Power Series Functions I

Motivation: How to represent a function as a Power Series by algebraic manipulation of a geometric series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1$$

So infact $f(x) = \dfrac{1}{1-x}$ is represented as a sum of a Geometric Power Series $\sum_{n=0}^{\infty} x^n$ for $|x|<1$

$|x|<1$ or $-1<x<1$ is the Interval of Convergence which means that $f(x) = \dfrac{1}{1-x}$ can be represented by the Power Series $\sum_{n=0}^{\infty} x^n$ on the interval $(-1,1)$

Ex1 Represent $f(x)=\dfrac{1}{1+x^3}$ as the sum of a Power series and find the interval of convergence.

let's start with the Geometric Sum Formula.

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots=\sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

Replace x by $-x^3$ in $\dfrac{1}{1-x}$ to obtain $\dfrac{1}{1-(-x^3)}$

$$f(x)=\frac{1}{1+x^3}=\frac{1}{1-(-x^3)}=\sum_{n=0}^{\infty}(-x^3)^n=\sum_{n=0}^{\infty}(-1)^n x^{3n}$$

Since $\dfrac{1}{1+x^3}$ is a geometric series it converges when $|-x^3| < 1 \Rightarrow |x^3| < 1 \Rightarrow |x|^3 < 1 \Rightarrow |x| < 1$

or $-1 < x < 1$, so the Interval of Convergence is $(-1, 1)$. This means that $f(x) = \dfrac{1}{1+x^3}$ can be represented as a sum of a power series $\sum_{n=0}^{\infty} (-1)^n x^{3n}$ on the Interval $(-1, 1)$

Ex2] Express $f(x) = \dfrac{x^2}{x+4}$ as the sum of a power series and find the Interval of Convergence.

$$f(x) = \frac{x^2}{x+4} = x^2 \cdot \frac{1}{x+4}$$

we need to factor 4 from x+4 in the Denominator

to fit the pattern of a geometric sum $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$x^2 \cdot \frac{1}{4+x} = x^2 \cdot \frac{1}{4\left(1+\frac{x}{4}\right)} = \frac{x^2}{4} \cdot \frac{1}{1+\frac{x}{4}}$$

now we can rewrite $\frac{1}{1+\frac{x}{4}}$ as $\frac{1}{1-\left(\frac{-x}{4}\right)}$

Replace x by $\frac{-x}{4}$ in the Geometric Sum Formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1-\left(\frac{-x}{4}\right)} = \sum_{n=0}^{\infty} \left(\frac{-x}{4}\right)^n$$

Multiply both sides of equation by $\frac{x^2}{4}$

$$\frac{x^2}{4}\cdot\frac{1}{1+\frac{x}{4}} = \frac{x^2}{4}\sum_{n=0}^{\infty}\left(\frac{-x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^2}{4}\left(\frac{-x}{4}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{x^2}{4}\,\frac{(-1)^n x^n}{4^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{4^{n+1}}$$

$$\therefore \frac{x^2}{4+x} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{4^{n+1}}$$

Now we need to find the Interval of Convergence.

Since $\dfrac{x^2}{4+x} = \dfrac{x^2}{4(1+x/4)} = \dfrac{x^2}{4(1-(-x/4))}$ is a geometric series with common ratio $r=-x/4$, it converges when $\left|\frac{-x}{4}\right|<1 \Rightarrow \left|\frac{x}{4}\right|<1 \Rightarrow |x|<4$

So Interval of Convergence of $\frac{x^2}{4+x}$ is $|x|<4$ or $(-4,4)$.

Key Concept: We used the Geometric Series sum formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ as a building block for developing the power series representation of the function $f(x) = \frac{x^2}{4+x}$.

Power Series Functions 2

Differentiation and Integration of Power Series

Assume that a power series $\sum_{n=0}^{\infty} C_n X^n$ has Radius of convergence $R>0$ and defines a differentiable (implies continuous) function on $(-R, R)$

$$f(x) = C_0 + C_1 X + C_2 X^2 + C_3 X^3 + \cdots = \sum_{n=0}^{\infty} C_n X^n$$

1] $f'(x) = C_1 + 2C_2 X + 3C_3 X^2 + \cdots = \sum_{n=1}^{\infty} n C_n X^{n-1}$

2] $\int f(x)\,dx = C + C_0 X + \frac{C_1 X^2}{2} + \frac{C_2 X^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{C_n X^{n+1}}{n+1}$

Key Concept: Power Series can be differentiated and integrated term by term and the resulting series have the same radius of convergence R as the original power series $\sum_{n=0}^{\infty} c_n x^n$; however convergence at endpoints may change, that is the original power series may converge at $x=R$ (right endpoint) wheras the differentiated power series may diverge at $x=R$, so we have 4 cases for the Interval of Convergence.

$(-R, R)$, $[-R, R)$, $(-R, R]$, $[-R, R]$

EX 1/A] Express $f(x) = \ln(1+x)$ as a power series and find its radius of convergence?

EX 1/B] Express $\ln 2$ as the sum of an infinite series by using result of EX 1/A]

1/A] Note $\frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int (1 - x + x^2 - x^3 + - \cdots)\,dx$$

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C$$

For $|x| < 1$

We can find C by plugging $x=0$ into both sides.

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}+C$$

Plug in $x=0$

$$\ln(1+0)=\sum_{n=0}^{\infty}\frac{(-1)^n 0^{n+1}}{n+1}+C \Rightarrow \ln 1 = 0+C \Rightarrow C=0$$

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+-\cdots$$

Since the orginal series $\frac{1}{1+x}=\frac{1}{1-(-x)}$ is the sum of a geometric series with common ratio $r=-x$ and converges when $|-x|<1 \Rightarrow |x|<1$.

The Radius of convergence of $\ln(1+x)$ is the same as the original series $\frac{1}{1+x}$: $R=1$

Ex 1/B] To find Ln2 simply plug in x=1 into Power series representation of Ln(1+x)

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-+\cdots$$

$$\ln(1+1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-+\cdots$$

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$$

Note: The series $\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$ Converges at

$x=1 \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ by the Alternating Series

Test and the sum of the series is Ln2

Compact Form of Differentiation and Integration of Power Series $\sum_{n=0}^{\infty} c_n x^n$

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n x^n\right] = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\int \left[\sum_{n=0}^{\infty} c_n x^n\right] dx = \sum_{n=0}^{\infty} \int \left[c_n x^n\right] dx = \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1} + C$$

⇒ Power Series can be differentiated or integrated term by term and resulting series has the same radius of convergence R as the original series.

Power Series Functions 3

EX 1/A] Find a power series representation of $f(x) = \tan^{-1}x$ and find the radius of convergence?

EX 1/B] Express π as the sum of an infinite series by using the result of EX 1/A]

EX 1/C] Evaluate $\int \tan^{-1}(x^2)\,dx$ as a power series.

1/A] Note $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx$$

Let's rewrite the integrand $\frac{1}{1+x^2}$ as the sum of a geometric series.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + - \cdots \qquad |x| < 1$$

$$\tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + - \cdots)\,dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \cdots$$

$$\tan^{-1}x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

To find C plug in $x=0$ into both sides.

$$\tan^{-1}x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow \tan^{-1}0 = C + \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+1}}{2n+1}$$

$$0 = C + 0 \Rightarrow C = 0$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Since the original series $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$ is the sum of a geometric series with $r=-x^2$ and converges when $|-x^2|<1 \Rightarrow |x^2|<1 \Rightarrow x^2<1$ $\Rightarrow \sqrt{x^2} < \sqrt{1} \Rightarrow |x|<1$. The Radius of convergence of the Power Series for $\tan^{-1}(x)$ is also $R=1$

Ex 1/B| To find π simply plug in $x=1$ into Power Series for $\tan^{-1}x$.

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \cdots$$

$$\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots\right) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Ex 1/C] Evaluate $\int \tan^{-1}(x^2)\,dx$ as a Power Series

Since we already know Power Series for $\tan^{-1}x$ simply plug in x^2 for x to find power series for the integrand $\tan^{-1}(x^2)$

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow \tan^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}$$

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}$$

$$\int \tan^{-1}(x^2)\,dx = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} \right) dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int x^{4n+2}\,dx$$

$$\int \tan^{-1}(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int x^{4n+2}\,dx$$

$$\int \tan^{-1}(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)} + C$$

$$\int \tan^{-1}(x^2)\,dx = C + \frac{x^3}{3} - \frac{x^7}{(7)(3)} + \frac{x^{11}}{(11)(5)} - \frac{x^{15}}{(15)(7)} + \cdots$$

The Power Series for $\int \tan^{-1}(x^2)\,dx$ has the same Radius of Convergence as the original series $\frac{1}{1+x^2}$ which is $R=1$.

Power Series Functions 4

EX 1/A] Given the power series for $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

Evaluate $\int x^2 \ln(1+x^4)\,dx$ as a power series.

EX 1/B] Use Part A to approximate $\int_0^{0.5} x^2 \ln(1+x^4)\,dx$

accurate to within 10^{-7}

EX 1/A] Since we are given the power series for

$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ simply plug in x^4 for x

to find power series for $\ln(1+x^4)$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Plug in x^4 for x

$$\ln(1+x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{n+1}$$

Now multiply both sides by x^2 to find the power series for the integrand $x^2 \ln(1+x^4)$

$$x^2 \ln(1+x^4) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4} x^2}{n+1}$$

$$x^2 \ln(1+x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{n+1}$$

Now Let's integrate term by term $\int x^2 \ln(1+x^4)\,dx$

$$\int x^2 \ln(1+x^4)\,dx = \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{n+1} \right] dx$$

$$\int x^2 \ln(1+x^4)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int x^{4n+6}\,dx$$

$$\int x^2 \ln(1+x^4)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+7}}{(n+1)(4n+7)} + C$$

$$\int x^2 \ln(1+x^4)\,dx = C + \frac{x^7}{(1)(7)} - \frac{x^{11}}{(2)(11)} + \frac{x^{15}}{(3)(15)} - \frac{x^{19}}{(4)(19)} + \cdots$$

EX 1/B] To evaluate the definite integral $\int_0^{0.5} x^2 \ln(1+x^4)\,dx$ we need to apply the Fundamental Theorem of Calculus, we can use any antiderivative so let's apply the antiderivative of EX 1/A] with $C=0$

$$\int_0^{0.5} x^2 \ln(1+x^4)\,dx = \left[\frac{x^7}{7} - \frac{x^{11}}{22} + \frac{x^{15}}{45} - \frac{x^{19}}{76} + \cdots \right]_0^{0.5}$$

$$\int_0^{0.5} x^2 \ln(1+x^4)\,dx = \left[\frac{(0.5)^7}{7} - \frac{(0.5)^{11}}{22} + \frac{(0.5)^{15}}{45} - \frac{(0.5)^{19}}{76} + \cdots \right]$$

Let's apply the Alternating Series Estimation Theorem to estimate the sum accurate to within 10^{-7}.

$$\int_0^{0.5} x^2 \operatorname{Ln}(1+x^4)\,dx = \left[\frac{(0.5)^7}{7} - \frac{(0.5)^{11}}{22} + \frac{(0.5)^{15}}{45} - \frac{(0.5)^{19}}{76} + \cdots\right]$$

By the Alternating Estimation Theorem if we add the first 3 terms the error is less than the $|$4th term$|$, Recall $|s - s_n| \leq |a_{n+1}|$

Since $\left|\frac{-(0.5)^{19}}{76}\right| \approx 0.000000025 < 10^{-7}$

$$\int_0^{0.5} x^2 \operatorname{Ln}(1+x^4)\,dx \approx \frac{(0.5)^7}{7} - \frac{(0.5)^{11}}{22} + \frac{(0.5)^{15}}{45}$$

$$\approx 0.00109455$$

$$\int_0^{0.5} x^2 \operatorname{Ln}(1+x^4)\,dx \approx \frac{(0.5)^7}{7} - \frac{(0.5)^{11}}{22} + \frac{(0.5)^{15}}{45}$$

$$\approx 0.00109455$$

$\int_0^{0.5} x^2 \operatorname{Ln}(1+x^4)\,dx \approx 0.00109455$ and this estimate is accurate to within 10^{-7}.

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Power Series Functions 5

Ex 1/A] Starting with the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$, Express $f(x) = \frac{1}{(1-x)^2}$ as a power series.

Ex 1/B] Find a power series representation for the function $f(x) = \frac{1}{(1-x)^3}$

Ex 1/C] Find a power series representation for the function $f(x) = \frac{x^3}{(1-x)^3}$

1/A] Given the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$ let's differentiate both sides.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1+x+x^2+x^3+\cdots) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots = \sum_{n=1}^{\infty} nx^{n-1} \qquad -1<x<1$$

Since the original series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is a geometric series with $R=1$, the radius of convergence of the Power Series for $\frac{1}{(1-x)^2}$ is also $R=1$.

1/B| We need to find power series for $f(x) = \dfrac{1}{(1-x)^3}$

Let's differentiate again to obtain:

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}\left(1+2x+3x^2+4x^3+\cdots\right) = \frac{d}{dx}\sum_{n=1}^{\infty} nx^{n-1}$$

$$\frac{d}{dx}(1-x)^{-2} = -2(1-x)^{-3}(-1) = \frac{2}{(1-x)^3}$$

$$\frac{2}{(1-x)^3} = 2+6x+12x^2+\cdots = \sum_{n=2}^{\infty} n(n-1)\,x^{n-2}$$

Note: $n=1$ gives 0, so start index at $n=2$

Multiply both sides by 1/2.

$$\frac{1}{(1-x)^3} = 1+3x+6x^2+\cdots = \sum_{n=2}^{\infty} \frac{n(n-1)\,x^{n-2}}{2}$$

We have obtained power series representation of

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)x^{n-2}}{2} \quad \text{for } -1 < x < 1$$

We can rewrite with index starting at $n=0$

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)x^{n}}{2} \quad \text{for } -1 < x < 1$$

1/C| Find Power Series for $f(x) = \dfrac{x^3}{(1-x)^3}$

$$f(x) = x^3 \cdot \frac{1}{(1-x)^3}$$

From 1/B| we found $\dfrac{1}{(1-x)^3} = \displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)x^{n-2}}{2}$

$$x^3 \cdot \frac{1}{(1-x)^3} = x^3 \sum_{n=2}^{\infty} \frac{n(n-1)x^{n-2}}{2} = \sum_{n=2}^{\infty} \frac{n(n-1)x^{n-2}x^3}{2}$$

$$\frac{x^3}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)x^{n+1}}{2} \quad \text{for} \quad -1 < x < 1$$

Expanding the power series we obtain:

$$\frac{x^3}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)x^{n+1}}{2}$$

$$= \frac{(2)(1)x^3}{2} + \frac{(3)(2)x^4}{2} + \frac{(4)(3)x^5}{2} + \cdots \quad \text{For } -1<x<1$$

Summary: Differentiate $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ twice

to find power series for $f(x) = \frac{1}{(1-x)^3}$

and then multiply by x^3 to find power series

for $f(x) = \frac{x^3}{(1-x)^3}$

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Power Series Functions 6

EX] Given the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$, Find the sum of the following series:

A] Find the sum of $\sum_{n=1}^{\infty} n x^n$ for $-1<x<1$

B] Find the sum of $\sum_{n=1}^{\infty} n^2 x^n$ for $-1<x<1$

C] Find the sum of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} n^2 (1.1)^n$

A] Start with $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ and differentiate both sides.

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\left(\frac{1}{1-x}\right) \Rightarrow \sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}(1-x)^{-1}$$

$$\sum_{n=1}^{\infty} nx^{n-1} = -1(1-x)^{-2}(-1) \Rightarrow \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2} \quad \text{for } -1<x<1$$

But we need $\sum_{n=1}^{\infty} nx^n$ so multiply both sides by x

$$x\sum_{n=1}^{\infty} nx^{n-1} = x.\frac{1}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2} \quad \text{for } |x| < 1$$

we have found the sum of the power series $\sum_{n=1}^{\infty} nx^n$ to be $\frac{x}{(1-x)^2}$ and since the original series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is a geometric series with $R=1$, the radius of convergence of the power series for $\sum_{n=1}^{\infty} nx^n$ is also $R=1$.

B] Now we need to find $\sum_{n=1}^{\infty} n^2 x^n$

Start with result of Part A] $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

Let's differentiate again to obtain:

$$\frac{d}{dx} \sum_{n=1}^{\infty} n x^n = \frac{d}{dx} \frac{x}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2(1) - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

Now multiply both sides by x

$$x \sum_{n=1}^{\infty} n^2 x^{n-1} = x \cdot \frac{(1+x)}{(1-x)^3} \Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = 1^2 x + 2^2 x^2 + 3^2 x^3 + \ldots = \frac{x(1+x)}{(1-x)^3} \qquad |x|<1$$

So we have found the sum of the power series for $\sum_{n=1}^{\infty} n^2 x^n$ to be $\frac{x(1+x)}{(1-x)^3}$ with $R=1$

C] Find $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} n^2 (1.1)^n$

From Part B we found $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$ for $|x|<1$

rewrite $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n$ ← $x = 1/2$

So plug in $x = 1/2$ to find sum of power series

since $x = 1/2$ is inside Interval of Convergence $(-1, 1)$

$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$ Plug in $x = 1/2$

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^3} = \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{\frac{1}{8}} = 6$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$

key Idea : $\sum_{n=1}^{\infty} n^2 \cdot \left(\frac{1}{2}\right)^n$ is the value of $\sum_{n=1}^{\infty} n^2 x^n$

Evaluated at $x = 1/2$ and since we have a closed form for the sum $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$ we can plug in $x = 1/2$ and find sum.

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n = 6$$

Now we need to find $\sum_{n=1}^{\infty} n^2 (1.1)^n$

$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$; We <u>cannot</u> plug in $x = 1.1$ to find the sum since $x = 1.1$ is outside the

Interval of convergence $-1 < x < 1$

$\therefore \sum_{n=1}^{\infty} n^2 (1.1)^n$ Diverges

Summary Notes

We can plug in $x = \frac{1}{2}$ into $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$

to find the sum of the power series since $x = 1/2$ is inside the Interval of Convergence $-1 < x < 1$

but we cannot plug in $x = 1.1$ to find the sum of the power series since $x = 1.1$ is outside the Interval of convergence $-1 < x < 1$

Power Series Functions 7

EX 1/A| Evaluate $\int \frac{x^3}{2+x^5}\,dx$ as a power series

EX 1/B| Use part A to estimate $\int_0^{0.5} \frac{x^3}{2+x^5}\,dx$ accurate to within 10^{-7}. (Hint: Apply AST Estimation theorem)

1/A] rewrite integrand as $\frac{x^3}{2+x^5} = x^3 \cdot \frac{1}{2+x^5}$

We need to factor 2 from $2+x^5$ in the denominator to fit the pattern of a geometric series sum:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$x^3 \cdot \frac{1}{2+x^5} = x^3 \cdot \frac{1}{2\left(1+\frac{x^5}{2}\right)} = \frac{x^3}{2}\left(\frac{1}{1-\left(\frac{-x^5}{2}\right)}\right)$$

replace x by $\frac{-x^5}{2}$ in the geometric sum formula:

$$\frac{x^3}{2} \cdot \frac{1}{1-\left(\frac{-x^5}{2}\right)} = \frac{x^3}{2} \sum_{n=0}^{\infty} \left(\frac{-x^5}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^3}{2} \left(\frac{-x^5}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{x^3}{2}\left(\frac{-x^5}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^3}{2}\,\frac{(-1)^n x^{5n}}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+3}}{2^{n+1}}$$

$$\frac{x^3}{2+x^5} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+3}}{2^{n+1}}$$

Now we integrate term by term to find $\int \frac{x^3}{2+x^5}\,dx$

$$\int \frac{x^3}{2+x^5}\,dx = \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+3}}{2^{n+1}}\right] dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \int x^{5n+3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n \quad x^{5n+4}}{2^{n+1}\,(5n+4)} + C$$

$$\int \frac{x^3}{2+x^5}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{5n+4}}{2^{n+1}(5n+4)} + C$$

$$\int \frac{x^3}{2+x^5}\,dx = C + \frac{x^4}{2\times 4} - \frac{x^9}{4\times 9} + \frac{x^{14}}{8\times 14} - \frac{x^{19}}{16\times 19} + - \cdots$$

Since the original series $\frac{x^3}{2} \cdot \frac{1}{1-\left(\frac{-x^5}{2}\right)}$ is a geometric series with $r = \frac{-x^5}{2}$, this new series converges for $\left|\frac{-x^5}{2}\right| < 1 \Rightarrow \frac{|x|^5}{2} < 1 \Rightarrow |x|^5 < 2 \Rightarrow |x| < 2^{1/5}$

$\Rightarrow -2^{1/5} < x < 2^{1/5}$ is the Interval of Convergence.

1/B| To evaluate the definite integral $\int_0^{0.5} \frac{x^3}{2+x^5}\,dx$ we need to apply the Fundamental Theorem of Calculus, so let's use the antiderivative of Ex 1/A| with C=0

$$\int_0^{0.5} \frac{x^3}{2+x^5}\,dx = \frac{x^4}{8} - \frac{x^9}{36} + \frac{x^{14}}{112} - \frac{x^{19}}{304} \Big|_0^{0.5}$$

$$\int_0^{0.5} \frac{x^3}{2+x^5}\,dx = \frac{(0.5)^4}{8} - \frac{(0.5)^9}{36} + \frac{(0.5)^{14}}{112} - \frac{(0.5)^{19}}{304} + \cdots$$

Let's apply the Alternating Series Estimation theorem to estimate the sum accurate to within 10^{-7}.

If we add the first 3 terms the error is less than the |4th term|, Recall $|S-S_n| \le |a_{n+1}|$

$$\int_0^{0.5} \frac{x^3}{2+x^5}\,dx = \frac{(0.5)^4}{8} - \frac{(0.5)^9}{36} + \frac{(0.5)^{14}}{112} - \frac{(0.5)^{19}}{304} + \cdots$$

since $\left|4\text{th term}\right| = \left|\frac{-(0.5)^{19}}{304}\right| \approx 0.000000006 < 10^{-7}$

$$\int_0^{0.5} \frac{x^3}{2+x^5}\,dx \approx \frac{(0.5)^4}{8} - \frac{(0.5)^9}{36} + \frac{(0.5)^{14}}{112}$$

$$\int_0^{0.5} \frac{x^3}{2+x^5}\,dx \approx \frac{(0.5)^4}{8} - \frac{(0.5)^9}{36} + \frac{(0.5)^{14}}{112}$$

$$\approx 0.00775879$$

$\int_0^{0.5} \frac{x^3}{2+x^5}\,dx \approx 0.00775879$ and this estimate

is accurate to within 10^{-7}.

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Power Series Functions 8

Ex 1/A] Express $f(x) = \ln\left(\frac{1+x}{1-2x}\right)$ as a power series and find radius of convergence.

Ex 1/B] Express $f(1/4)$ as the sum of an infinite series by using result of Ex 1/A]

Ex 1/C] Can we express $f(3/4)$ as the sum of an infinite series ?

1/A] Note that $\frac{d}{dx}\mathrm{Ln}(1+x) = \frac{1}{1+x}$

$$\mathrm{Ln}\left(\frac{1+x}{1-2x}\right) = \mathrm{Ln}(1+x) - \mathrm{Ln}(1-2x)$$

$$\mathrm{Ln}\left(\frac{1+x}{1-2x}\right) = \int \frac{1}{1+x}\,dx - (-2)\int \frac{1}{1-2x}\,dx$$

$$\mathrm{Ln}\left(\frac{1+x}{1-2x}\right) = \int \frac{1}{1-(-x)}\,dx + 2\int \frac{1}{1-2x}\,dx$$

Now that we have written both integrands as the sum of a geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \;;\, |r|<1$

we can expand the sum and integrate term by term.

$$\ln\left(\frac{1+x}{1-2x}\right) = \int \frac{1}{1-(-x)}\,dx + 2\int \frac{1}{1-2x}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x)^n\,dx + 2\int \sum_{n=0}^{\infty} (2x)^n\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx + 2\int \sum_{n=0}^{\infty} 2^n x^n\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^n\,dx + \sum_{n=0}^{\infty} 2^{n+1} \int x^n\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + \sum_{n=0}^{\infty} 2^{n+1} \frac{x^{n+1}}{n+1} + C$$

$$\ln\left(\frac{1+x}{1-2x}\right) = \sum_{n=0}^{\infty} \frac{((-1)^n + 2^{n+1})\,x^{n+1}}{n+1} + C$$

$$\ln\left(\frac{1+x}{1-2x}\right)=\sum_{n=0}^{\infty}\frac{((-1)^n+2^{n+1})\,x^{n+1}}{n+1}+C$$

To find C plug in $x=0$ into both sides

$$\ln\left(\frac{1+0}{1-0}\right)=\sum_{n=0}^{\infty}\frac{((-1)^n+2^{n+1})\,(0)^{n+1}}{n+1}+C$$

$$0 = 0+C \Rightarrow C=0$$

$$\ln\left(\frac{1+x}{1-2x}\right)=\sum_{n=0}^{\infty}\frac{((-1)^n+2^{n+1})\,x^{n+1}}{n+1}$$

Now let's find the radius of convergence for

$$f(x)=\ln\left(\frac{1+x}{1-2x}\right)$$

$$Ln\left(\frac{1+x}{1-2x}\right) = \int \frac{1}{1-(-x)}\,dx + 2\int \frac{1}{1-2x}\,dx$$

The first integrand $\frac{1}{1-(-x)}$ is the sum of a geometric series with common ratio $r=-x$ and converges for $|-x|<1 \Rightarrow |x|<1 \Rightarrow$ Radius $=1$ for $Ln(1+x)$

The second integrand $\frac{1}{1-2x}$ is the sum of a geometric series with $r=2x$ and converges for $|2x|<1$ $\Rightarrow |x|<\frac{1}{2} \Rightarrow$ Radius $=\frac{1}{2}$ for $Ln(1-2x)$

Since we are adding these 2 power series we should take the minimum of the 2 values of R

$\therefore$ Radius $=$ Min$(1/2, 1) = 1/2$

$$f(x) = Ln\left(\frac{1+x}{1-2x}\right) = \sum_{n=0}^{\infty} \frac{\left((-1)^n + 2^{n+1}\right) x^{n+1}}{n+1}$$

$R = \frac{1}{2}$ Radius of Convergence

$\frac{-1}{2} < x < \frac{1}{2}$ Interval of Convergence

$$f(x) = Ln\left(\frac{1+x}{1-2x}\right) = 3x + \frac{3x^2}{2} + 3x^3 + \frac{15}{4}x^4 + \cdots$$

1/B] To express $f(1/4)$ as the sum of an infinite series simply plug in $x = 1/4$ into power series for

$$f(x) = \ln\left(\frac{1+x}{1-2x}\right) = \sum_{n=0}^{\infty} \frac{((-1)^n + 2^{n+1})\, x^{n+1}}{n+1} \quad \text{for } |x| < \frac{1}{2}$$

$$f(1/4) = \ln(5/2) = \sum_{n=0}^{\infty} \frac{((-1)^n + 2^{n+1})(1/4)^{n+1}}{n+1}$$

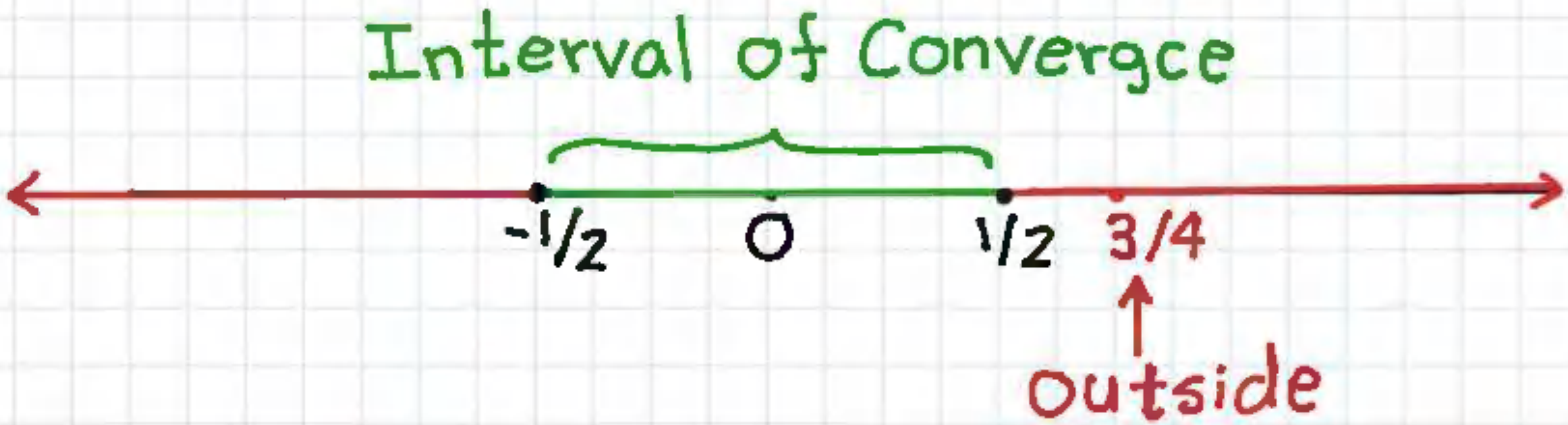
Note: Since $x = 1/4$ is inside the Interval of Convergence $(-1/2, 1/2)$ we can plug in $x = 1/4$

$$f(1/4) = \ln(5/2) = 3(1/4) + \frac{3}{2}(1/4)^2 + 3(1/4)^3 + \frac{15}{4}(1/4)^4 + \ldots$$

1/C) Can we express $f(3/4)$ as the sum of an Infinite Series?

No! We cannot plug in $x = 3/4$ into the Power Series for $f(x) = \ln\left(\frac{1+x}{1-2x}\right) = \sum_{n=0}^{\infty} \frac{((-1)^n + 2^{n+1})x^{n+1}}{n+1}$

Since $x = 3/4$ is <u>outside</u> the Interval of Convergence $\frac{-1}{2} < x < \frac{1}{2}$



Power Series Functions 9

Given $f(x) = \dfrac{2x+4}{(x-1)(2x+1)}$, Express the function as the sum of a power series by applying partial fractions. Also find the interval of convergence.

Solution: Let's apply partial fraction decomposition.

$$\frac{2x+4}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1} = \frac{A(2x+1)+B(x-1)}{(x-1)(2x+1)}$$

Multiply both sides by $(x-1)(2x+1)$ to clear out the denominators.

$$2x+4 = A(2x+1)+B(x-1)$$

Plug $x=1 \Rightarrow 2(1)+4 = A(2+1)+B(0) \Rightarrow 6 = 3A \Rightarrow A=2$

Plug $x=-1/2 \Rightarrow 2(-1/2)+4 = A(0)+B(-\frac{1}{2}-1)$

$\Rightarrow 3 = 0 + B(-3/2) \Rightarrow B = \cancel{3}\left(\frac{-2}{\cancel{3}}\right) = -2$

$$\frac{2x+4}{(x-1)(2x+1)} = \frac{2}{x-1} + \frac{-2}{2x+1}$$

Now fit each fraction into pattern of a geometric sum $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad ; |x|<1$

$$\frac{2}{x-1} = \frac{2}{-1(1-x)} = -2 \cdot \frac{1}{1-x} = -2\sum_{n=0}^{\infty} x^n$$

$$\frac{2}{x-1} = -2 \cdot \frac{1}{1-x} = -2 \sum_{n=0}^{\infty} x^n$$

This is a geometric series which converges for $|x| < 1$, so the interval of convergence is $-1 < x < 1$ and Radius of convergence is $R = 1$

For the second fraction $\frac{-2}{2x+1}$

$$\frac{-2}{2x+1} = \frac{-2}{1-(-2x)} = -2 \cdot \frac{1}{1-(-2x)} = -2 \sum_{n=0}^{\infty} (-2x)^n$$

$$\frac{-2}{2x+1} = -2 \sum_{n=0}^{\infty} (-2)^n x^n = \sum_{n=0}^{\infty} (-2)^{n+1} x^n$$

$$\frac{-2}{2x+1} = -2 \cdot \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2)^{n+1} x^n$$

This is a geometric series with $r=-2x$, it converges when $|-2x| < 1 \Rightarrow |2x| < 1 \Rightarrow |x| < \frac{1}{2}$

So the interval of convergence for $\frac{-2}{2x+1}$ is $(-1/2, 1/2)$ and the Radius of convergence is $R = 1/2$.

Since we are adding these 2 power series we should take the minimum of the 2 values of R

$R = \text{Min}(1/2, 1) = 1/2$

Putting it all together:

$$\frac{2x+4}{(x-1)(2x+1)} = \frac{2}{x-1} - \frac{2}{2x+1} = -2\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-2)^{n+1} x^n$$

$$\frac{2x+4}{(x-1)(2x+1)} = \sum_{n=0}^{\infty} \left(-2+(-2)^{n+1}\right) x^n$$

$R = 1/2$ Radius of Convergence

$-1/2 < x < 1/2$ Interval of Convergence

$$f(x) = \frac{2x+4}{(x-1)(2x+1)} = -4+2x-10x^2+14x^3-+\cdots$$

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Power Series Functions 10

Ex| Express $f(x) = \ln x$ as a power series in powers of $x-e$ and also find the Radius and interval of convergence.

Hint: $\ln(1+u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{n+1}}{n+1}$; $|u|<1$

let's rewrite $\ln x = \ln(e+(x-e))$

$$\ln x = \ln\left(e\left(1+\frac{x-e}{e}\right)\right) = \ln e + \ln\left(1+\frac{x-e}{e}\right)$$

let $u=(x-e)/e \Rightarrow \ln(1+u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{n+1}}{n+1}$

$$\ln x = 1 + \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-e}{e}\right)^{n+1} \Big/ (n+1)$$

$$\ln x = 1 + \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-e}{e}\right)^{n+1} \Big/ (n+1)$$

$$f(x) = \ln x = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n (x-e)^{n+1}}{e^{n+1}\,(n+1)}$$

Power Series of $f(x) = \ln x$ in powers of $x-e$

Now let's find the radius and interval of convergence

$\ln(1+u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{n+1}}{n+1}$ has radius $R=1$

and interval of convergence $|u|<1$, so power series for $\ln\left(1+\left(\frac{x-e}{e}\right)\right)$ converges for

$$\left|\frac{x-e}{e}\right| < 1 \Rightarrow |x-e| < e \Rightarrow -e < x-e < e \Rightarrow 0<x<2e$$

$0<x<2e$ Interval of Convergence.

$$f(x) = \ln x = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n (x-e)^{n+1}}{e^{n+1}(n+1)}$$ has Radius $R = e$

and interval of convergence $0 < x < 2e$

$$\ln x = 1 + \frac{x-e}{e} - \frac{(x-e)^2}{2e^2} + \frac{(x-e)^3}{3e^3} - \frac{(x-e)^4}{4e^4} + - \cdots$$

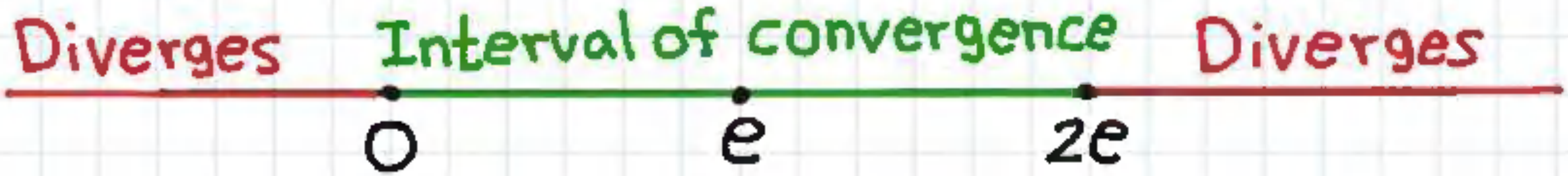
Summary:

$$\ln x = \ln(e + (x-e)) = \ln\left(e\left(1 + \frac{x-e}{e}\right)\right)$$

$$\ln x = \ln e + \ln\left(1 + \frac{x-e}{e}\right) = 1 + \ln\left(1 + \frac{x-e}{e}\right)$$

Apply hint: $\ln(1+u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{n+1}}{n+1}$; $|u| < 1$

Diagram for Radius and Interval of Convergence



Recall: Radius of convergence is the distance from an endpoint to the center of the power series.

Radius $= 2e - e = e$ or Radius $= e - 0 = e$