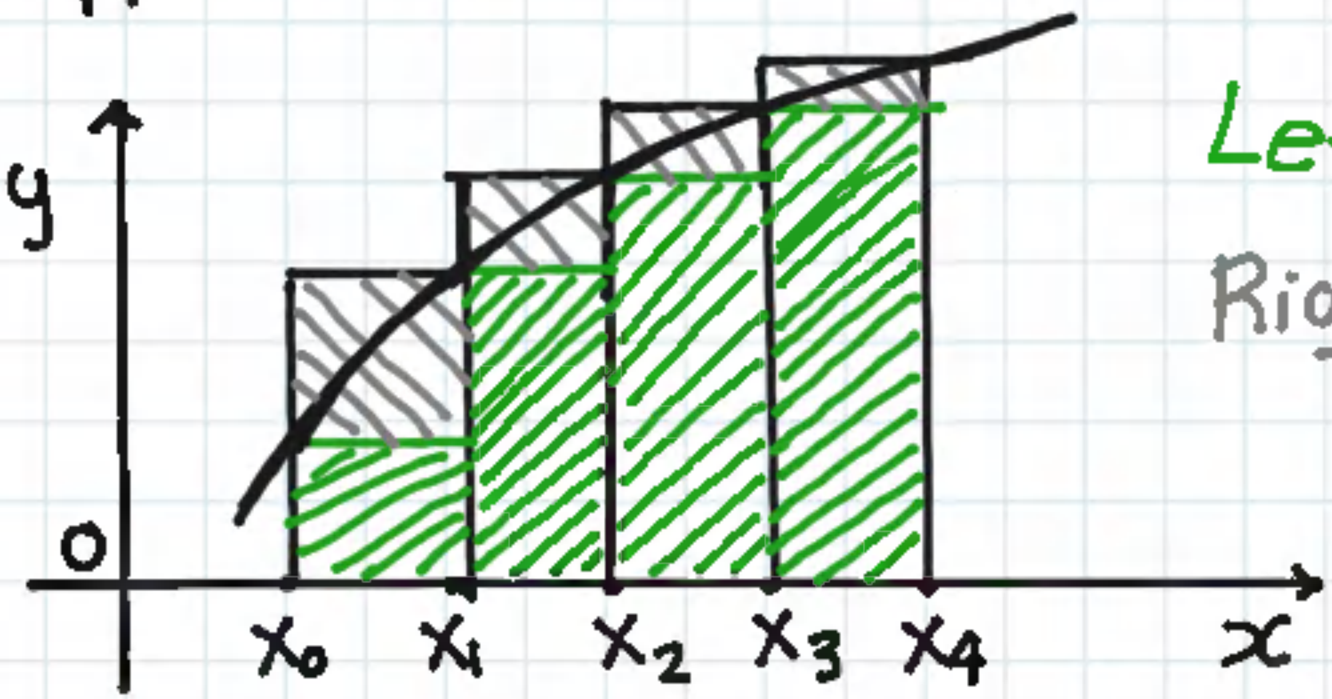
Left hand: $\int_a^b f(x)\,dx \cong L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$

Right hand: $\int_a^b f(x)\,dx \cong R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$

Midpoint: $\int_a^b f(x)\,dx \cong M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$

where $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a + i\,\Delta x \quad i = 1, \dots, n$



Left hand L_4

Right hand R_4

Derivation of Trapezoidal rule by averaging the left and right endpoint Riemann sum

Trapezoidal Rule

Lets define $T_n = \frac{(L_n + R_n)}{2}$ as the average of the Left endpoint and Right endpoint Riemann sums.

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right]$$

$$= \frac{\Delta x}{2}\left[\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right)\right.$$

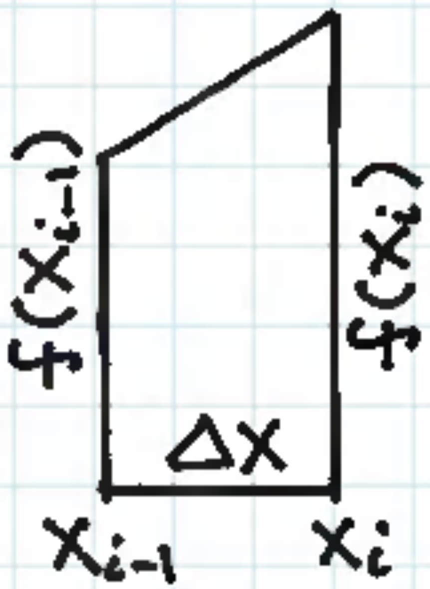
$$= \frac{\Delta x}{2}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\right]$$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x \quad i = 0, 1, 2, \ldots, n$

Trapezoidal rule explained with a diagram as the area of a typical trapezoid

Let's illustrate the Area of a typical Trapezoid



$$A_i = \frac{\Delta x}{2}\left[f(x_{i-1}) + f(x_i) \right]$$

If we add up all these trapezoids we obtain the general Trap Rule formula

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x \quad i = 0, 1, 2, \ldots, n$

Ex| Apply a) The trapezoidal Rule and b) the midpoint Rule with $n=5$ to approximate $\int_0^1 x^2\,dx$

a) Given $n=5$, $a=0$, $b=1$ with $\Delta x = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

$f(x) = x^2$

0 0.2 0.4 0.6 0.8 1.0

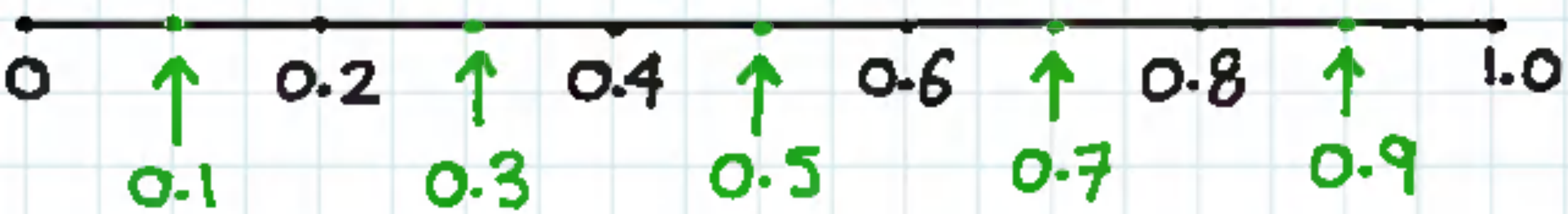
$$\int_0^1 x^2\,dx \cong T_5 = \frac{0.2}{2}\Big[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1) \Big]$$

$$T_5 = 0.1\left[0^2 + 2(0.2)^2 + 2(0.4)^2 + 2(0.6)^2 + 2(0.8)^2 + 1^2 \right]$$

$$T_5 = 0.34$$

b) Midpoint Rule with $n=5$, $a=0$, $b=1$, $\Delta x = \frac{1}{5} = 0.2$

$f(x) = x^2$ $\qquad M_5 \cong \int_0^1 x^2\,dx$



$$\int_0^1 x^2\,dx \cong M_5 = \Delta x\left[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)\right]$$

$$M_5 = 0.2\left[(0.1)^2 + (0.3)^2 + (0.5)^2 + (0.7)^2 + (0.9)^2\right]$$

$$M_5 = 0.33$$

Note: We chose the definite integral $\int_0^1 x^2\,dx$ so that we can examine how accurate these estimates T_5 and M_5 are as compared to the exact value of the integral $\int_0^1 x^2\,dx$ which can be easily computed.

Applying F.T.C: $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$T_5 = 0.34$; $M_5 = 0.33$ and $\int_0^1 x^2\,dx = 1/3$

Let's compare errors:

$E_T = \int_0^1 x^2\,dx - T_5 = \frac{1}{3} - 0.34 = -0.00\overline{66}$

$E_M = \int_0^1 x^2\,dx - M_5 = \frac{1}{3} - 0.33 = 0.00\overline{33}$

$$\boxed{\text{Observe: } |E_M| = \frac{1}{2}|E_T|}$$

Numerical Integration introduction to the Simpson's Rule

Numerical Integration 2 (Simpson's Rule)

The trapezoidal rule approximation applies straight line segments and adds up all the trapezoids to estimate the definite integral $\int_a^b f(x)\,dx$, however a more accurate technique known as Simpson's rule fits a quadratic function to equally spaced data points.

Start with $[a,b]$; $\Delta x = \frac{b-a}{n}$; $x_i = a + i\Delta x$

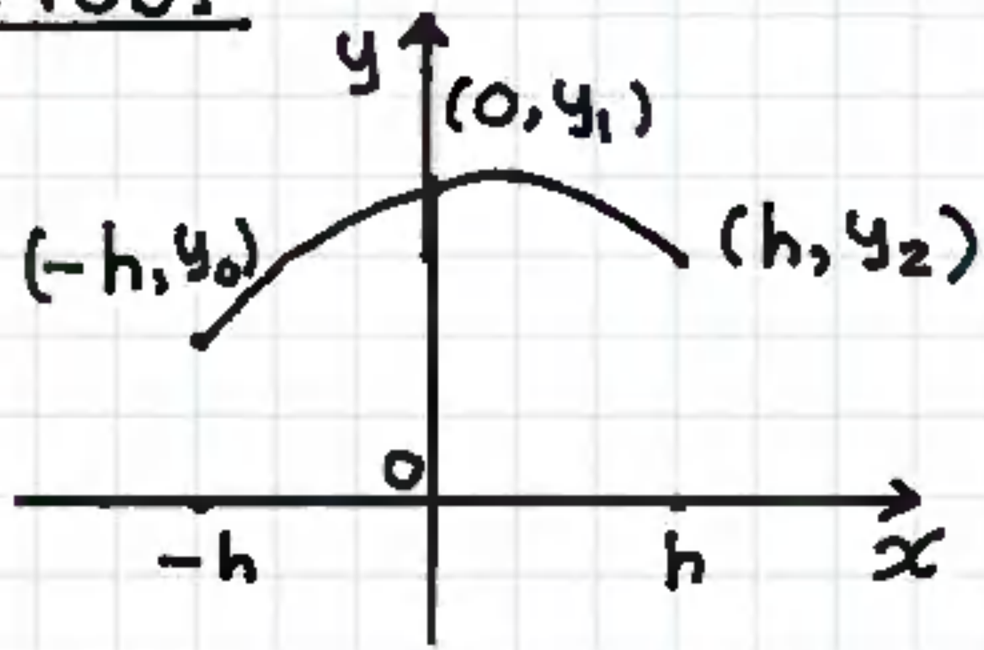
n even number $\qquad i = 0, 1, \ldots, n$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad \ldots \quad \ldots \quad x_{n-2} \quad x_{n-1} \quad x_n$

$a \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad b$

Proof of Simpson's rule by fitting a quadratic y=Ax^2+Bx+C thru 3 points

Proof



$y = Ax^2 + Bx + C$

Lets compute A, B and C such that parabola goes thru these 3 points.

$x = -h \quad y = y_0 \Rightarrow \quad y_0 = Ah^2 - Bh + C$ (1)

$x = 0 \quad y = y_1 \Rightarrow \quad y_1 = A(0)^2 + B(0) + C \Rightarrow \quad C = y_1$ (2)

$x = h \quad y = y_2 \Rightarrow \quad y_2 = Ah^2 + Bh + C$ (3)

Add equations (1) and (3) we get $y_0 + y_2 = 2Ah^2 + 2C$

$2Ah^2 = y_0 + y_2 - 2C = y_0 + y_2 - 2y_1 = y_0 - 2y_1 + y_2$

$C = y_1$

Now we compute the area under the fitted parabola from $x=-h$ to $x=h$.

$$\int_{-h}^{h}(Ax^2+Bx+C)\,dx = \frac{Ax^3}{3}+\frac{Bx^2}{2}+Cx\Big|_{-h}^{h}$$

$$= \frac{Ah^3}{3}+\frac{Bh^2}{2}+Ch-\left(\frac{-Ah^3}{3}+\frac{Bh^2}{2}-Ch\right)$$

$$= \frac{2Ah^3}{3}+2Ch$$

$$\therefore \int_{-h}^{h}(Ax^2+Bx+C)\,dx = \frac{2Ah^3}{3}+2Ch = \frac{h}{3}\left[2Ah^2+6C\right]$$

From before: $C=y_1$ and $2Ah^2=y_0-2y_1+y_2$

$$\int_{-h}^{h}(Ax^2+Bx+C)\,dx=\frac{h}{3}\left[2Ah^2+6C\right]=\frac{h}{3}\left[y_0-2y_1+y_2+6y_1\right]$$

$$=\frac{h}{3}\left[y_0+4y_1+y_2\right]$$

Note: $2Ah^2 = y_0-2y_1+y_2$
and $C=y_1$

Now let's add up areas under quadratic functions over all pairs $(x_0,x_2),(x_2,x_4),(x_4,x_6),\ldots,(x_{n-2},x_n)$

$$\int_{x_0}^{x_2}f(x)\,dx+\int_{x_2}^{x_4}f(x)\,dx+\cdots+\int_{x_{n-2}}^{x_n}f(x)\,dx$$

$$=\frac{h}{3}\left[y_0+4y_1+y_2\right]+\frac{h}{3}\left[y_2+4y_3+y_4\right]+\cdots+\frac{h}{3}\left[y_{n-2}+4y_{n-1}+y_n\right]$$

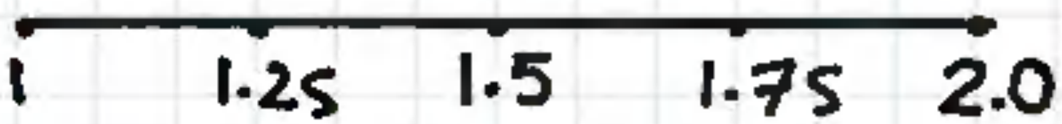
$$=\frac{h}{3}\left[y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right]$$

$$=\frac{\Delta x}{3}\left[y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right]$$

Ex| Apply Simpson's Rule with $n=4$ to estimate

$$\int_1^2 \frac{1}{x}\,dx$$

Solution: $f(x) = 1/x$; $n=4$; $\Delta x = (2-1)/4 = 0.25$



$$\int_1^2 \frac{1}{x}\,dx \cong S4 = \frac{0.25}{3}\Big[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2) \Big]$$

$$= \frac{0.25}{3}\Big[1 + \frac{4}{1.25} + \frac{2}{1.5} + \frac{4}{1.75} + \frac{1}{2} \Big] = 0.69325$$

Simpson's Rule S4 approximation to $\int_1^2 \frac{1}{x}\,dx$

$S4 = 0.69325$

Notice the definite integral $\int_1^2 \frac{1}{x}\,dx$ can be computed exactly by applying F.T.C

$$\int_1^2 \frac{1}{x}\,dx = \operatorname{Ln}x\Big|_1^2 = \operatorname{Ln}2 - \overset{0}{\overset{\shortparallel}{\operatorname{Ln}1}} = \operatorname{Ln}2 \cong 0.69315$$

Let's compute error for Simpson's rule S_4

$$S_4 = 0.69325 \qquad \int_1^2 \frac{1}{x}\,dx \cong 0.69315$$

$$E_{simpsons} = \int_1^2 \frac{1}{x}\,dx - S_4 = 0.69315 - 0.69325 = -0.0001$$

$$|E_{simpsons}| = 0.0001$$

Notice this error of approximation is very small even with $n=4$ subintervals.

Review of Simpson's rule formula we need even number of partitions ,n is even.

Review

Simpson's Rule:

$$\int_a^b f(x)\,dx = S_n = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

where n is even and $\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$ where $i = 0, 1, 2, \ldots, n$

Notice the pattern: 1 4 2 4 2 4 2 4 1

$$S_n = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

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Numerical Integration Error estimates for Trapezoidal rule and midpoint rule

Numerical Integration 3 Error estimates for Trapezoidal and Midpoint Rule

Assume $|f''(x)| \leq K$ for $a \leq x \leq b$, E_T and E_m are defined as an upper bound of the error of Trapezoidal Rule and the midpoint rule then:

$$|E_T| = \left| \int_a^b f(x)\,dx - T_n \right| \leq \frac{K(b-a)^3}{12n^2}$$

$$|E_m| = \left| \int_a^b f(x)\,dx - M_n \right| \leq \frac{K(b-a)^3}{24n^2}$$

Key Concept: $K = \text{Max}\,|f''(x)|$ on $a \leq x \leq b$

1/a Apply Trapezoidal Rule and Midpoint Rule with $n=4$ to estimate the integral $\int_1^2 \frac{1}{x}\,dx$

1/b Estimate the error bound and compare to exact error by applying F.T.C.

a) Solution: $f(x) = \frac{1}{x}$; $n=4$; $\Delta x = \frac{(2-1)}{4} = \frac{1}{4} = 0.25$

Trap. Rule

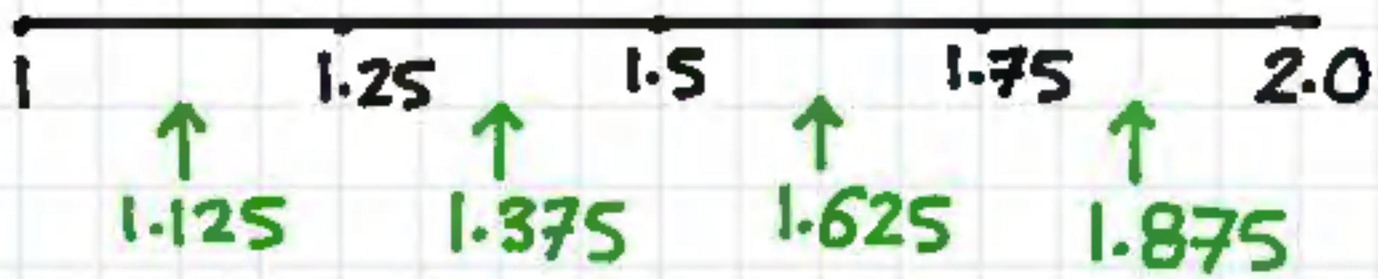
1 1.25 1.5 1.75 2.0

$$\int_1^2 \frac{1}{x}\,dx \cong T_4 = \frac{0.25}{2}\left[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2) \right]$$

$$T_4 = \frac{0.25}{2}\left[\frac{1}{1} + \frac{2}{1.25} + \frac{2}{1.5} + \frac{2}{1.75} + \frac{1}{2} \right] \cong 0.69702$$

1/a cont. $f(x)=\frac{1}{x}$; $n=4$; $\Delta x=(2-1)/4=0.25$

Midpoint rule



$$\int_1^2 \frac{1}{x}\,dx \cong M_4 = 0.25\left[f(1.125)+f(1.375)+f(1.625)+f(1.875)\right]$$

$$M_4 = 0.25\left[\frac{1}{1.125}+\frac{1}{1.375}+\frac{1}{1.625}+\frac{1}{1.875}\right] \cong 0.69122$$

Review: $\left.\begin{array}{l} T_4 \cong 0.69702 \\ M_4 \cong 0.69122 \end{array}\right\}$ $\int_1^2 \frac{1}{x}\,dx$ Approximation

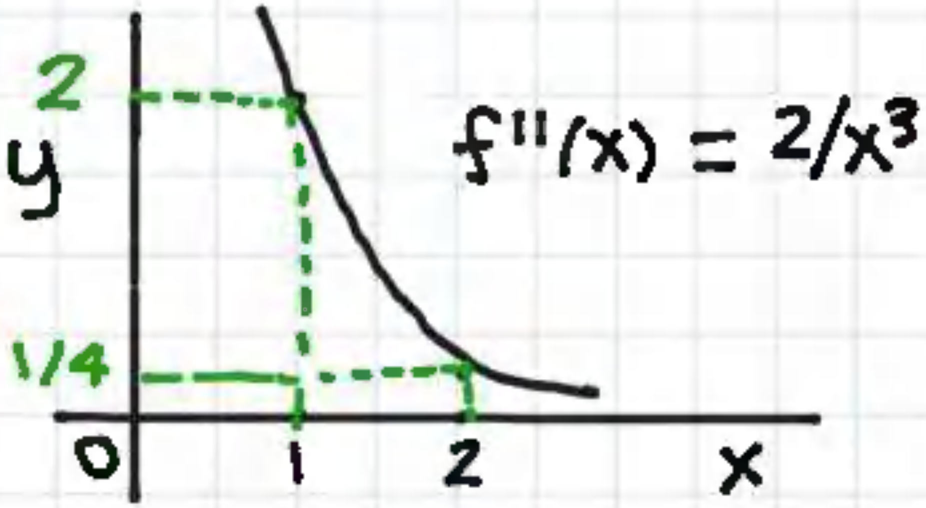
1/b Error estimate for Trapezoidal and Midpoint Rule

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Lets find $K = \max|f''(x)|$ on $1 \le x \le 2$

$f(x) = 1/x$; $f'(x) = -1/x^2 = -x^{-2}$ and $f''(x) = 2/x^3$

It is clear that $f''(x) = 2/x^3$ is strictly decreasing on $[1,2]$ and therefore $|f''(x)| \le 2$ on $[1,2]$



Note: Since $f''(x) = 2/x^3$ is strictly decreasing on $[1,2]$, the maximum value of $f''(x)$ is obtained at the Left endpoint $x=1$, and $\underline{f''(1) = 2}$

1/b) cont. $E_T \le \dfrac{k(b-a)^3}{12n^2}$ $E_M \le \dfrac{k(b-a)^3}{24n^2}$

For both Trap. rule and Midpoint Rule:

$k = 2$; $a = 1, b = 2$; $n = 4$

$$E_T \le \frac{2(2-1)^3}{12(4)^2} \cong 0.01042$$

$$E_M \le \frac{2(2-1)^3}{24(4)^2} \cong 0.00521$$

Now Let's apply F.T.C to compute exact error

$$\int_1^2 \frac{1}{x}\,dx = \ln x \Big|_1^2 = \ln 2 - \overset{=0}{\ln 1} = \ln 2 \cong 0.69315$$

1|b| Cont. Let's now compare Exact error and Error estimate for both Trap Rule and midpoint rule

$$E_T(\text{exact}) \cong \left| \int_1^2 \frac{1}{x}\,dx - T_4 \right| \cong |0.69315 - 0.69702|$$

$$E_T(\text{exact}) \cong |-0.00387| = 0.00387$$

$$E_{MID}(\text{exact}) \cong \left| \int_1^2 \frac{1}{x}\,dx - M_4 \right| \cong |0.69315 - 0.69122|$$

$$E_{MID}(\text{exact}) \cong 0.00193$$

From Error estimate formula:

$E_T \leq 0.01042$ and $E_M \leq 0.00521$

Note in both cases the error estimates are larger than actual error computed by applying F.T.C and this is why we call it an upper bound estimate for the error of approximation.

Numerical Integration 4 Error estimate for Simpson's Rule

Assume $|f^4(x)| \le K$ for $a \le x \le b$, E_s is defined as the upper bound of the error of applying Simpson's Rule to estimate the definite integral $\int_a^b f(x)\,dx$

$$|E_s| = \left| \int_a^b f(x)\,dx - S_n \right| \le \frac{K(b-a)^5}{180\,n^4}$$

Key Concept: $K = \text{Max}\,|f^4(x)|$ on $a \le x \le b$

1/a) Apply Simpson's Rule with $n=4$ to estimate

$$\int_0^1 e^{x^2}\,dx$$

1/b) Estimate the Simpson's error bound of this approximation?

1/c) How large should n be such that the simpson's rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate within 0.00001?

1/a) Solution: $f(x) = e^{x^2}$; $n=4$; $\Delta x = \frac{1-0}{4} = 0.25$

Simpson's Rule

1/a Simpson's Rule

0   0.25   0.5   0.75   1.0

$$\int_0^1 e^{x^2}\,dx \cong S_4 = \frac{0.25}{3}\left[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1) \right]$$

$$S_4 = \frac{0.25}{3}\left[e^{0^2} + 4e^{(0.25)^2} + 2e^{(0.5)^2} + 4e^{(0.75)^2} + e^{1^2} \right]$$

$$S_4 \cong 1.46272$$

$$\int_0^1 e^{x^2}\,dx \cong S_4 \cong 1.46272$$

Simpson's rule error estimation finding K=max|f^4(x)| solved example

1/b) Simpson's Rule error estimate

$$|E_s| \leq \frac{K(b-a)^5}{180n^4}$$

Let's find $K = \max|f^4(x)|$ on $0 \leq x \leq 1$

Given $f(x) = e^{x^2}$, we can compute $f^4(x)$ to be:

$f^4(x) = (12 + 48x^2 + 16x^4)e^{x^2}$ and since $0 \leq x \leq 1$

It is clear that $f^4(x)$ is increasing on $[0, 1]$

$$|f^4(x)| = (12 + 48x^2 + 16x^4)e^{x^2} \leq (12 + 48 + 16)e^1 = 76e$$

Plugging in $K = 76e$; $a = 0, b = 1$ and $n = 4$

$$E_s \leq \frac{K(b-a)^5}{180n^4} \Rightarrow E_s \leq \frac{76e(1-0)^5}{180(4)^4} \cong 0.00448$$

1/C] Find n such that $|E_S| = \left| \int_0^1 e^{x^2}dx - S_n \right| \le 0.00001$

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

we found $K = 76e$; $a=0, b=1$ Solve for n?

$$|E_S| \le \frac{76e(1-0)^5}{180n^4} \le 0.00001$$

$$\frac{180n^4}{76e} \geq \frac{1}{0.00001} \Rightarrow n^4 \geq \frac{100000(76e)}{180}$$

$$\Rightarrow n^4 \geq 114771.9 \Rightarrow n \geq 18.406$$

Round n up to next even integer

$n = 20$ will suffice.

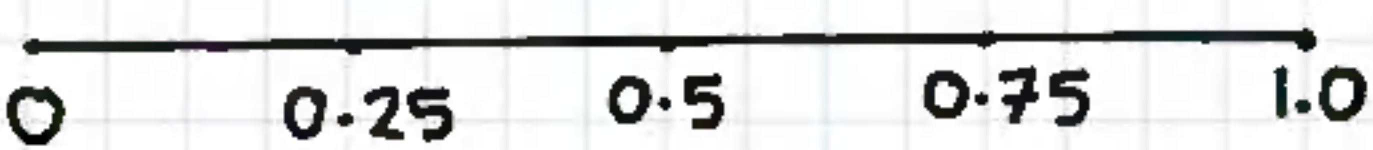
Numerical Integration 5 (Error estimate examples)

1/a Apply the trapezoidal rule with $n=4$ to estimate the integral $\int_0^1 \sin(x^2)\,dx$

1/b Find an upper bound for the error of approximating $\int_0^1 \sin(x^2)\,dx$ using the trapezoidal rule with $n=4$

1/a Solution : $f(x)=\sin(x^2)$; $n=4$; $\Delta x=\frac{1-0}{4}=0.25$

Trapezoidal Rule



0 0.25 0.5 0.75 1.0

$$T_4 = \frac{0.25}{2}\left[\sin(0^2) + 2\sin(0.25)^2 + 2\sin(0.5)^2 + 2\sin(0.75)^2 + \sin(1^2)\right]$$

$$T_4 \cong 0.315975$$

$$\int_0^1 \sin(x^2)\,dx \cong T_4 \cong 0.315975$$

1/b Trapezoidal Rule error estimate

$$|E_T| \le \frac{K(b-a)^3}{12n^2}, \text{ where } |f''(x)| \le K$$

Let's find $K = \max|f''(x)|$ on $0 \le x \le 1$

1/b Cont.

$f(x) = \sin(x^2)$; $f'(x) = \cos(x^2) \cdot 2x$

$f''(x) = -\sin(x^2) \cdot 2x \cdot 2x + \cos(x^2) \cdot 2$

$f''(x) = -4x^2 \sin(x^2) + 2\cos(x^2)$

$|f''(x)| = |-4x^2 \sin(x^2) + 2\cos(x^2)| \leq |-4x^2| \, |\sin(x^2)| + 2|\cos(x^2)|$

$0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq 1$

$|\cos(x^2)| \leq 1$ and $|\sin(x^2)| \leq 1$

$|f''(x)| \leq 4(1) + 2(1) = 6 \Rightarrow k = 6$

$\text{Max}\, |f''(x)| = k = 6$

1/b] Cont.

$$|E_T| \le \frac{k(b-a)^3}{12n^2}$$

$a=0$, $b=1$, $n=4$, $k=6$

$$\left| \int_0^1 \sin(x^2)\,dx - T_4 \right| \le \frac{6(1-0)^3}{12(4)^2} = 0.03125$$

2] Find an upper bound for the error of approximating the integral $\int_0^3 (e^{-2x} + x^3)\,dx$

Using the midpoint rule with $n=10$

$$|I - M_n| \le \frac{K(b-a)^3}{24n^2} \quad \text{where } I = \int_a^b f(x)\,dx$$

2/Cont. Let's find $K = \max|f''(x)|$ for $0 \le x \le 3$

$f(x) = e^{-2x} + x^3$; $f'(x) = e^{-2x}(-2) + 3x^2$

$f''(x) = e^{-2x}(-2)(-2) + 6x = 4e^{-2x} + 6x$

On the interval $0 \le x \le 3$, $0 \le e^{-2x} \le 1$ and $0 \le 6x \le 18$

$|f''(x)| = |4e^{-2x} + 6x| = 4e^{-2x} + 6x \le 4(1) + 18 = 22$

Therefore $K = \text{Max}|f''(x)| = 22$ for $0 \le x \le 3$

We have $K = 22$; $a = 0, b = 3$; $n = 10$

$$|I - M_{10}| \le \frac{K(b-a)^3}{24n^2} \Rightarrow |E_M| \le \frac{22(3-0)^3}{24(10)^2}$$

$$|I - M_{10}| = |E_M| \le 0.2475$$

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Numerical Integration 6

Ex 1/a The velocity of an airplane is recorded during a 1 hour flight at 10 minute intervals in km/hr. Apply Simpson's Rule to estimate the distance travelled by the airplane.

t	0	10	20	30	40	50	60
$v(t)$	500	550	575	605	640	642	539

Solution: The distance travelled by the airplane is given by $\int_0^1 v(t)\,dt$ where t is in hours.

Ex.1/a cont.

$\Delta t = 10 \text{ min} = \frac{1}{6}$ hour $\quad a=0 \quad b=1 \quad n=6$

Simpson's Rule

$$S_6 = \frac{\Delta t}{3}\left[V(0) + 4V(1/6) + 2V(2/6) + 4V(3/6) + 2V(4/6) + 4V(5/6) + V(1)\right]$$

$$S_6 = \frac{1/6}{3}\left[500 + 4(550) + 2(575) + 4(605) + 2(640) + 4[642] + 539\right]$$

$$S_6 = \frac{1}{18}\left[10657\right] \cong 592.06 \text{ km}$$

The distance travelled by the airplane during the 1 hour flight is approximately 592.06 km.

Ex.1/b For the same airplane recorded velocities assume that $0 \le V^4(t) \le 2.5$ Estimate the error of Simpson's approximation to the distance travelled?

Solution:

$$|E_s| \le \frac{K(b-a)^5}{180n^4} \quad \text{where} \quad K = \max|V^4(t)|$$

here: $|V^4(t)| \le 2.5$ $\quad a=0 \quad b=1 \quad n=6 \quad K=2.5$

$$\left|\int_0^1 v(t)\,dt - S_6\right| = |E_s| \le \frac{2.5(1-0)^5}{180(6)^4} \cong 0.00001$$

Numerical Integration 7

1/a| Apply Trapezoidal Rule and the midpoint rule with $n=5$ to estimate the integral $\int_1^2 xe^x dx$

1/b| Estimate the error bound and compare to actual error by applying F.T.C

1/C| Illustrate with a diagram that the trapezoidal rule overestimates the definite integral $\int_1^2 xe^x dx$ and the midpoint rule underestimates the integral given by $\int_1^2 xe^x dx$

1/a) Solution: $f(x) = xe^x$; $n = 5$; $a = 1, b = 2$; $\Delta x = \frac{2-1}{5} = 0.2$

Trapezoidal Rule

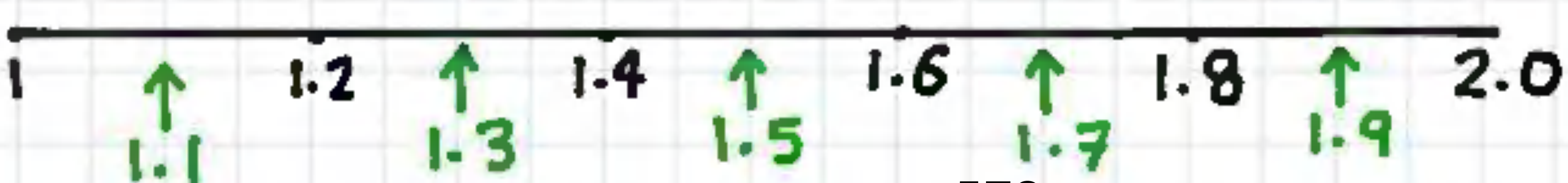
$$\int_1^2 xe^x\,dx \cong T_5 = \frac{0.2}{2}\left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right]$$

$$T_5 = \frac{0.2}{2}\left[1e^1 + 2.(1.2)e^{1.2} + 2(1.4)e^{1.4} + 2(1.6)e^{1.6} + 2(1.8)e^{1.8} + 2e^2\right]$$

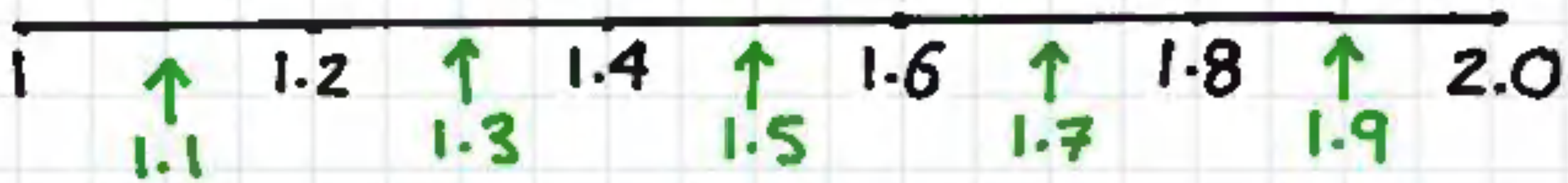
$T_5 = 7.44477$

1/a) cont. $f(x) = xe^x$; $n = 5$; $a = 1, b = 2$; $\Delta x = \frac{2-1}{5} = 0.2$

Midpoint Rule



1/a) cont.



$$\int_1^2 xe^x\,dx \cong M_5 = 0.2\left[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\right]$$

$$M_5 = 0.2\left[1.1e^{1.1}+1.3e^{1.3}+1.5e^{1.5}+1.7e^{1.7}+1.9e^{1.9}\right]$$

$$M_5 \cong 7.36122$$

Review: $\left.\begin{array}{l} T_5 \cong 7.44477 \\ M_5 \cong 7.36122 \end{array}\right\} \int_1^2 xe^x\,dx$ Approximation

1/b Error estimate for Trapezoidal and Midpoint Rule

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Let's find $K = \max|f''(x)|$ on $1 \leq x \leq 2$

$f(x) = xe^x$; $f'(x) = 1e^x + xe^x$; $f''(x) = e^x + 1e^x + xe^x$

$f''(x) = 2e^x + xe^x = e^x(2+x)$

It is clear that $f''(x) = e^x(2+x)$ is strictly increasing on $[1,2]$ and therefore $|f''(x)| \leq e^2 \cdot 4$

Note: Since $f''(x) = e^x(2+x)$ is strictly increasing on $[1,2]$, the maximum value of $f''(x)$ is obtained at the right endpoint $x = 2$ and $f''(2) = 4e^2 = K$

1/b cont. $E_T \le \frac{K(b-a)^3}{12n^2}$ $E_M \le \frac{K(b-a)^3}{24n^2}$

For both Trapezoidal rule and midpoint rule:

$K = 4e^2$; $a=1, b=2$; $n=5$

$$|E_T| \le \frac{4e^2(2-1)^3}{12(5)^2} \cong 0.09852$$

$$|E_M| \le \frac{4e^2(2-1)^3}{24(5)^2} \cong 0.04926$$

Now, Let's apply F.T.C to compute actual error

$$\int_1^2 xe^x\,dx = xe^x \Big|_1^2 - \int_1^2 e^x\,dx$$

Integrate by Parts

$u = x$ $dv = e^x dx$

$du = dx$ $v = e^x$

$$= xe^x - e^x \Big|_1^2 = 2e^2 - e^2 - (1e^1 - e)$$

$$= e^2 \cong 7.38906$$

1/b] Cont. Let's now compare actual error and error estimate for both Trapezoidal rule and midpoint rule

$$E_T(\text{actual}) = \int_1^2 xe^x\,dx - T_5 = 7.38906 - 7.44477$$

$$E_T(\text{actual}) = -0.05571$$

$$E_M(\text{actual}) = \int_1^2 xe^x\,dx - M_5 = 7.38906 - 7.36122$$

$$E_{MID}(\text{actual}) = 0.02784$$

From error estimate formula

$|E_T| \leq 0.09852$ $\quad$ $|E_T(\text{actual})| = 0.05571$

$|E_M| \leq 0.04926$ $\quad$ $|E_M(\text{actual})| = 0.02784$

1/b) cont.

Note in both cases the error estimate upper bounds are larger than actual error computed by applying F.T.C and this is why we call it an upper bound estimate for the error of approximation.

1/c) Illustrate with a diagram why Trapezoidal Rule is an overestimate and the midpoint Rule is an underestimate.

Solution: Let's start by comparing actual error estimates $E_T(\text{actual}) = -0.05571$

$E_{MID}(\text{actual}) = 0.02784$

OPPOSITE SIGNS

Illustrating with a diagram that Trapezoidal rule overestimates when f(x) is concave up

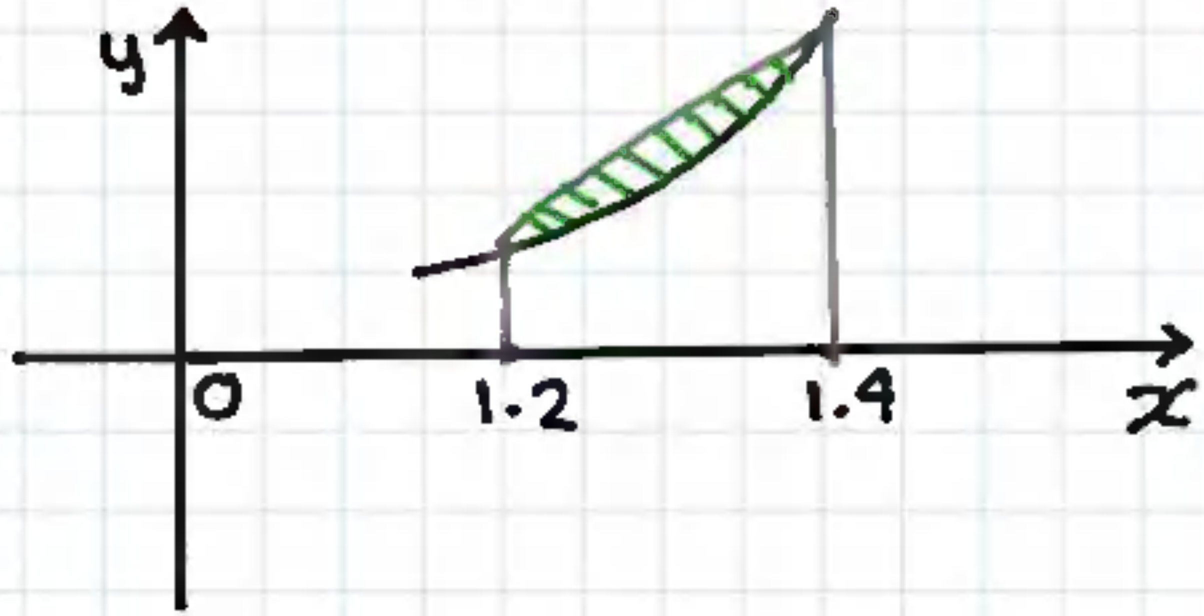
Since $E_T = \int_1^2 xe^x dx - T_5 = -0.05571 < 0$

Therefore $T_5 > \int_1^2 xe^x dx \Rightarrow T_5$ overestimates

Let's illustrate with a diagram:

We know that $f''(x) = e^x(2+x) \geq 0$ for $1 \leq x \leq 2$

Therefore $f(x) = xe^x$ is concave up on $[1,2]$ and the top base of the trapezoid (secant line) Overestimates



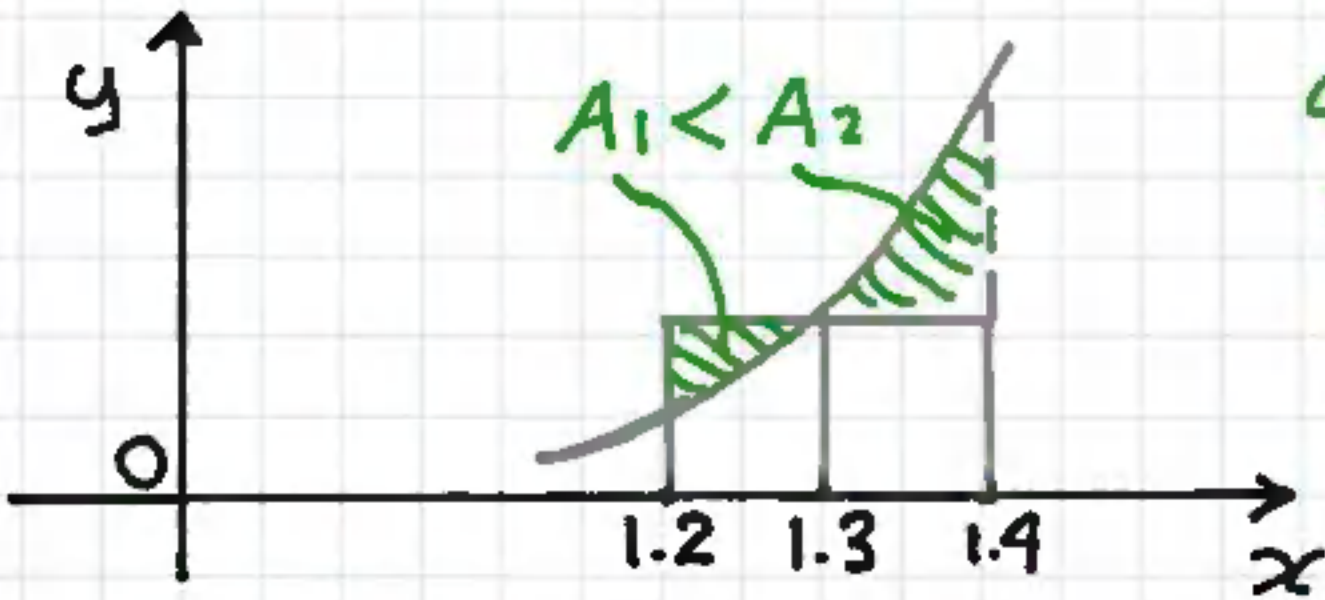
Illustrating with a diagram that the midpoint rule underestimates when f(x) is concave up

1/C For the midpoint rule

$$E_{MID} = \int_1^2 xe^x\,dx - M_5 = 0.02784 > 0$$

Therefore $M_5 < \int_1^2 xe^x\,dx \Rightarrow M_5$ underestimates

Lets illustrate with a diagram.



Since f(x) is concave up $A_1 < A_2$ and Midpoint rule Underestimates

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