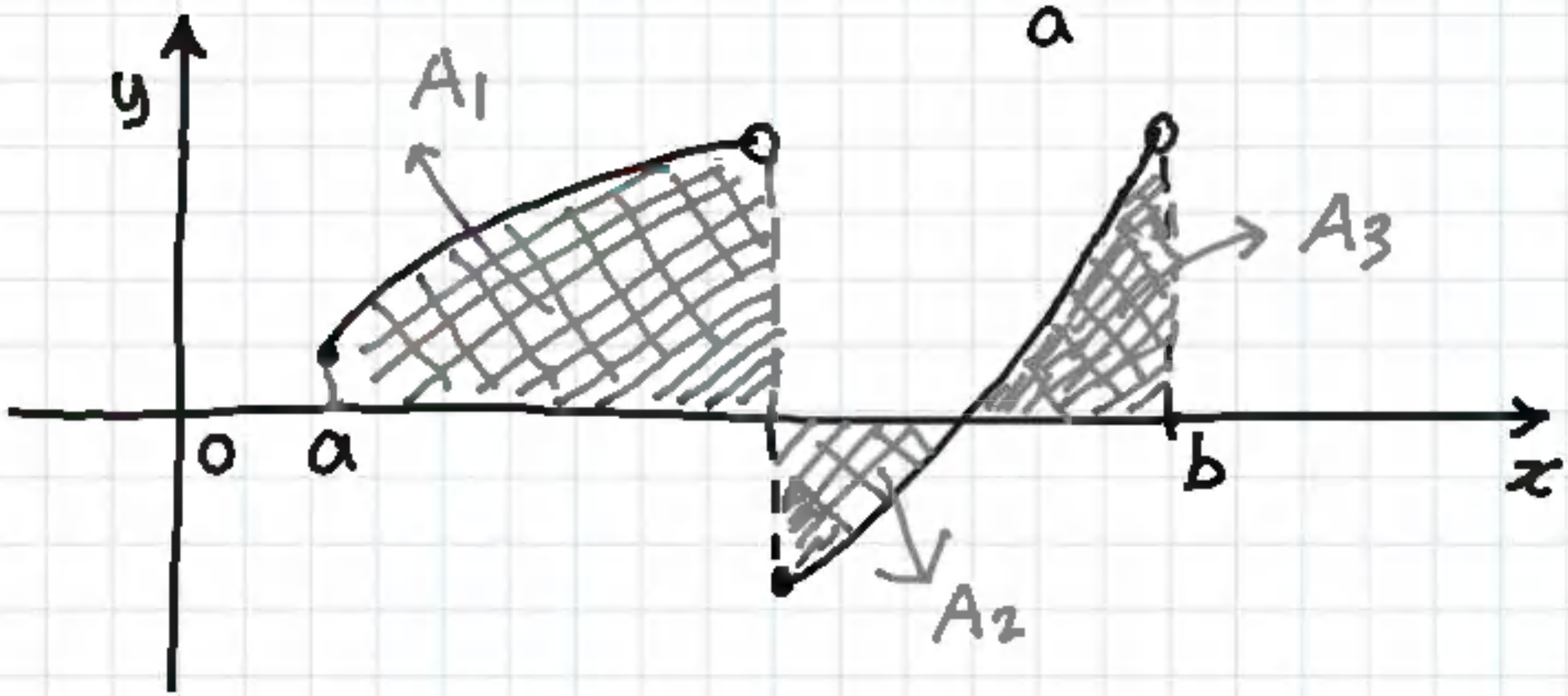
Definite Integrals 1

let's start with some Theory!

Theorem: If $f(x)$ is continuous on $[a,b]$ or piecewise continuous on $[a,b]$ then $f(x)$ is integrable on $[a,b]$, therefore the definite integral exists and is given by $\int_a^b f(x)\,dx$



$\int_a^b f(x)\,dx$ is the net Area which means area above the x axis minus the area below the x axis (Geometric Idea)

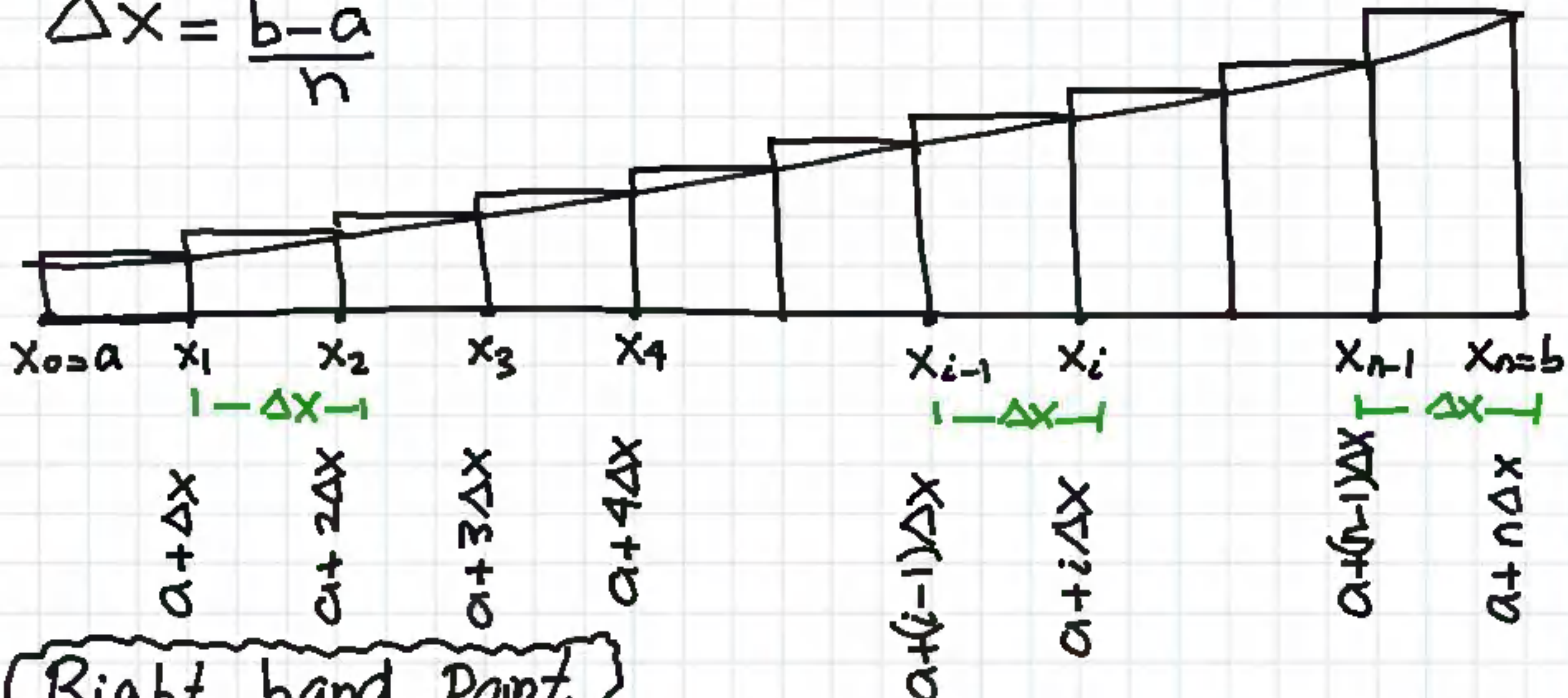
$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

Theorem: If $f(x)$ is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

We are going to introduce the theory of a definite integral for right hand, left hand and midpoint Riemann Sums.

Divide the interval $[a,b]$ into n equal subintervals

$$\Delta x = \frac{b-a}{n}$$



Right hand Point

$$R_n = f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \cdots + f(a+n\Delta x)\Delta x$$

$$R_n = \sum_{i=1}^{n} f(a+i\Delta x)\,\Delta x$$

Right hand Riemann Sum

Take limit as number of rectangles n approaches ∞

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x = \int_a^b f(x)\,dx$$

Right hand Riemann sum

$$\lim_{n \to \infty} L_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + (i-1)\Delta x)\Delta x = \int_a^b f(x)\,dx$$

Left hand Riemann Sum

$$\lim_{n \to \infty} M_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + (i - \tfrac{1}{2})\Delta x)\Delta x = \int_a^b f(x)\,dx$$

Midpoint Riemann sum

Summary Riemann Sum

$$\Delta x = \frac{b-a}{n}$$

Right hand Point: $x_i = a + i\Delta x$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$$

Left hand Point: $x_{i-1} = a+(i-1)\Delta x$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+(i-1)\Delta x)\Delta x$$

Midpoint: $x_{mid} = a+(i-1/2)\Delta x$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+(i-1/2)\Delta x)\Delta x$$

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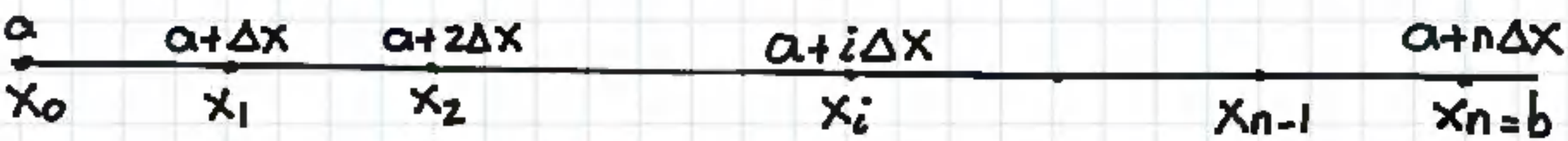
Definite Integrals 2

2| Prove that if $f(x)$ is continuous on $[a,b]$, and the interval $[a,b]$ is divided into n equal partitions with $\Delta x = \frac{b-a}{n}$ then the difference between the Right hand Riemann sum and the Left hand Riemann sum approaches zero as the number of subintervals n approaches infinity.

Solution: Divide the interval $[a,b]$ into n equal partitions $\Delta x = \frac{b-a}{n}$



$$x_n = a + n\Delta x = a + \cancel{n}\frac{(b-a)}{\cancel{n}} = b$$

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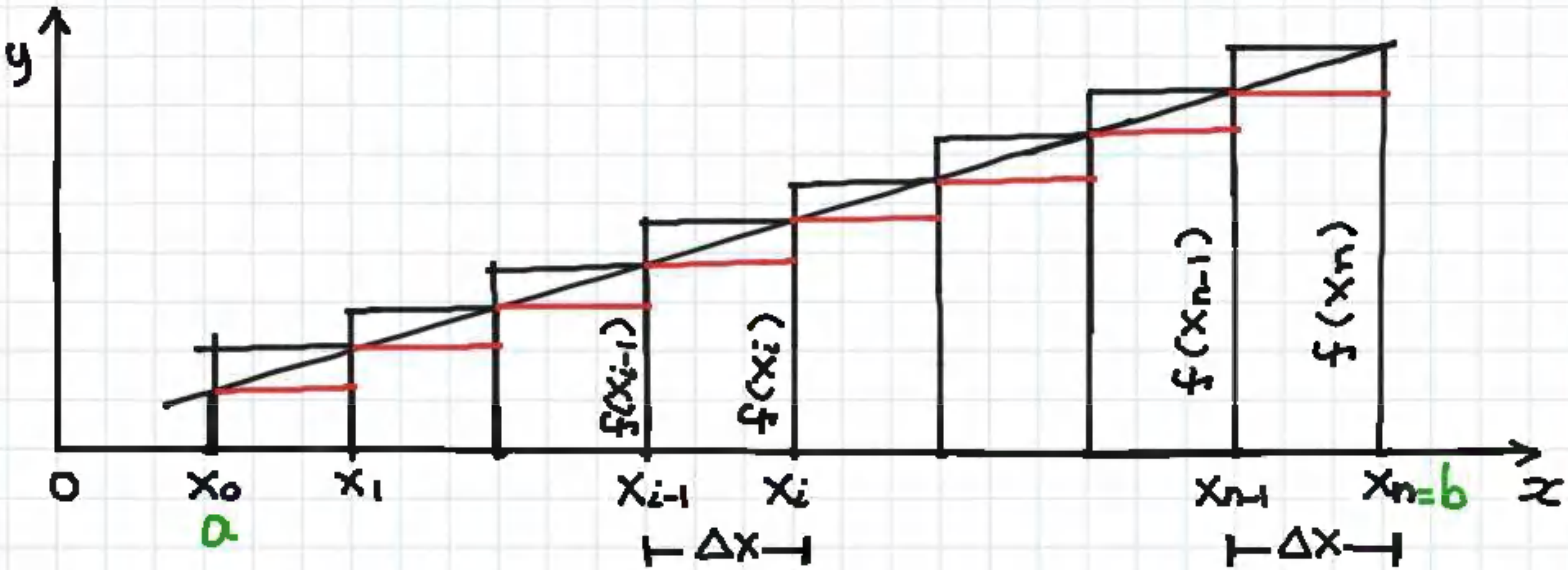


$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_i)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x$$

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_i)\Delta x + \cdots + f(x_{n-1})\Delta x$$

$$R_n - L_n = f(x_n)\Delta x - f(x_0)\Delta x$$

$$R_n - L_n = \Delta x\left[f(x_n) - f(x_0)\right] = \frac{b-a}{n}\left[f(b) - f(a)\right]$$

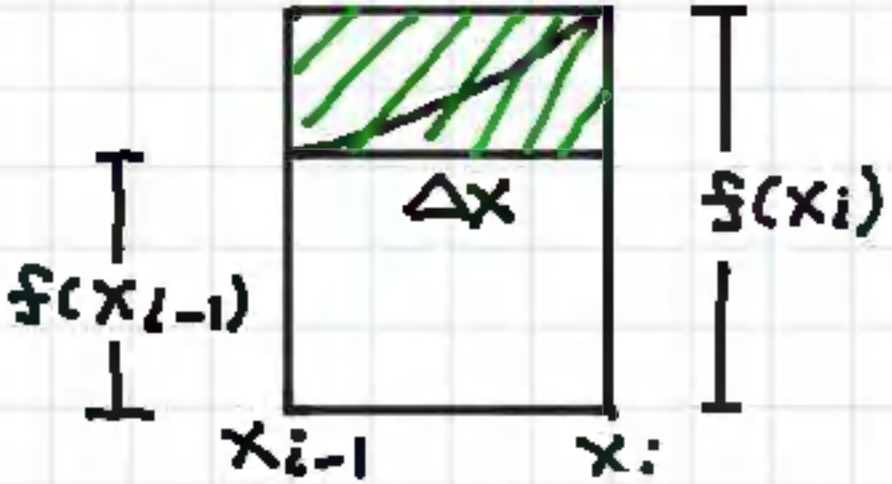
$$R_n - L_n = \frac{b-a}{n}\left[f(b) - f(a)\right]$$

Now take limit as $n \to \infty$

$$\lim_{n\to\infty} R_n - L_n = \lim_{n\to\infty} \frac{b-a}{n}\left[f(b)-f(a)\right] = 0$$

Note: $b-a$ and $f(b)-f(a)$ are constants

So we have shown that the difference between left hand and right hand Riemann Sums approaches zero as $n \to \infty$.



$[f(x_i) - f(x_{i-1})]\,\Delta x$ is the difference between Right hand and left hand Rule.

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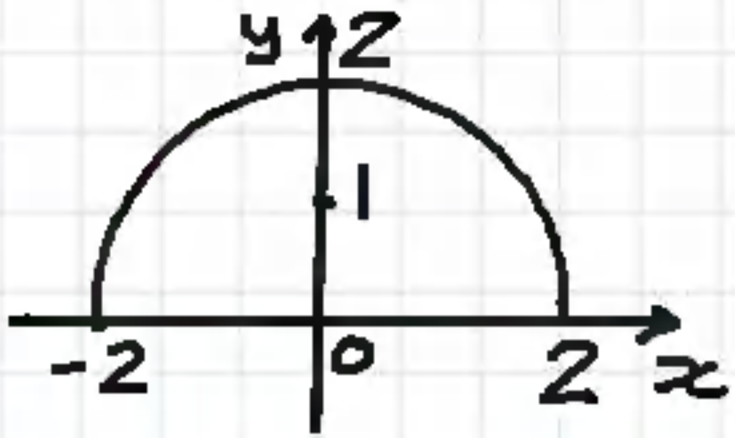
Definite Integrals 3 (Lower and upper Sums)

3) Calculate Lower and Upper estimates for the area above the x axis and below the function $f(x)=\sqrt{4-x^2}$ with 4 rectangles $n=4$



Solution:

$y=\sqrt{4-x^2}$

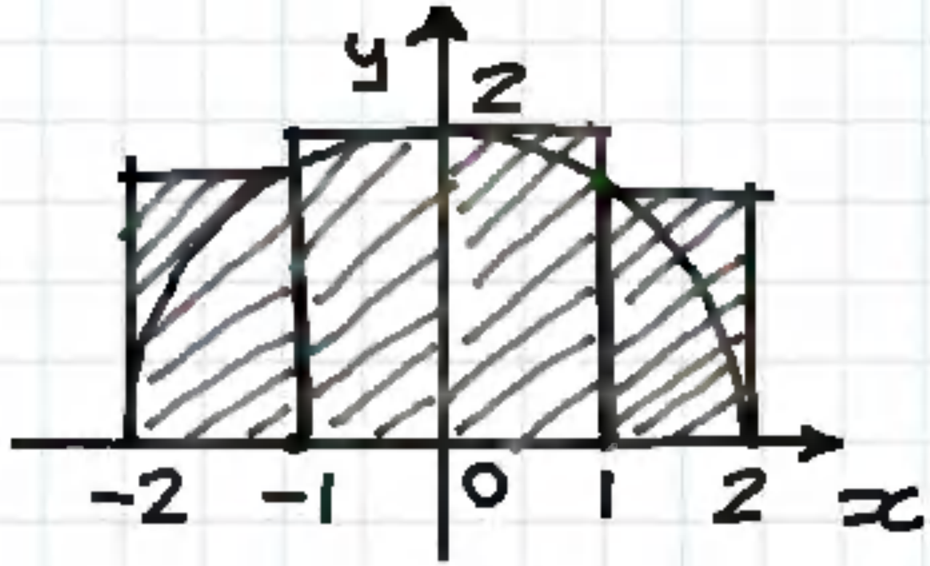
$y^2=4-x^2$ Square both sides

$x^2+y^2=4$ Equation of circle with radius 2

$A=\dfrac{\pi(r)^2}{2}$ Area of semi-circle

$A=\dfrac{\pi(2)^2}{2}=2\pi$ Exact area

Upper Sum



$n = 4$ rectangles

$$\Delta x = \frac{b-a}{n} = \frac{2 - -2}{4} = 1$$

$f(x) = \sqrt{4-x^2}$ $\quad f(-1) = \sqrt{3}$ $\quad f(0) = 2$ $\quad f(1) = \sqrt{3}$

$$\text{UPPER SUM} = f(-1)(1) + f(0)(1) + f(0)(1) + f(1)(1)$$
$$= \sqrt{3}(1) + 2(1) + 2(1) + \sqrt{3}(1)$$

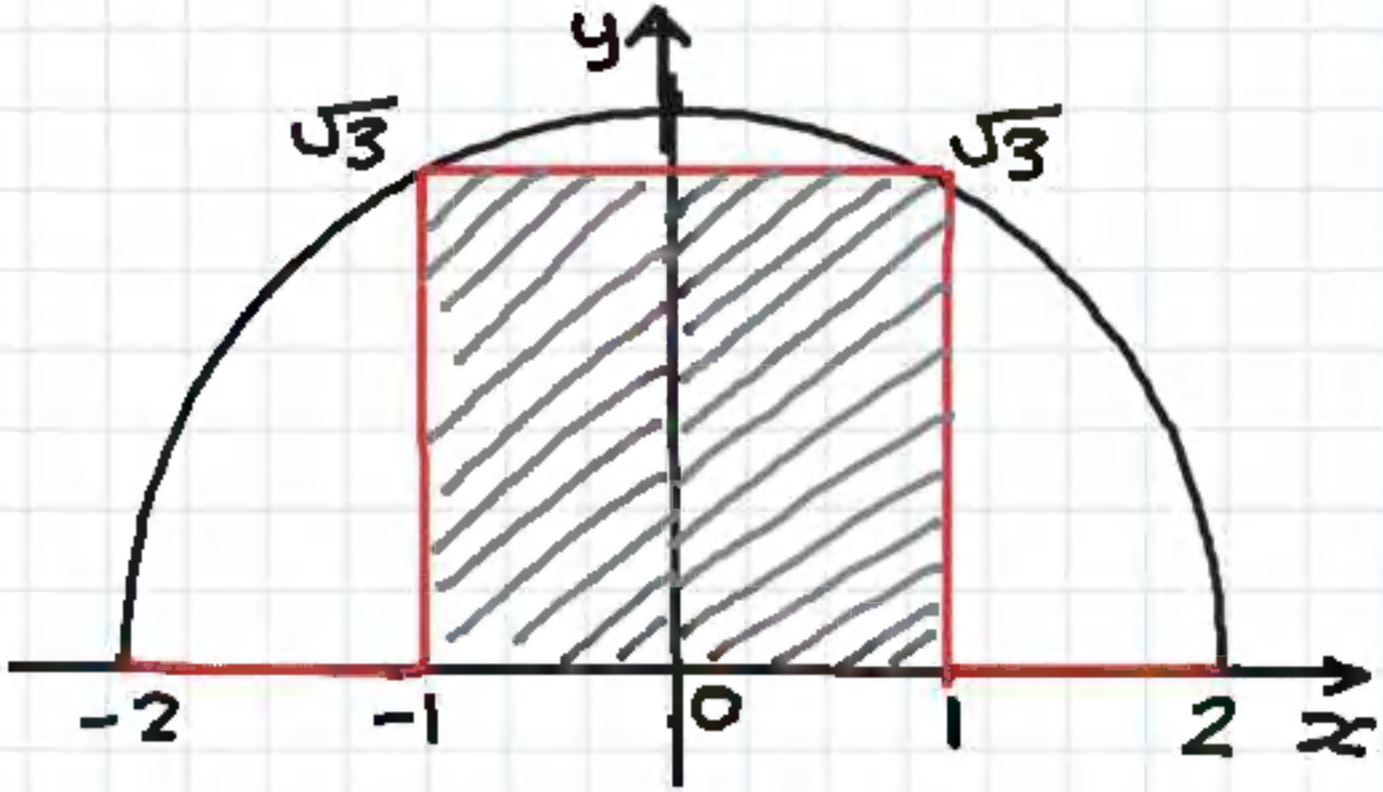
$$\boxed{\text{UPPER SUM} = 2\sqrt{3} + 4}$$

$$\text{Exact area} = \frac{\pi(2)^2}{2} = \boxed{2\pi}$$

Lower Sum

$n=4$ rectangles

$$\Delta x = \frac{b-a}{n} = \frac{2--2}{4} = 1$$



$f(x)=\sqrt{4-x^2}$ $\quad f(-2)=0$ $\quad f(-1)=\sqrt{3}$ $\quad f(1)=\sqrt{3}$ $\quad f(2)=0$

Lower Sum $= f(-2)(1)+f(-1)(1)+f(1)(1)+f(2)(1)$

Lower Sum $= 0(1)+\sqrt{3}(1)+\sqrt{3}(1)+0(1) = \boxed{2\sqrt{3}}$

Lower Sum $= 2\sqrt{3}$

Exact area $= \dfrac{\pi(2)^2}{2} = \boxed{2\pi}$

Summary

Lower Sum $= 2\sqrt{3}$ Upper Sum $= 4 + 2\sqrt{3}$

Lower Sum $\cong 3.464$ Upper Sum $\cong 7.464$

Exact Area $= 2\pi \cong 6.283$

$$\boxed{3.464 < 6.283 < 7.464}$$

Lower Sum < exact area < Upper Sum

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Definite Integrals 4 (Riemann Sum to Integral)

4) Express the following limit as a definite integral

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{n^2} \cos\left(\frac{j^2}{n^2}\right)$$

Solution:

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j) \overbrace{\left(\frac{b-a}{n}\right)}^{\Delta x} = \int_a^b f(x)\,dx$$

let's find x_j, a, b, $f(x_j)$ and Δx

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{n} \cos\left(\left(\frac{j}{n}\right)^2\right) \underbrace{\frac{1}{n}}_{\Delta x} = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j) \underbrace{\left(\frac{b-a}{n}\right)}_{\Delta x}$$

$$x_j = \frac{j}{n} \Rightarrow x_1 = \frac{1}{n} \quad x_2 = \frac{2}{n} \quad x_3 = \frac{3}{n} \quad \dots \quad x_n = \frac{n}{n} = 1 = b$$

$$\Delta x = \frac{b-a}{n} = \frac{1}{n} \Rightarrow b-a=1 \Rightarrow a=0 \text{ since } b=1$$

Since $x_j = \dfrac{j}{n}$; $f(x_j) = \dfrac{j}{n}\cos\left(\left(\dfrac{j}{n}\right)^2\right) = x_j\cos(x_j^2)$

$f(x) = x\cos(x^2)$

We now have all the pieces of the puzzle!

$a=0$ $\quad b=1$ $\quad x_j = j/n$ $\quad f(x_j) = x_j\cos(x_j^2)$ $\quad \Delta x = \dfrac{1}{n}$

$f(x) = x\cos(x^2)$

Now we can transform the Riemann sum to a definite integral

$$\lim_{n\to\infty}\sum_{j=1}^{n}\underbrace{\frac{j}{n}\cos\left(\left(\frac{j}{n}\right)^2\right)}_{f(x_j)}\cdot\underbrace{\frac{1}{n}}_{\Delta x} = \int_0^1 \underbrace{x\cos(x^2)}_{f(x)}\,dx$$

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Definite Integrals 5 (Riemann Sum to Integral)

5] The limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{4+\left(\frac{i}{n}\right)^2} \cdot \frac{2}{n}$ represents an integral $\int_a^b f(x)\,dx$, Find the integral?

Solution:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n} = \int_a^b f(x)\,dx$$

let's find x_i, a, b, $f(x_i)$ and Δx

Given: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{4+\left(\frac{i}{n}\right)^2} \cdot \frac{2}{n}$

$$x_i = \frac{i}{n} \Rightarrow \quad x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \quad x_3 = \frac{3}{n}, \ldots, x_n = \frac{n}{n} = 1 = b$$

$$\Delta x = \frac{b-a}{n} = \frac{1}{n} \qquad \Delta x = x_i - x_{i-1} = \frac{i}{n} - \frac{(i-1)}{n} = \frac{1}{n}$$

$$\Delta x = \frac{\boxed{b-a}}{n} = \frac{\boxed{1}}{n} \Rightarrow b-a=1 \Rightarrow a=0 \text{ since } b=1$$

Since $x_i = i/n$; $f(x_i) = \dfrac{1}{4+\left(\frac{i}{n}\right)^2} = \dfrac{1}{4+x_i^2}$

We now have all the pieces of the puzzle!

$a=0$, $b=1$, $f(x_i) = \dfrac{1}{4+x_i^2}$, $f(x) = \dfrac{1}{4+x^2}$

Lets factor out 2 from Riemann Sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{4+\left(\frac{i}{n}\right)^2} \cdot \frac{2}{n} = 2 \lim_{n\to\infty} \sum_{i=1}^{n} \overbrace{\frac{1}{4+\left(\frac{i}{n}\right)^2}}^{f(x_i)} \cdot \overbrace{\frac{1}{n}}^{\Delta x}$$

$$2 \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{1}{4+\left(\frac{i}{n}\right)^2}}_{f(x_i)} \cdot \underbrace{\frac{1}{n}}_{\Delta x} = 2 \int_0^1 \underbrace{\frac{1}{4+x^2}}_{f(x)} \, dx$$

Summary

Riemann Sum

$\Downarrow$

Definite Integral

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{4+\left(\frac{i}{n}\right)^2}\cdot\frac{2}{n}$$

$$\Downarrow$$

$$2\lim_{n\to\infty}\sum_{i=1}^{n}\overbrace{\frac{1}{4+\left(\frac{i}{n}\right)^2}}^{f(x_i)}\cdot\overbrace{\frac{1}{n}}^{\Delta x}$$

$$\Downarrow$$

$$2\int_0^1\frac{1}{4+x^2}\,dx$$

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Definite Integrals 6 (Riemann Sum to Integral)

6| Express the following limit as a definite integral $\int_a^b f(x)\,dx$

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\sqrt{\frac{3}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)$$

Solution: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\overbrace{\left(\frac{b-a}{n}\right)}^{\Delta x} = \int_a^b f(x)\,dx$

let's find x_i, a, b, $f(x_i)$ and Δx

It is clear that the general term is $\sqrt{\frac{i}{n}}$

So we can write the above expression in a more compact form $\sum_{i=1}^{n} f(x_i)\,\Delta x$

lets rewrite $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\sqrt{\frac{3}{n}}+\dots+\sqrt{\frac{n}{n}}\right)$

into $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} \cdot \frac{1}{n}$

Let $x_i = i/n$

$x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, $x_3 = \frac{3}{n} \dots$ $x_i = \frac{i}{n} \dots$ $x_n = \frac{n}{n} = 1 = b$

To find Δx simply subtract $x_2 - x_1 = \frac{2}{n} - \frac{1}{n} = \frac{1}{n}$

or in general $\Delta x = x_i - x_{i-1} = \frac{i}{n} - \left(\frac{i-1}{n}\right) = \frac{1}{n}$

$\Delta x = \frac{\boxed{b-a}}{n} = \frac{\boxed{1}}{n} \Rightarrow b-a=1 \Rightarrow a=0$ since $b=1$

Since $x_i = \frac{i}{n}$; $f(x_i) = \sqrt{\frac{i}{n}} = \sqrt{x_i} \Rightarrow f(x) = \sqrt{x}$

We now have all the pieces of the puzzle!

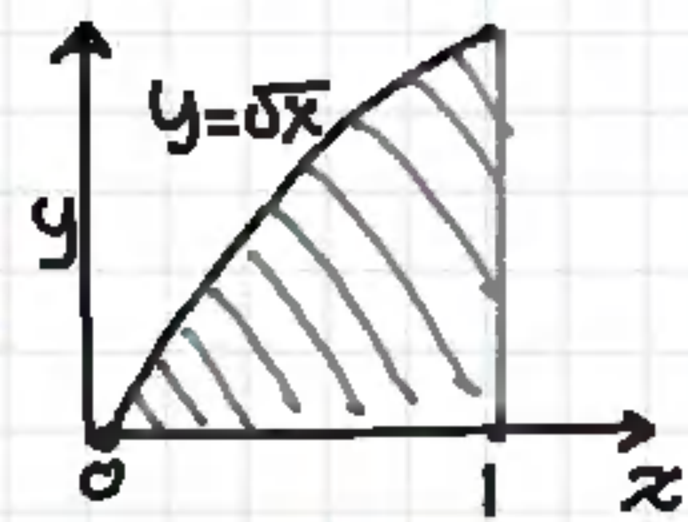
$a=0$, $b=1$, $f(x)=\sqrt{x}$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\sqrt{\frac{i}{n}}}_{f(x_i)}\cdot\underbrace{\frac{1}{n}}_{\Delta x}=\int_0^1\underbrace{\sqrt{x}}_{f(x)}\,dx$$

BIG PICTURE

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}\cdot\frac{1}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{x_i}\,\Delta x=\underbrace{\int_0^1\sqrt{x}\,dx}_{\text{Area under } y=\sqrt{x} \text{ from 0 to 1}}$$



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Definite Integrals 7 (Riemann Sum to Integral)

7] Express $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi}{2}\left(\frac{i-1}{n}\right)\right) \cdot \frac{1}{n}$

as a definite integral $\int_0^1 f(x)\,dx$ and $\int_0^{\pi/2} f(x)\,dx$

Solution: $$\underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x}_{\text{Left hand Riemann Sum}} = \underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}}$$

$$\int_{a=0}^{b=1} f(x)\,dx \qquad x_{i-1} = \frac{i-1}{n} \qquad i = 1, 2, \ldots, n$$

$$x_0 = 0 \quad x_1 = \frac{1}{n} \quad x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}$$

$\Downarrow$

$a = 0$

$$b = x_{n-1} + \Delta x = \frac{n-1}{n} + \frac{1}{n} = 1$$

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$a = 0 \quad b = 1 \quad \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \quad x_{i-1} = \frac{i-1}{n}$

$f(x_{i-1}) = \sin\left(\frac{\pi}{2}(x_{i-1})\right) \Rightarrow f(x) = \sin\left(\frac{\pi x}{2}\right)$

Now, we have all the pieces of the puzzle!

$a = 0 \quad b = 1 \quad f(x) = \sin\left(\frac{\pi x}{2}\right)$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\sin\left[\frac{\pi}{2}\left(\frac{i-1}{n}\right)\right]}_{f(x_{i-1})} \cdot \underbrace{\frac{1}{n}}_{\Delta x} = \int_0^1 \underbrace{\sin\left(\frac{\pi x}{2}\right)}_{f(x)} \underbrace{dx}_{dx}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin\Big[\underbrace{\frac{\pi}{2}\Big(\frac{i-1}{n}\Big)}_{x_{i-1}}\Big]\cdot\frac{1}{n} = \int_{a=0}^{b=\frac{\pi}{2}} f(x)\,dx$$

$$\text{Let } x_{i-1} = \frac{\pi}{2}\Big(\frac{i-1}{n}\Big) \quad i = 1,2,3,\ldots,n$$

$$x_0 = 0 \quad x_1 = \frac{\pi}{2n} \quad x_2 = \frac{\pi}{n} \quad x_3 = \frac{3\pi}{2n} \quad \ldots \quad x_{n-1} = \frac{\pi}{2}\Big(\frac{n-1}{n}\Big)$$

$$a = 0 \quad b = \frac{\pi}{2} \quad f(x_{i-1}) = \sin(x_{i-1}) \Rightarrow f(x) = \sin x$$

$$\Delta x = \frac{b-a}{n} = \frac{\pi/2 - 0}{n} = \frac{\pi}{2n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin\Big[\frac{\pi}{2}\Big(\frac{i-1}{n}\Big)\Big]\cdot\frac{\pi/2}{n}\cdot\frac{1}{\pi/2}$$

$$\frac{2}{\pi}\lim_{n\to\infty} \sum_{i=1}^{n} \sin\Big[\frac{\pi}{2}\Big(\frac{i-1}{n}\Big)\Big]\cdot\pi/2n$$

$$\frac{2}{\pi}\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\sin\left(\frac{\pi}{2}\left(\frac{i-1}{n}\right)\right)}_{f(x_{i-1})}\cdot\underbrace{\frac{\pi}{2n}}_{\Delta x}=\frac{2}{\pi}\int_0^{\pi/2}\underbrace{\sin x}_{f(x)}\,\underbrace{dx}_{dx}$$

Summary

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\frac{\pi}{2}\left(\frac{i-1}{n}\right)\right)\cdot\frac{1}{n}\Rightarrow\int_0^1\sin\left(\frac{\pi x}{2}\right)dx$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\frac{\pi}{2}\left(\frac{i-1}{n}\right)\right)\cdot\frac{1}{n}\Rightarrow\frac{2}{\pi}\int_0^{\pi/2}\sin x\,dx$$

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Definite Integrals 8 (Riemann Sum to Integral)

8] Express $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \log\left(1+i\left(\frac{e+1}{n}\right)\right)\left(\frac{e+1}{n}\right)$

as a definite integral with $a=-1$ $\int_a^b f(x)\,dx$

Solution: Recall Right hand Riemann Sum Formula

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) = \int_a^b f(x)\,dx$$

$$x_i = a+i\left(\frac{b-a}{n}\right) \Rightarrow x_i = -1+i\left(\frac{b--1}{n}\right) = -1+i\left(\frac{b+1}{n}\right)$$

The tricky part is to rewrite $\log\left(1+i\left(\frac{e+1}{n}\right)\right)$ in such a way so that $x_i = -1+i\left(\frac{b+1}{n}\right)$ is inside $\log(\qquad)$

$$\log\left(1+i\left(\frac{e+1}{n}\right)\right) \Rightarrow \log\left(2+\overbrace{-1+i\left(\frac{e+1}{n}\right)}^{x_i}\right)$$

$$x_i = -1+i\left(\frac{e+1}{n}\right) \qquad f(x_i) = \log(2+x_i)$$

$i = 1, 2, \ldots, n$

$$x_1 = -1+\frac{e+1}{n} \qquad x_2 = -1+\frac{2(e+1)}{n} \quad \ldots \quad x_n = -1+\cancel{n}\left(\frac{e+1}{\cancel{n}}\right) = e = b$$

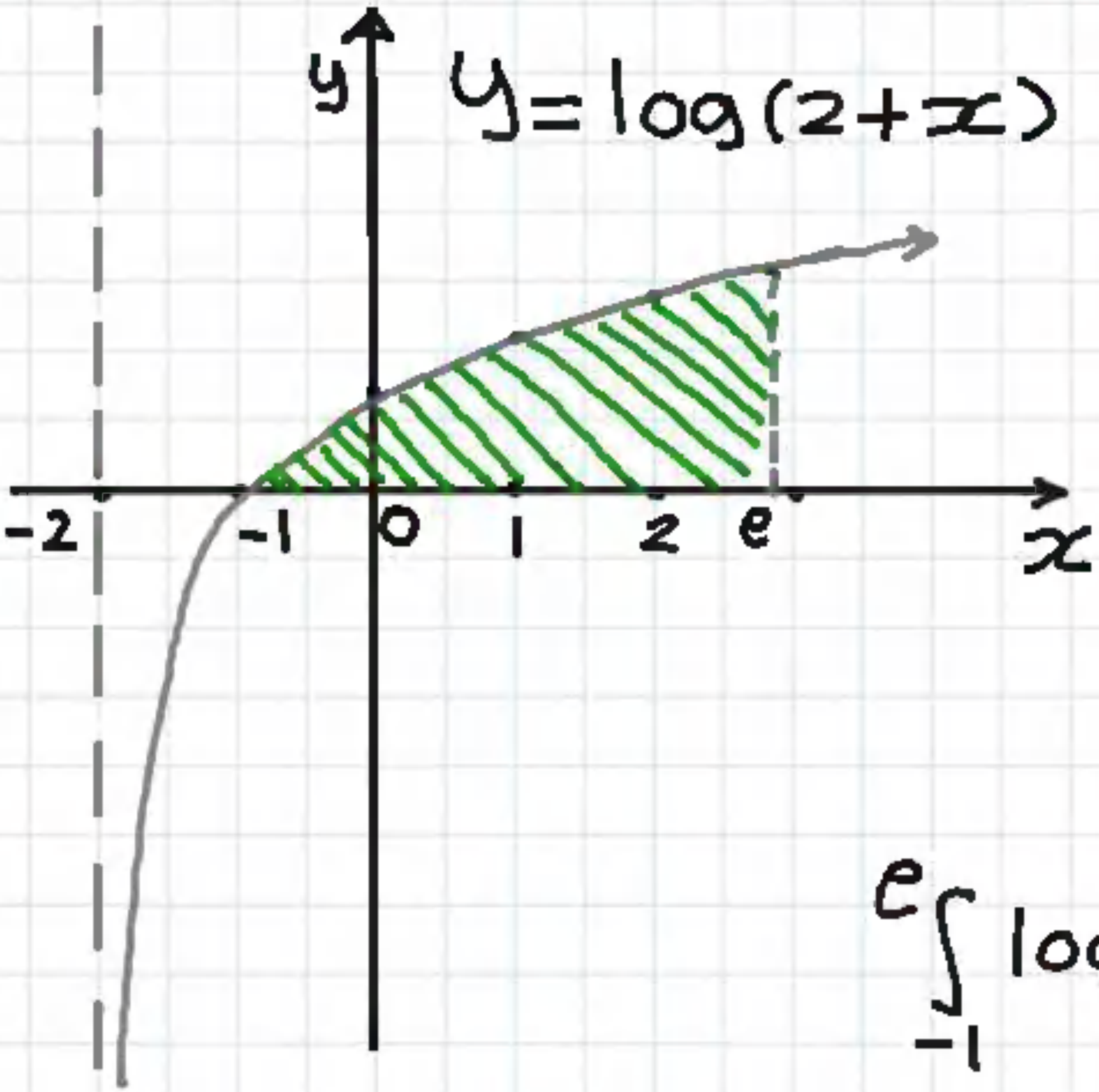
$$a=-1 \qquad b=e \qquad \Delta x = \frac{b-a}{n} = \frac{e--1}{n} = \frac{e+1}{n}$$

$$f(x_i) = \log(2+x_i) \Rightarrow f(x) = \log(2+x)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\log\left(1+i\left(\frac{e+1}{n}\right)\right)\left(\frac{e+1}{n}\right) = \int_{-1}^{e}\log(2+x)\,dx$$

Riemann Sum $\Longrightarrow$ Def. Integral

Diagram



$$\int_{-1}^{e} \log(2+x)\,dx = \text{Shaded Area}$$

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Definite Integrals 9 (Riemann Sum to Integral)

9| The limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{n^2-i^2}}{n^2}$ represents an integral $\int_a^b f(x)\,dx$ what is the integral? Evaluate the integral using Geometry?

Solution: Recall Right hand Riemann Sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx \quad ; \quad \Delta x = \frac{b-a}{n}$$

Goal is to rewrite Riemann Sum in such a way that $\Delta x = \frac{1}{n}$ appears inside the summation.

Let's Do some Algebra!

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{\frac{n^2}{n^2}-\frac{i^2}{n^2}}}{\frac{n^2}{n}}$$

$\sqrt{n^2}=n \quad n>0$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{1-\left(\frac{i}{n}\right)^2}}{n}=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}\cdot\underbrace{\frac{1}{n}}_{\Delta x}$$

$x_i=i/n \quad i=1,2,\ldots,n \qquad \Delta x=1/n$

$x_1=1/n\ ,\ x_2=2/n\ ,\ x_3=3/n\ \ldots\ x_i=i/n\ \ldots\ x_n=\frac{n}{n}=1=b$

lets find a?

$a=x_1-\Delta x=\underbrace{\frac{1}{n}}_{x_1}-\underbrace{\frac{1}{n}}_{\Delta x}=0$ OR $\lim_{n\to\infty}x_1=\lim_{n\to\infty}\frac{1}{n}=0$

$a=0$

We now have all the pieces of the puzzle!

$a = 0 \quad b = 1 \quad f(x_i) = \sqrt{1 - x_i^2} \Rightarrow f(x) = \sqrt{1 - x^2}$

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

Let's Transform Riemann Sum to Integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{n^2 - i^2}}{n^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\sqrt{1-(i/n)^2}}_{f(x_i)} \cdot \underbrace{\frac{1}{n}}_{\Delta x} = \int_{a=0}^{b=1} \underbrace{\sqrt{1-x^2}}_{f(x)}\, \underbrace{dx}_{dx}$$

Part 2 of the question is to compute

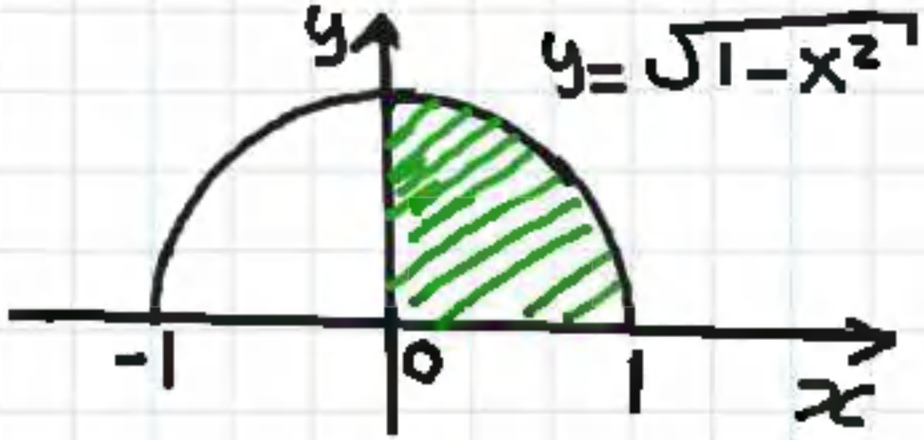
$\int_0^1 \underbrace{\sqrt{1-x^2}}_{f(x)}\, dx$ by applying Geometry?

$$\int_0^1 \sqrt{1-x^2}\,dx$$

$$y=\sqrt{1-x^2} \Rightarrow y^2=\left(\sqrt{1-x^2}\right)^2=1-x^2$$

$x^2+y^2=1$ equation of a circle with radius 1 and center at the origin.

$$\int_0^1 \sqrt{1-x^2}\,dx=\frac{\pi(1)^2}{4}=\frac{\pi}{4}$$



$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2}=\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4}$$

Calculus 2 ∫ Evaluating a definite integral of a piecewise continuous function by interpreting it in terms of Areas under a curve example.

Definite Integrals 10 (Piecewise continuous functions)

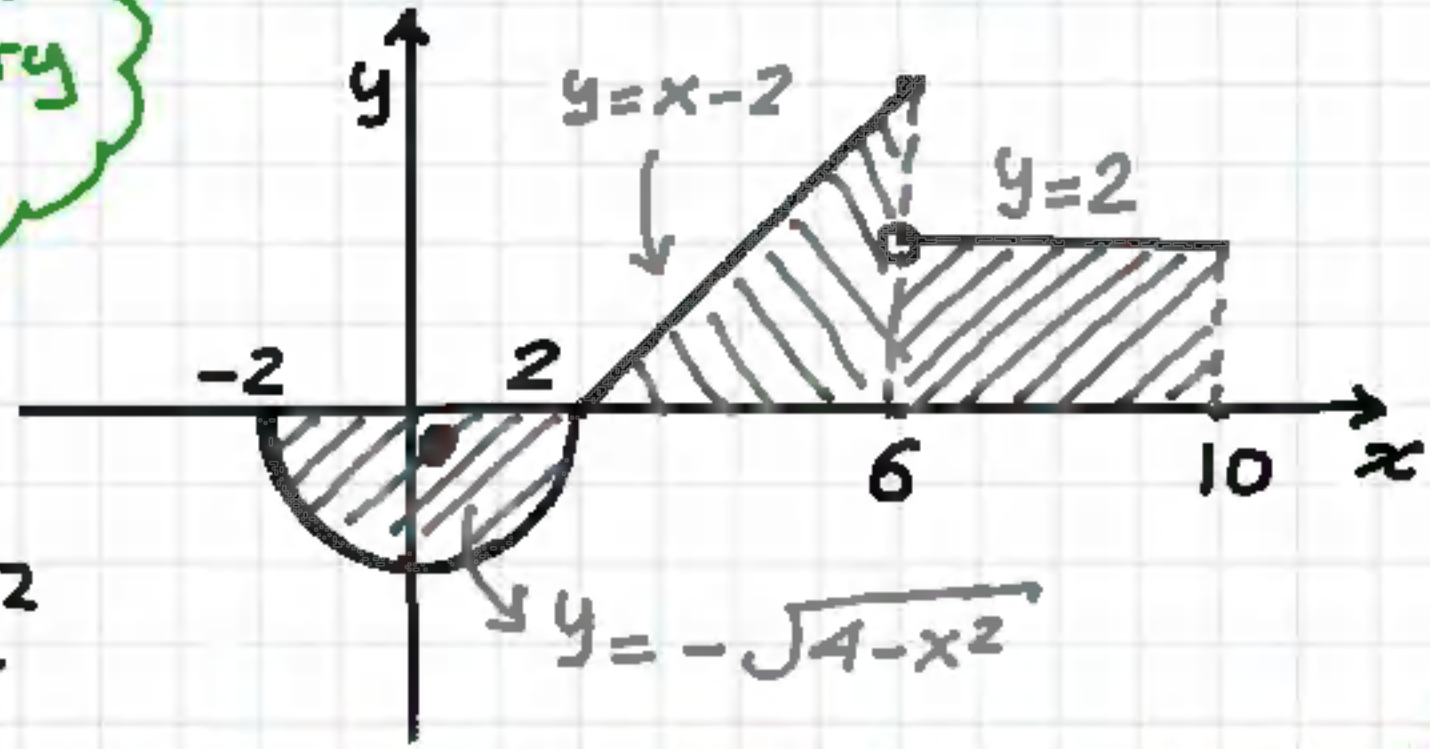
Evaluate $\int_{-2}^{10} f(x)\,dx$ by interpreting it in terms of Areas.

$$f(x) = \begin{cases} -\sqrt{4-x^2} & -2 \le x \le 2 \\ x-2 & 2 < x \le 6 \\ 2 & 6 < x \le 10 \end{cases}$$

Solution: Let's Decompose $\int_{-2}^{10} f(x)\,dx$ into the sum of three separate Integrals.

$$\int_{-2}^{10} f(x)\,dx = \int_{-2}^{2} -\sqrt{4-x^2}\,dx + \int_{2}^{6} x-2\,dx + \int_{6}^{10} 2\,dx$$

let's sketch f(x) and apply basic geometry to compute every integral.



$$\int_{-2}^{2} -\sqrt{4-x^2}\,dx = \frac{-\pi(2)^2}{2} = -2\pi$$

minus the area of a semi-circle

$$\int_{2}^{6} (x-2)\,dx = \frac{1}{2}(6-2)(4) = 8$$

Area of a triangle

$$\int_{6}^{10} 2\,dx = (10-6)(2) = 8$$

Area of a rectangle

Now Let's add up the integrals

$$\int_{-2}^{10} f(x)\,dx = \int_{-2}^{2} -\sqrt{4-x^2}\,dx + \int_{2}^{6} x-2\,dx + \int_{6}^{10} 2\,dx$$

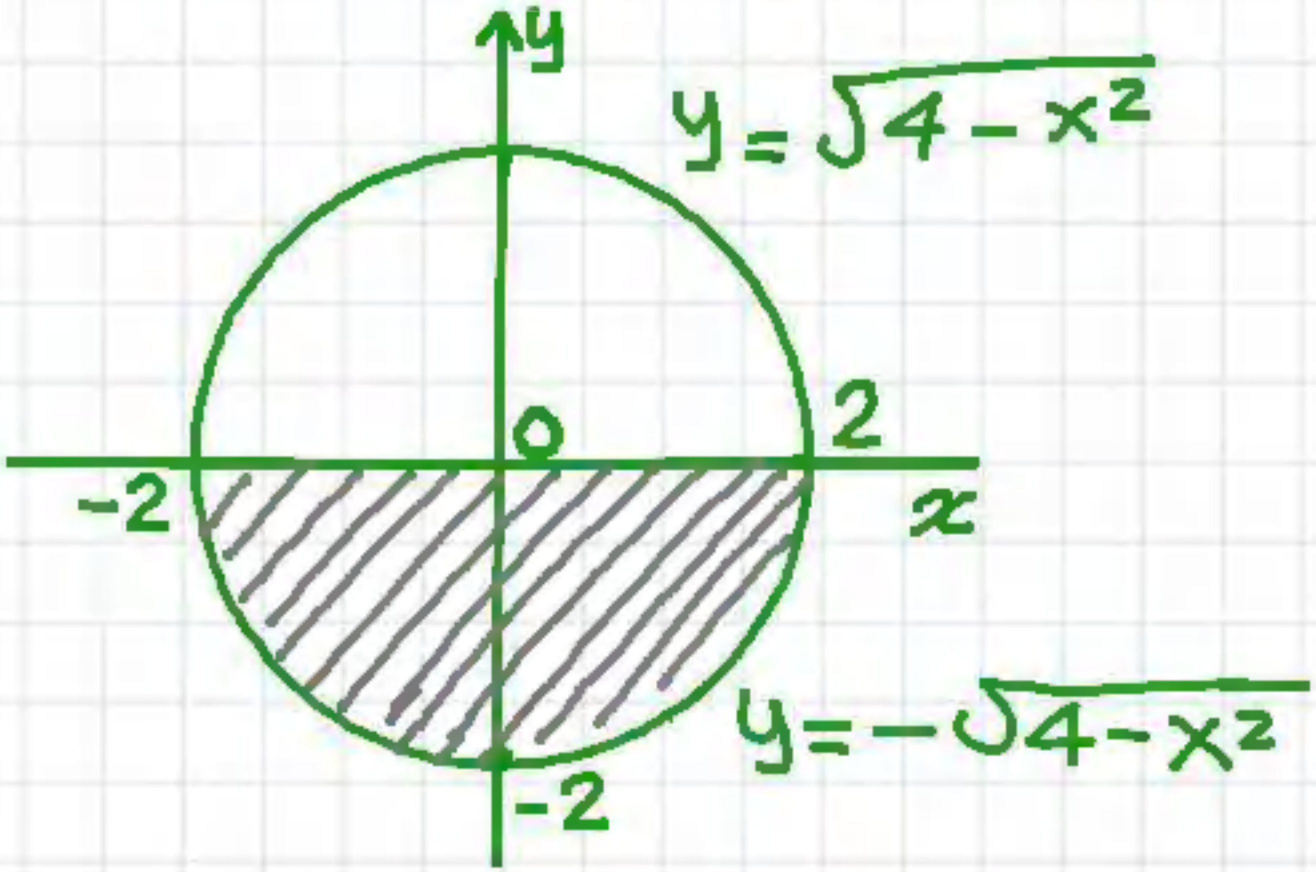
$$= -2\pi + 8 + 8$$

$$\int_{-2}^{10} f(x)\,dx = \boxed{16-2\pi}$$

Recall $y = -\sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2$

$x^2 + y^2 = 4$ circle of radius 2 with center at $(0,0)$. Therefore $y = -\sqrt{4-x^2}$ is the bottom half of circle of radius 2

$$y = -\sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4$$



$$\int_{-2}^{2} -\sqrt{4-x^2}\,dx = -\frac{\pi(2)^2}{2} = \boxed{-2\pi}$$

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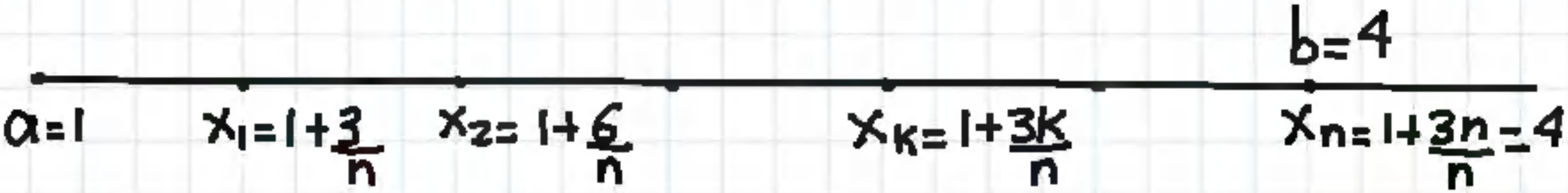
Definite Integrals II (Integral to Riemann Sum)

III Evaluate $\int_1^4 2 - x^2 \, dx$ by applying the definition of Right hand Riemann Sum?

Solution: Let's divide the interval $[a, b] = [1, 4]$ into n equal partitions $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$ which gives us the grid points:

$$x_k = a + k\Delta x = a + k\left(\frac{b-a}{n}\right) = 1 + \frac{3k}{n}$$

$k = 1, 2, 3, \ldots, n$



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \boxed{\sum_{k=1}^{n} \underbrace{f\left(a+k\left(\frac{b-a}{n}\right)\right)}_{f(x_k)} \underbrace{\left(\frac{b-a}{n}\right)}_{\Delta x}}$$

Rn (Right hand Riemann SUM)

$a=1 \quad b=4 \quad \Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$

$x_k = a + k\left(\frac{b-a}{n}\right) = 1 + k\left(\frac{4-1}{n}\right) = 1 + \frac{3k}{n}$

$f(x) = 2 - x^2 \Rightarrow f(x_k) = 2 - x_k^2 = 2 - \left(1 + \frac{3k}{n}\right)^2$

$$\int_1^4 (2 - x^2)\,dx = \lim_{n\to\infty} \boxed{\sum_{k=1}^{n} \left[2 - \left[1 + \frac{3k}{n}\right]^2\right] \cdot \frac{3}{n}}$$

Rn

let's simplify R_n and then evaluate $\int_1^4 (2-x^2)\,dx$ by taking $\lim\limits_{n\to\infty} R_n = \int_1^4 (2-x^2)\,dx$

$$R_n = \sum_{k=1}^{n} \left[2 - \left[1 + \frac{3k}{n}\right]^2 \right] \cdot \frac{3}{n}$$

We need some tools first!

$$\boxed{\sum_{k=1}^{n} 1 = n} \quad \boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}} \quad \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

$$\boxed{\sum_{k=1}^{n} c\,a_k = c \sum_{k=1}^{n} a_k} \quad \boxed{\sum_{k=1}^{n} a_k + b_k = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k}$$

$$R_n = \frac{3}{n} \sum_{k=1}^{n} \left[2 - \left(1 + \frac{6k}{n} + \frac{9k^2}{n^2}\right) \right] = \frac{3}{n} \sum_{k=1}^{n} \left(1 - \frac{6k}{n} - \frac{9k^2}{n^2}\right)$$

$$R_n = \frac{3}{n}\left[\sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \frac{6k}{n} - \sum_{k=1}^{n} \frac{9k^2}{n^2}\right]$$

$$R_n = \frac{3}{n}\left[\sum_{k=1}^{n} 1 - \frac{6}{n}\sum_{k=1}^{n} k - \frac{9}{n^2}\sum_{k=1}^{n} k^2\right]$$

$$R_n = \frac{3}{n}\left[n - \frac{6}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{2} - \frac{9}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$R_n = \frac{3}{n}\left[n - 3(n+1) - \frac{3}{2}\left[\frac{(n^2+n)(2n+1)}{n^2}\right]\right]$$

$$R_n = \frac{3\cancel{n}}{\cancel{n}} - \frac{9(n+1)}{n} - \frac{9}{2n^3}\left[2n^3 + 3n^2 + n\right]$$

$$R_n = 3 - 9\left(1 + \frac{1}{n}\right) - \frac{9}{2}\left[\frac{2\cancel{n^3}}{\cancel{n^3}} + \frac{3n^2}{n^3} + \frac{n}{n^3}\right]$$

$$R_n = 3 - 9\left(1 + \frac{1}{n}\right) - \frac{9}{2}\left[2 + \frac{3}{n} + \frac{1}{n^2}\right]$$

$$R_n = 3 - 9\left(1 + \frac{1}{n}\right) - \frac{9}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

Now take $\lim_{n \to \infty} R_n$ to evaluate $\int_1^4 (2 - x^2)\,dx$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} 3 - 9\Big(1 + \underbrace{\frac{1}{n}}_{0}\Big) - \frac{9}{2}\Big(2 + \underbrace{\frac{3}{n}}_{0} + \underbrace{\frac{1}{n^2}}_{0}\Big)$$

$$\lim_{n\to\infty} R_n = 3 - 9 - 9 = \boxed{-15}$$

Recall:

$\lim_{n\to\infty} \frac{1}{n} = 0$

and $\lim_{n\to\infty} \frac{1}{n^2} = 0$

$$\int_1^4 (2 - x^2)\,dx = \boxed{-15}$$

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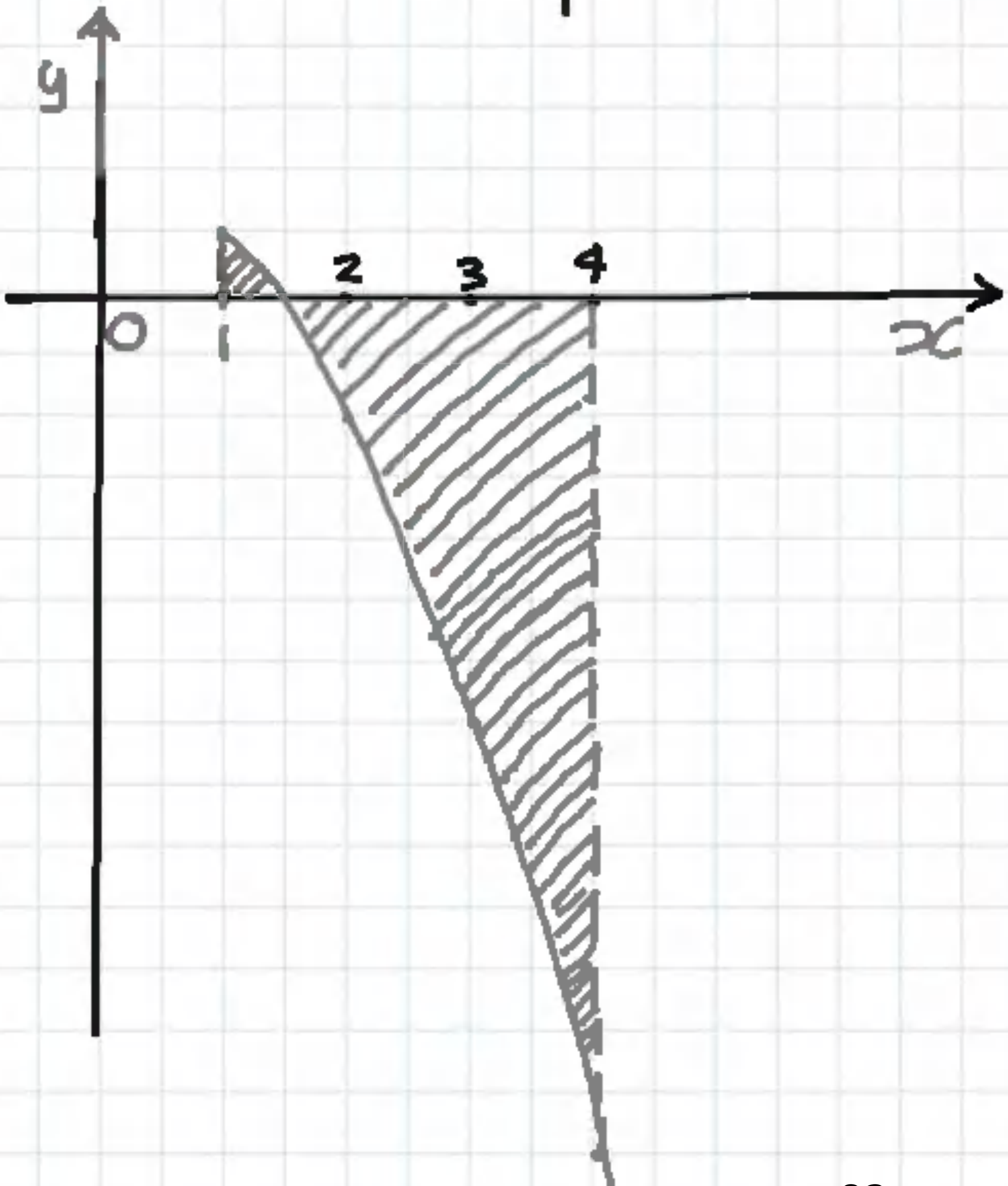
Let's graph $\int_1^4 2-x^2\,dx = -15 =$ net Area



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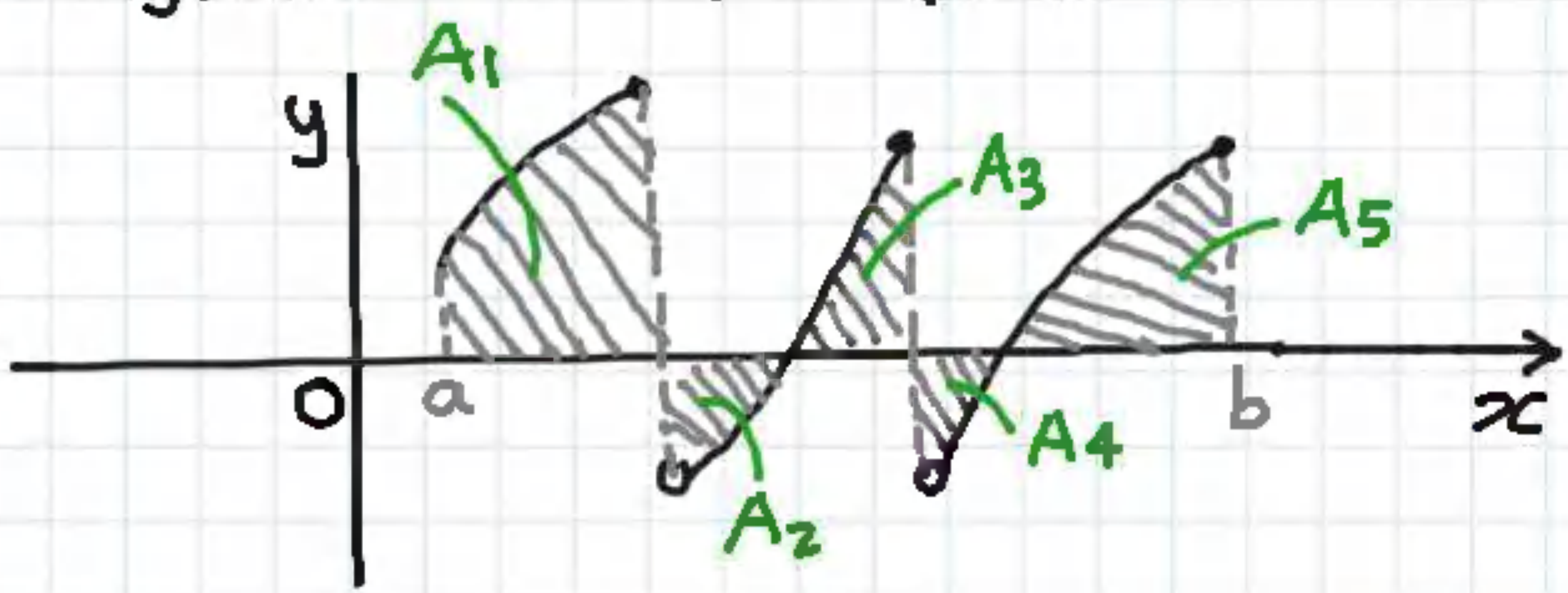
Definite Integrals 12 (Properties of Definite Integral)

Theorem: If $f(x)$ is continuous on $[a, b]$ or if $f(x)$ has only a finite number of jump discontinuities then $f(x)$ is integrable on $[a, b]$, therefore the definite integral $\int_a^b f(x)\,dx$ exists.

Example: Integrable function (Piecewise Continuous)

Figure 1



$$\int_a^b f(x)\,dx = \text{Net Area} = \text{Area above } x \text{ axis minus Area below the } x \text{ axis}$$

$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4 + A_5$$ See Figure 1

Theorem: If $f(x)$ is integrable on $[a,b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x = a + i\left(\frac{b-a}{n}\right)$

Note: Definite integral is a number so it does not matter which variable of integration you use.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Definite Integrals Properties

1] $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

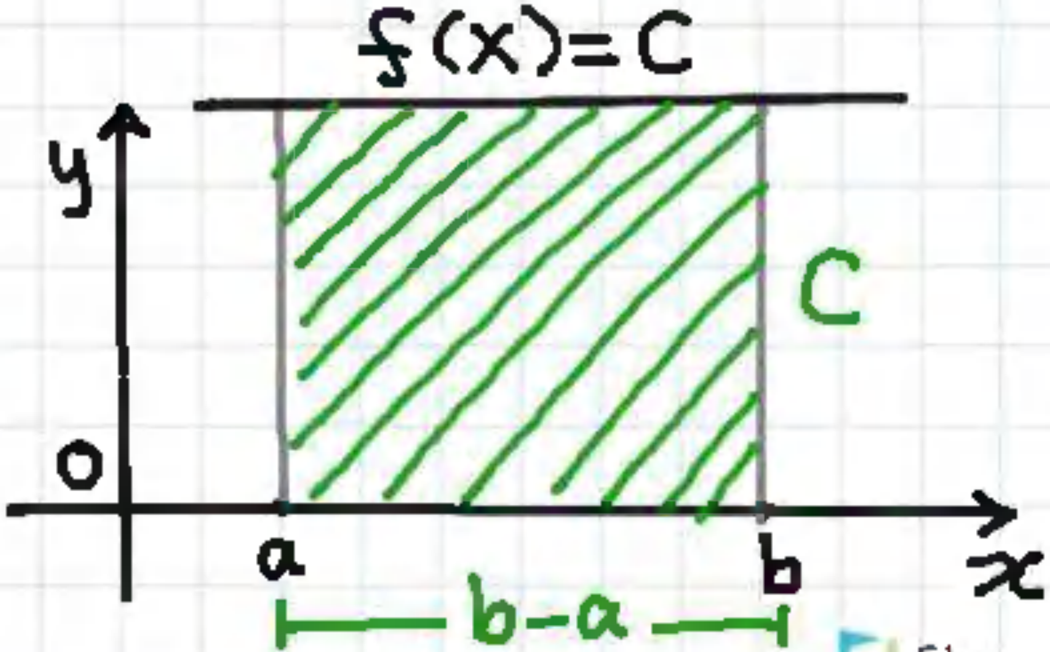
Example: $\int_2^1 [x^2+x]\,dx = -\int_1^2 [x^2+x]\,dx$

2] $\int_a^a f(x)\,dx = 0$

Example: $\int_1^1 \sin(x^2)\,dx = 0$

3] $\int_a^b C\,dx = C(b-a)$

Shaded area $= C(b-a)$



4] $\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

Example: $\int_1^2 [x^3 + x^2]\, dx = \int_1^2 x^3\, dx + \int_1^2 x^2\, dx$

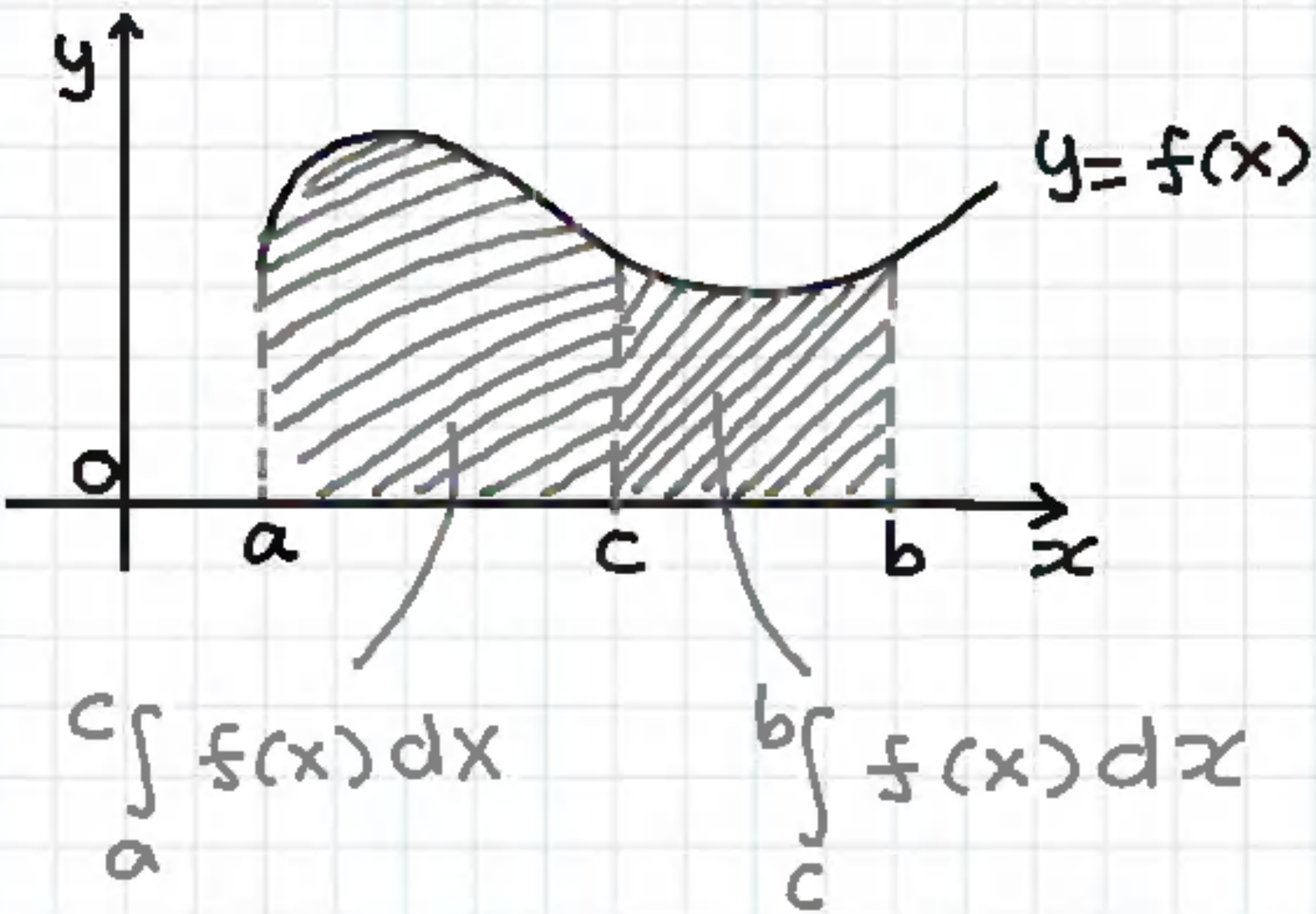
5] $\int_a^b C f(x)\, dx = C \int_a^b f(x)\, dx$ C is any constant

Example: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4x\cos x\, dx = 4 \int_{-\frac{\pi}{2}}^{\pi/2} x \cos x\, dx$

6] $\int_a^b [C_1 f(x) + C_2 g(x)]\, dx = C_1 \int_a^b f(x)\, dx + C_2 \int g(x)\, dx$

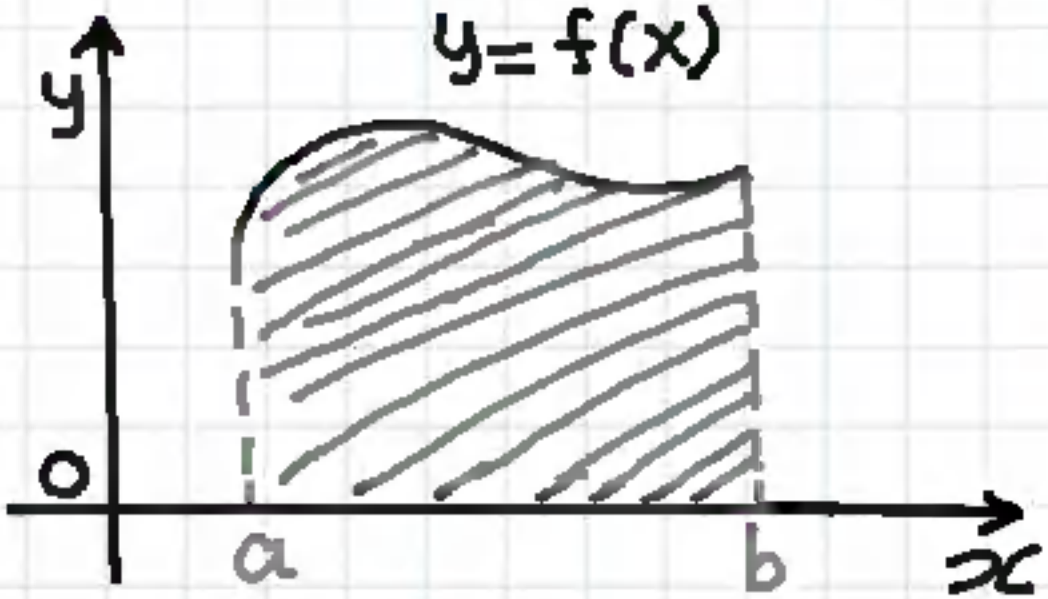
Ex: $\int_0^1 [2e^x + 3\sin x]\, dx = 2\int_0^1 e^x\, dx + 3\int_0^1 \sin x\, dx$

$\boxed{7}$ $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



Assuming $f(x) \geq 0$ for $a \leq x \leq b$ then Area under $f(x)$ from a to c plus the Area from c to b is equal to the entire area from a to b.

8] If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$



$dA = f(x)\,dx \geq 0$

$\int_a^b dA = A \geq 0$

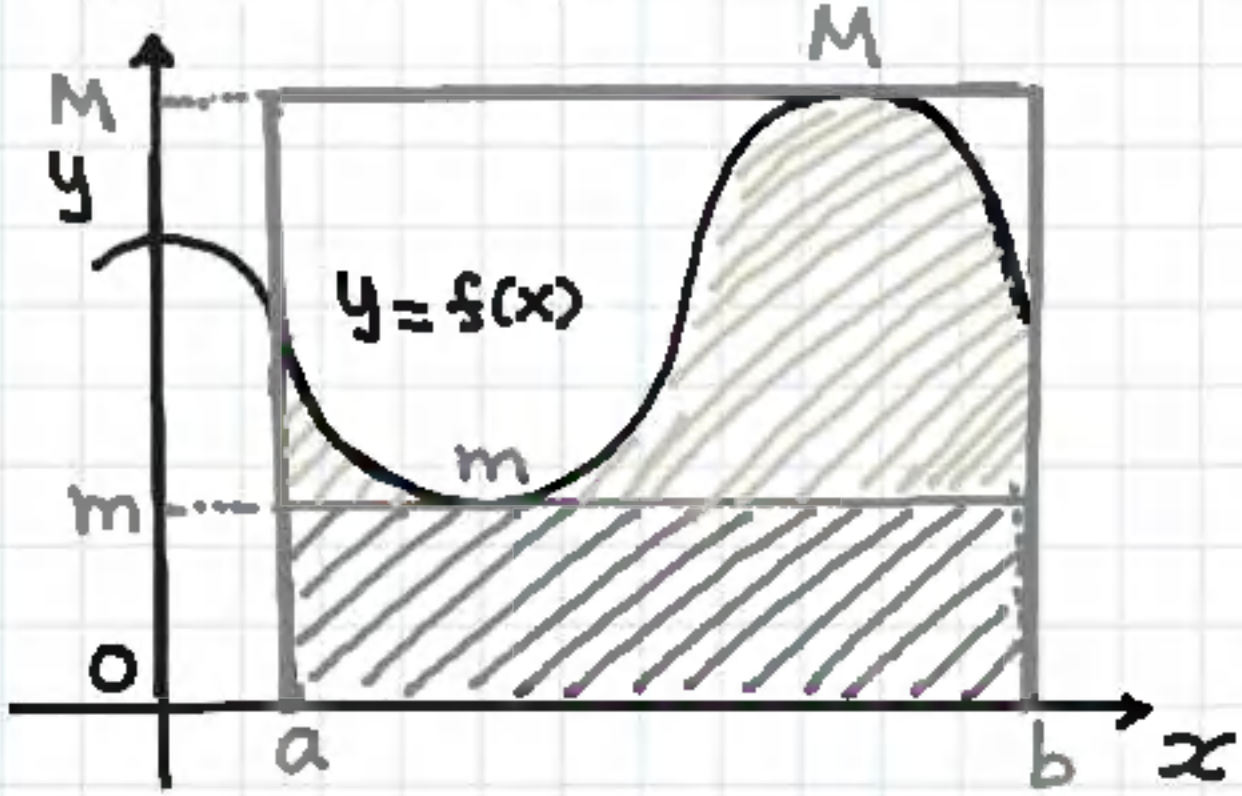
If $f(x) \geq 0$ then $\int_a^b f(x)\,dx$ = Area under $y = f(x)$ is also positive

9] If $f(x) \geq g(x)$ for $a \leq x \leq b$ then

$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

10 If $m \le f(x) \le M$ for $a \le x \le b$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$



Proof:

$$m \le f(x) \le M \qquad \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

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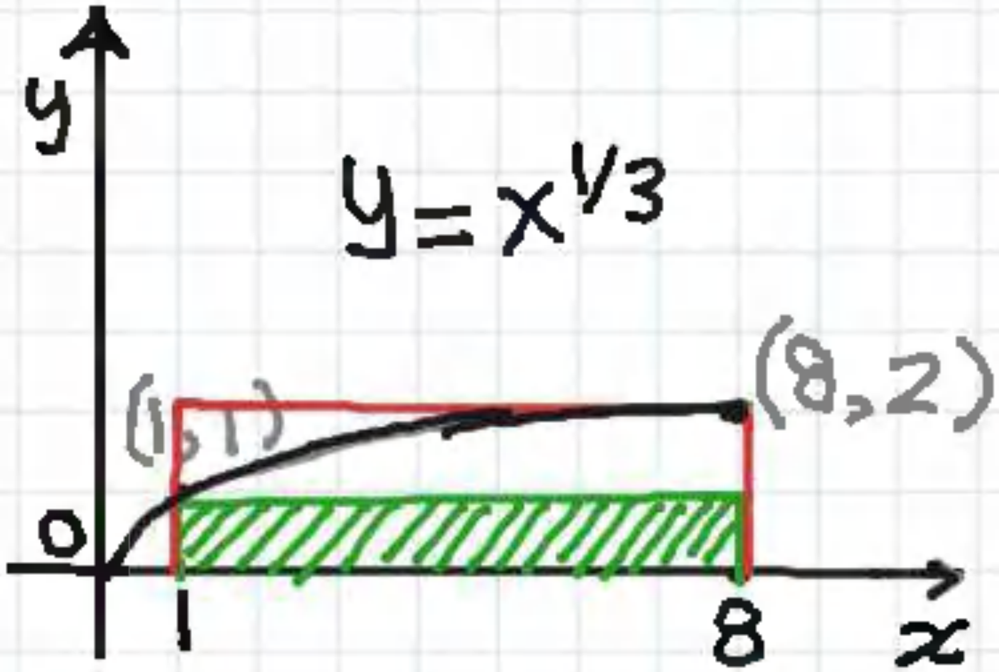
Definite Integrals 12 (Estimating a definite Integral)

Recall if $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Estimate $\int_1^8 \sqrt[3]{x}\,dx$ using above property

Solution: $f(x) = \sqrt[3]{x}$ is an increasing function on $[1, 8]$ therefore it will obtain its Absolute Maximum at $x=8$ and Absolute Minimum at $x=1$.



x	$y = \sqrt[3]{x}$	
1	1	Abs. Min.
8	2	Abs. Max.

$m = f(1) = \sqrt[3]{1} = 1$

$M = f(8) = \sqrt[3]{8} = 2$

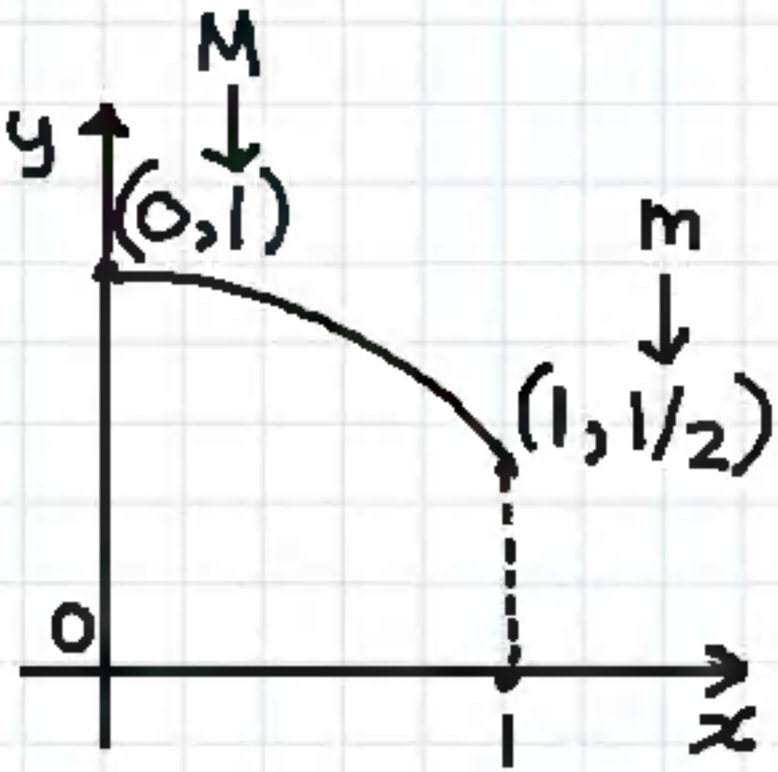
$1 \le x^{1/3} \le 2 \quad \text{for} \quad 1 \le x \le 8$

$$1(8-1) \le \int_1^8 x^{1/3}\,dx \le 2(8-1)$$

$$\boxed{7} \le \int_1^8 \sqrt[3]{x}\,dx \le \boxed{14}$$

Estimate $\int_0^1 2^{-x^2}\,dx$

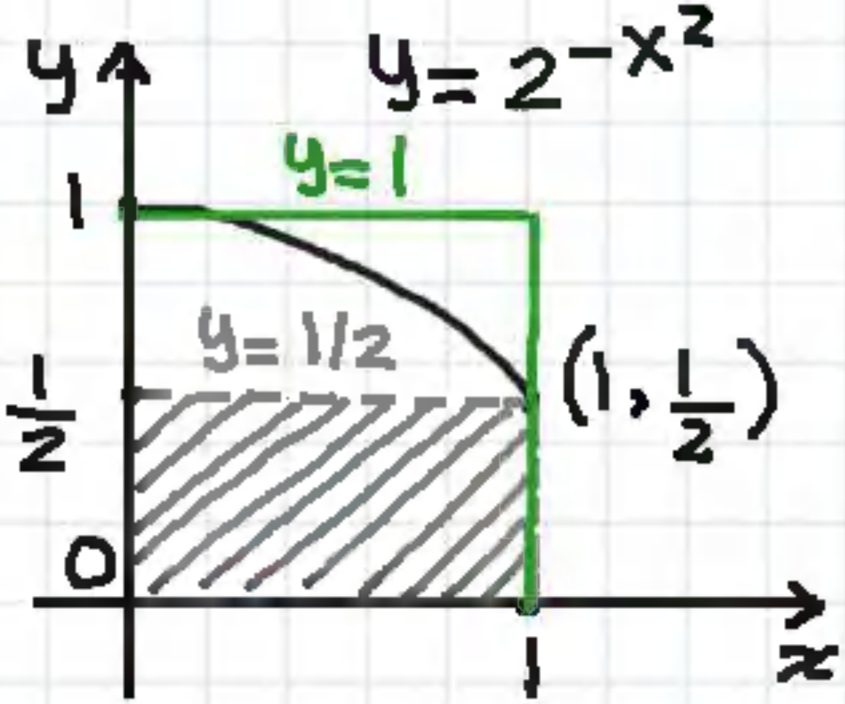
Solution: $f(x) = 2^{-x^2}$ is a decreasing function on $[0,1]$, therefore $f(x)$ obtains its Absolute Maximum at $x=0$ and its Absolute Minimum at $x=1$



x	$y=2^{-x^2}$	
0	$y=2^{-0}=1$	Abs. Max
1	$y=2^{-1}=1/2$	Abs. Min

$m = f(1) = 1/2$ Abs. Min.

$M = f(0) = 1$ Abs. Max.



Key Concept

$$m \le f(x) \le M \quad \text{for } \boxed{a \le x \le b}$$

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$1/2 \le 2^{-x^2} \le 1 \quad \text{for } 0 \le x \le 1$$

$$\frac{1}{2}(1-0) \le \int_0^1 2^{-x^2}\,dx \le 1(1-0)$$

$$\frac{1}{2} \le \int_0^1 2^{-x^2}\,dx \le 1$$

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