Separable equations I

Consider a First-order differential equation of the form $\frac{dy}{dx} = F(x,y)$, if we can factor $\frac{dy}{dx}$ as a as a product of a function of x and a function of y then we can write $\frac{dy}{dx} = f(x) \cdot g(y)$; then we call this a separable differential equation.

Given an initial condition $y(x_0) = y_0$ we obtain a separable differential equation with an initial condition.

$$\frac{dy}{dx} = f(x)\,g(y) \qquad y(x_0) = y_0$$

General Strategy:

Given $\dfrac{dy}{dx} = F(x, y)$ $\quad y(x_0) = y_0$

step 1] Factor $\dfrac{dy}{dx} = f(x)\,g(y)$

step 2] $\dfrac{dy}{g(y)} = f(x)\,dx$ separate $f(x)$ and $g(y)$

step 3] $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$ Integrate both sides

step 4] Solve for y as a function of x (explicit solution) or leave answer in implicit form.

step 5] Apply initial condition $y(x_0) = y_0$ to find arbitary constant C of integration.

Ex] Find the solution of the differential equation $\frac{dy}{dx} = x^3 y$ with the initial condition $y(0) = 2$

Solution: $\frac{dy}{y} = x^3\,dx$ separate the variables

$\int \frac{dy}{y} = \int x^3\,dx$ integrate both sides

$\ln|y| = \frac{x^4}{4} + C$

$e^{\ln|y|} = e^{x^4/4 + C} = e^{x^4/4} \cdot e^C = ke^{x^4/4}$

$|y| = ke^{x^4/4}$ where $k = e^C$

exponentiate both sides and solve for y

$$|y| = ke^{x^4/4} \Rightarrow y = \pm k e^{x^4/4} \Rightarrow y = Be^{x^4/4}$$

where $B = \pm k$ is determined from $y(0) = 2$

$$y(x) = Be^{x^4/4} \; ; \; y(0) = 2 \Rightarrow x = 0 \, , y = 2$$

$$2 = Be^0 \Rightarrow B = 2 \Rightarrow y(x) = 2e^{x^4/4}$$

Summary:

$$\frac{dy}{dx} = x^3 y \quad ; \; y(0) = 2$$

Solution: $y(x) = 2e^{x^4/4}$

Separable equations 2

Ex] Solve the differential equation:

$y' = \dfrac{x^3 \ln x}{y}$ with initial condition $y(1) = -1$

Solution:

$\dfrac{dy}{dx} = \dfrac{x^3 \ln x}{y}$ write y' as $\dfrac{dy}{dx}$

$y\,dy = x^3 \ln x \, dx$ separate variables

$\int y\,dy = \int x^3 \ln x \, dx$ integrate both sides

$$\int y\,dy = \int x^3 \ln x\,dx$$

Apply by Parts

$u = \ln x \quad du = \frac{1}{x}dx$

$dv = x^3 dx \quad v = x^4/4$

$\int u\,dv = uv - \int v\,du$

$$\frac{y^2}{2} = \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \ln x \cdot \frac{x^4}{4} - \frac{1}{4}\int x^3\,dx$$

$$\frac{y^2}{2} = \ln x \cdot \frac{x^4}{4} - \frac{1}{4}\frac{x^4}{4} + C$$

$$y^2 = \frac{\ln x \cdot x^4}{2} - \frac{1}{8}x^4 + 2C$$

$$y^2 = \frac{\ln x \cdot x^4}{2} - \frac{1}{8}x^4 + C_1$$

$C_1 = 2C$

$$y = \pm\sqrt{\frac{\ln x \cdot x^4}{2} - \frac{1}{8}x^4 + C_1}$$

$$y = \pm\sqrt{\frac{\ln x \cdot x^4}{2} - \frac{1}{8}x^4 + C_1}$$

$y(1) = -1$

$$y(x) = -\sqrt{\frac{\ln x \cdot x^4}{2} - \frac{1}{8}x^4 + C_1}$$

Choose $y(x) = -\sqrt{}$ to match up $y(1) = -1$

$$-1 = -\sqrt{\frac{\ln 1 \cdot 1}{2} - \frac{1}{8} + C_1}$$

Apply initial condition $y(1) = -1$

$$-1 = -\sqrt{0 - \frac{1}{8} + C_1}$$

Recall: $\ln 1 = 0$

$$1 = -\frac{1}{8} + C_1 \Rightarrow C_1 = \frac{9}{8}$$

Square both sides and solve for C_1

$$y(x) = -\sqrt{\frac{\ln x \cdot x^4}{2} - \frac{1}{8}x^4 + C_1}$$

$$C_1 = 9/8$$

$$y(x) = -\sqrt{\frac{\ln x \cdot x^4}{2} - \frac{1}{8}x^4 + \frac{9}{8}}$$

Solution of $\frac{dy}{dx} = \frac{x^3 \ln x}{y}$ $y(1) = -1$

Partial check: $y(1) = -\sqrt{\frac{\ln 1}{2} - \frac{1}{8} + \frac{9}{8}} = -\sqrt{0+1} = -1$ ✓

Doing a partial check using initial condition $y(1) = -1$ is a good strategy for discovering mistakes.

Separable equations 3

Ex] Solve the differential equation:

$\frac{dy}{dx} = 2(1+y^2)\cos^3 x$ that satisfies $y(\pi)=1$

Solution:

$$\frac{dy}{(1+y^2)} = 2\cos^3 x \, dx$$ separate the variables

$$\int \frac{dy}{(1+y^2)} = \int 2\cos^3 x \, dx$$ Integrate both sides

$$\int \frac{1}{(1+y^2)}\,dy = 2\int \cos^3 x\,dx$$

Apply U-Subst.

$U=\sin x \quad du=\cos x\,dx$

$$\tan^{-1}y = 2\int (1-\sin^2 x)\cos x\,dx$$

trig. identity

$\cos^2 x + \sin^2 x = 1$

$$\tan^{-1}y = 2\int (1-u^2)\,du$$

$$\tan^{-1}y = 2\left[u - \frac{u^3}{3}\right] + C$$

Substitute

$U = \sin x$

$$\tan^{-1}y = 2\left[\sin x - \frac{\sin^3 x}{3}\right] + C$$

$$y = \tan\left[2\sin x - \frac{2}{3}\sin^3 x + C\right]$$

$$y = \tan\left(2\sin x - \frac{2}{3}\sin^3 x + C\right)$$ Solve for C

Now apply initial condition $y(\pi)=1 \Rightarrow x=\pi, y=1$

$$1 = \tan\left(2\underbrace{\sin\pi}_{=0} - \frac{2}{3}(\underbrace{\sin\pi}_{=0})^3 + C\right)$$

$$1 = \tan(0 - 0 + C) = \tan C \Rightarrow \tan C = 1 \Rightarrow C = \pi/4$$

Solution: $y = \tan\left(2\sin x - \frac{2}{3}\sin^3 x + \frac{\pi}{4}\right)$

Question: $\frac{dy}{dx} = 2(1+y^2)\cos^3 x \qquad y(\pi)=1$

Separable equations 4

Ex] A tank contains 30 kg of dissolved salt in 1000 Litres of water. A brine solution with salt concentration of 10 g/L is pumped into the tank at the rate of 20L/min. The solution is well mixed and drains from the tank at the same rate of 20 L/min. a) How much salt is in the tank after t minutes? ; b) How long will it take for the concentration of salt in the tank to be reduced to 20 g/L?

a) solution

$x(t)$ = mass (Kg) of salt in the tank after t minutes.

$x(0) = 30$ kg

rate of salt flowing in $= 10\,g/L \times 20L/min = 200\,g/min$

rate of salt flowing in $= 0.2\,kg/min$

Volume of tank $= 1000\,L$ at all times t

rate of salt out $= \dfrac{x}{1000}\,\dfrac{Kg}{L} \cdot \dfrac{20L}{min} = 0.02x\,\dfrac{kg}{min}$

rate out = concentration of salt at time t multiplied by amount of solution leaving the tank.

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dx}{dt} = 0.2 - 0.02X \qquad x(0) = 30\,\text{kg}$$

$$\frac{dx}{0.2-0.02X} = \frac{dt}{1}$$ separate the variables

$$\int \frac{dx}{0.2-0.02X} = \int dt$$ Integrate both sides

Apply U-Subst.

$U = 0.2 - 0.02X \qquad du = -0.02\,dx$

$dx = du/-0.02$

$$\int \frac{1}{u}\frac{du}{-0.02} = t + C \Rightarrow -50\int \frac{1}{u}\,du = t + C$$

$$-50\,Ln|u| = t + C \Rightarrow Ln|u| = \frac{-1}{50}t + C/-50$$

$$e^{Ln|u|} = e^{\frac{-1}{50}t - \frac{1}{50}C} = e^{-\frac{1}{50}t} \cdot e^{-\frac{1}{50}C}$$

$$|u| = k\,e^{-0.02t} \quad \text{where } k = e^{-\frac{1}{50}C}$$

$$|0.2 - 0.02x| = k\,e^{-0.02t}$$

substitute

$u = 0.2 - 0.02x$

$$0.2 - 0.02x = \pm k\,e^{-0.02t}$$

$$0.2 - 0.02x = B\,e^{-0.02t}$$

$B = \pm k$

$$-0.02x = B\,e^{-0.02t} - 0.2$$

$$x = \frac{Be^{-0.02t} - 0.2}{-0.02} = -50Be^{-0.02t} + 10$$

$$x(t) = 10 - 50Be^{-0.02t}$$

Solve for B

Now apply initial condition $x(0) = 30\,kg$

$$30 = 10 - 50Be^{0} \Rightarrow 20 = -50B \Rightarrow B = -2/5$$

$$x(t) = 10 - 50\left(-\frac{2}{5}\right)e^{-0.02t}$$

$$x(t) = 10 + 20e^{-0.02t}$$

solution for part a)

b) How long will it take for the concentration of salt to be reduced to 20 g/L ?

b) Salt concentration = 20 g/L = 0.02 Kg/L

Since the volume of solution in the tank will always be 1000 Litres we can reformulate the question and find the time when mass of salt in tank will be 0.02 kg/L .1000 L = 20 kg

hence solve $x(t) = 20$ for t

$$x(t) = 10 + 20e^{-0.02t}$$

$$20 = 10 + 20e^{-0.02t} \Rightarrow 10 = 20e^{-0.02t}$$

$$\frac{1}{2} = e^{-0.02t} \Rightarrow \ln\left(\frac{1}{2}\right) = \ln e^{-0.02t}$$

$$\ln(1/2) = -0.02t \Rightarrow t = \ln(1/2)/_{-0.02} \quad \text{minutes}$$

$$t = \frac{\ln(1/2)}{-0.02} = \frac{\ln 1 - \ln 2}{-0.02} = \frac{-\ln 2}{-0.02} = 50 \ln 2 \text{ minutes}$$
$$\cong 34.66 \text{ min.}$$

Hence it will take approximately 34.66 minutes for the salt concentration in the tank to become 20 g/L or 0.02 kg/L.

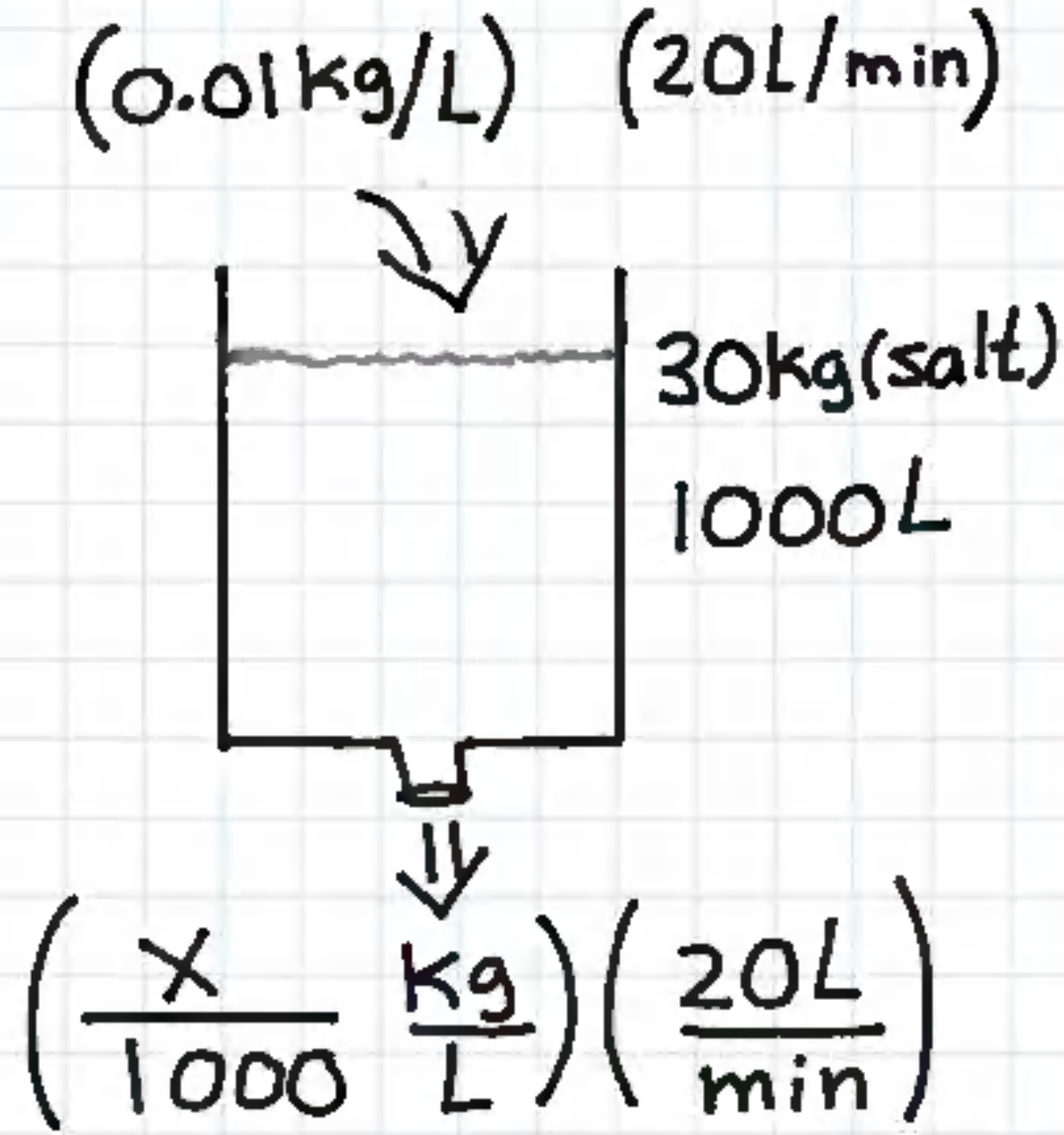
Summary Diagram

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dx}{dt} = (0.01)(20) - \frac{x}{1000}(20)$$

$$\frac{dx}{dt} = 0.2 - 0.02x$$

$$x(0) = 30\text{ kg}$$



Solution a) $x(t) = 10 + 20e^{-0.02t}$

Separable equations 5

Ex] According to Newton's Law the force of gravity on an object of mass m that has been projected vertically upward is inversely proportional to the square of distance from the center of the earth and is given by $F(y) = \frac{-K}{(y+R)^2}$, k is a constant

$y = y(t)$ is the object's distance above earth surface

g = acceleration due to gravity on earth.

R = Radius of earth.

a) Show that $m\frac{dv}{dt} = \frac{-mgR^2}{(y+R)^2}$

b] A rocket is fired vertically upward with an initial velocity V_0, Let h be the maximum height above earth surface, Find the initial velocity for rocket to reach a height of h.

Hint: By Chain Rule, $m\frac{dv}{dt} = mv\frac{dv}{dy}$

c] Find the escape velocity V_e such that rocket does not return to earth.

D] Use $R = 6371$ km, $g = 9.8\ m/s^2$ to calculate V_e in m/sec and km/hr

a) Show that $m\dfrac{dv}{dt} = \dfrac{-mgR^2}{(y+R)^2}$

Solution: $F(y) = \dfrac{-k}{(y+R)^2}$

at earth surface $y=0$ $\quad F(0) = -mg$

$-mg = \dfrac{-k}{R^2} \Rightarrow k = mgR^2$

$F(y) = \dfrac{-mgR^2}{(y+R)^2} \Rightarrow F = m \times a = m\dfrac{dv}{dt} = \dfrac{-mgR^2}{(y+R)^2}$

$m\dfrac{dv}{dt} = \dfrac{-mgR^2}{(y+R)^2}$

b) Find the initial velocity for rocket to reach a maximum height of h.

Solution: By Chain Rule $\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v\frac{dv}{dy}$

$$m\frac{dv}{dt} = mv\frac{dv}{dy} = \frac{-mgR^2}{(y+R)^2}$$

differential equation

$$v\,dv = \frac{-gR^2}{(y+R)^2}\,dy$$

separate the variables
cancel out mass m

$$\int v\,dv = \int \frac{-gR^2}{(y+R)^2}\,dy$$

Integrate both sides

$$\int v\,dv = \int \frac{-gR^2}{(y+R)^2}\,dy$$

Apply U-Subst.

$U = y+R \quad du = dy$

$$\frac{V^2}{2} = \frac{gR^2}{y+R} + C$$

$$V^2 = \frac{2gR^2}{y+R} + 2C$$

Take square root of both sides

$$V = \pm\sqrt{\frac{2gR^2}{y+R} + C_1}$$

$C_1 = 2C$

$$V = \sqrt{\frac{2gR^2}{y+R} + C_1}$$

Choose plus sign since rocket is moving up.

$$V = \sqrt{\frac{2gR^2}{y+R} + C_1}$$

Velocity as a function of y

since $y=0$ when $t=0$, the initial condition at $t=0$

$$V(t=0) = V(y=0) = V_0$$

initial velocity

$$V_0 = \sqrt{\frac{2gR^2}{R} + C_1} \Rightarrow V_0^2 = 2gR + C_1$$

$$C_1 = V_0^2 - 2gR$$

$$V(y) = \sqrt{\frac{2gR^2}{y+R} + V_0^2 - 2gR}$$

b] cont To reach a maximum height of h set $V=0$ and $y=h$ and solve for initial velocity V_0

$$V(y)=\sqrt{\frac{2gR^2}{y+R}+V_0^2-2gR}$$

key concept
maximum height
reached when $V=0$

$$0=\sqrt{\frac{2gR^2}{h+R}+V_0^2-2gR}$$

Set $V=0$ and $y=h$
and solve for V_0

$$0=\frac{2gR^2}{h+R}+V_0^2-2gR$$

$$V_0^2=2gR-\frac{2gR^2}{h+R} \Rightarrow V_0=\sqrt{2gR-\frac{2gR^2}{h+R}}$$

c] Find escape velocity Ve

$$V_0 = \sqrt{2gR - \frac{2gR^2}{h+R}}$$

initial velocity to reach maximum height h

Take limit of V_0 as height $h \to \infty$

$$\lim_{h\to\infty} V_0 = \sqrt{2gR - \frac{2gR^2}{h+R}} = \sqrt{2gR}$$

$\Rightarrow 0$ as $h \to \infty$

$$V_{escape} = \sqrt{2gR}$$

D] Find Vescape for rocket leaving earth.

$R = 6371 \text{ Km}$; $g = 9.8 \text{ m/s}^2$

$$V_e = \sqrt{2gR}$$

$$V_e = \sqrt{2(9.8)(6371 \times 10^3)} = 11174.6 \text{ m/sec}$$

$$V_e = 11174.6 \frac{\cancel{m}}{\cancel{sec}} \cdot \frac{1 \text{Km}}{1000 \cancel{m}} \cdot \frac{3600 \cancel{sec}}{1 \text{ hr}}$$

$$V_e \cong 40228.5 \text{ Km/hour}$$

Summary Diagram

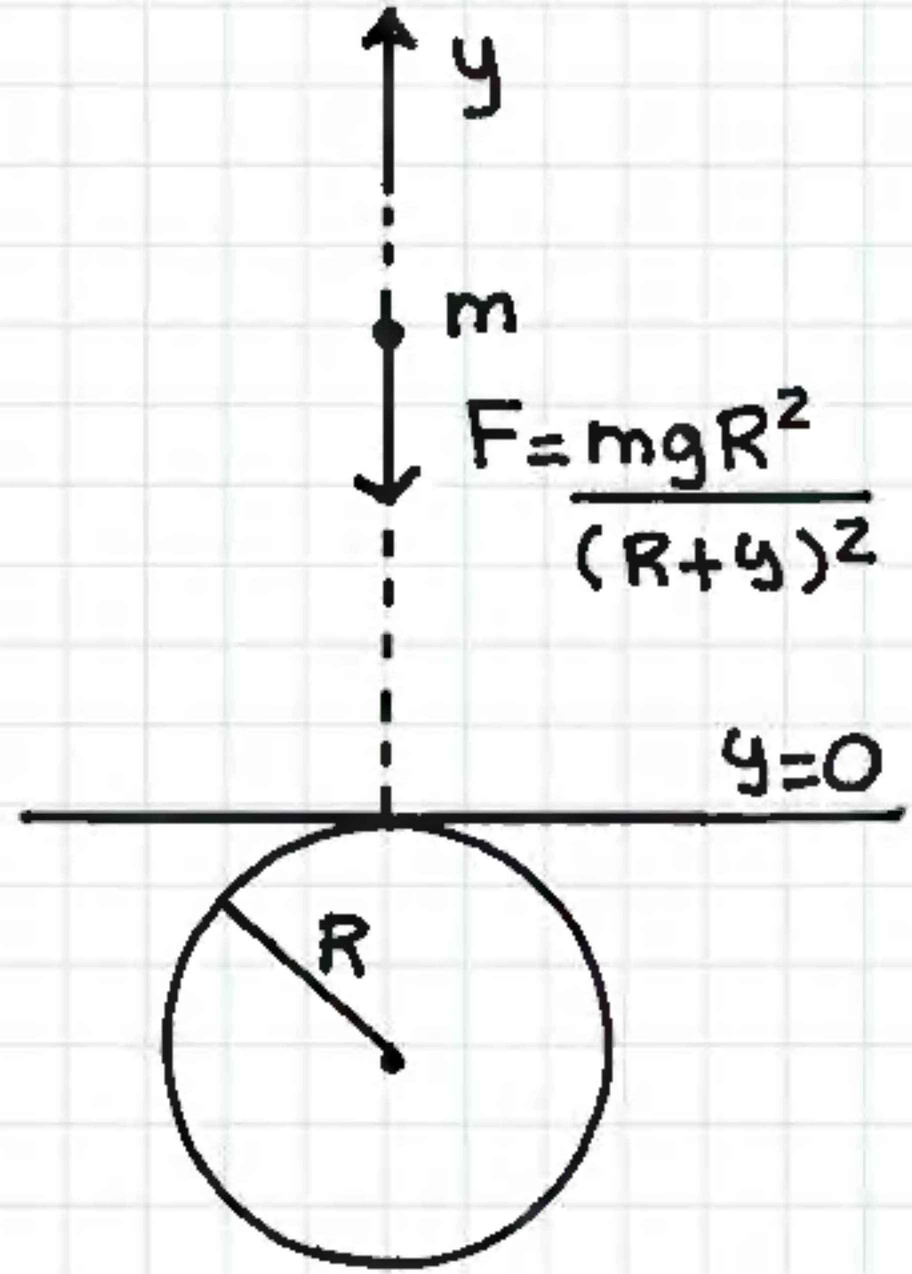
$$m\frac{dv}{dt} = \frac{-mgR^2}{(y+R)^2}$$

$$mv\frac{dv}{dy} = \frac{-mgR^2}{(y+R)^2}$$

$$V(y) = \sqrt{\frac{2gR^2}{y+R} + V_0^2 - 2gR}$$

set $V=0$ $\quad y=h$

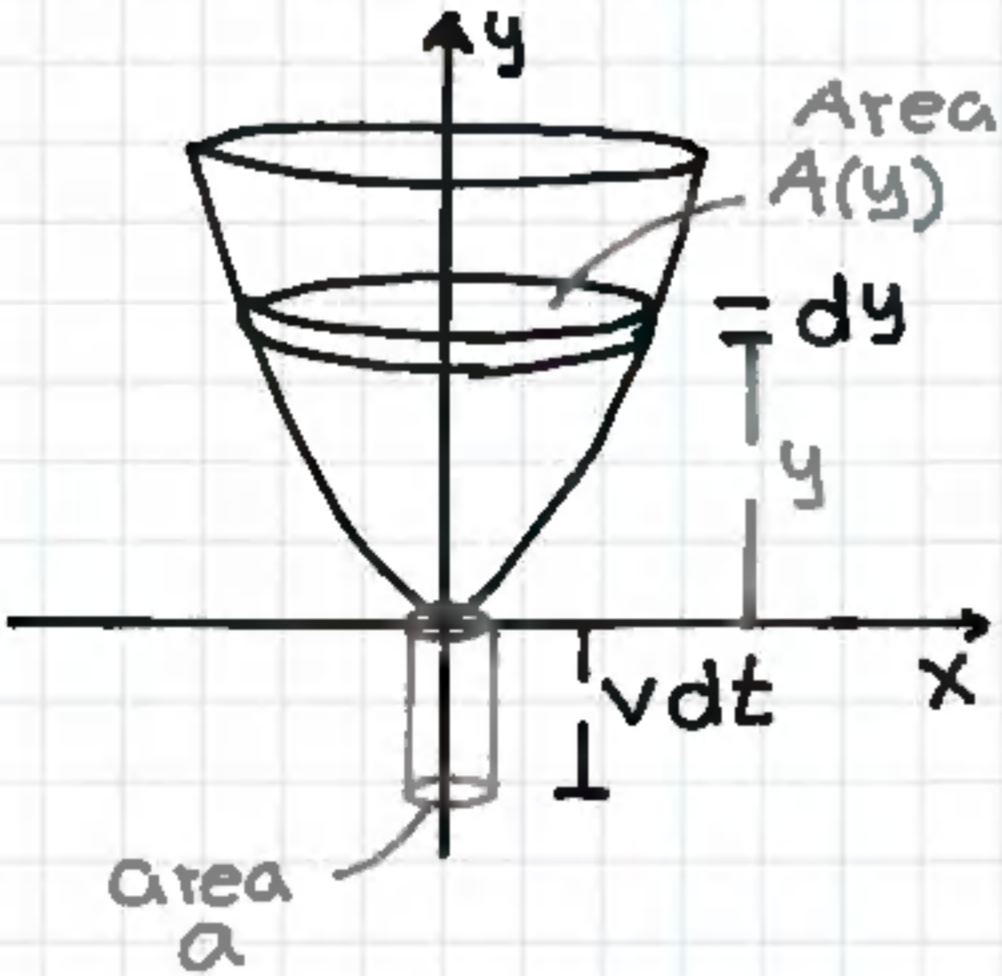
$$V_0 = \sqrt{2gR - \frac{2gR^2}{h+R}}$$



$$V_{esc} = \lim_{h\to\infty} V_0 = \sqrt{2gR}$$

Separable equations 6

Torricelli's Law:



6] $A(y)$ is area of slice of liquid at height y.

7] $dV = A(y)\,dy$

Model Assumptions

1] Water tank with hole at bottom with area a

2] $y(t)$ is depth of water in tank in metres at time t seconds.

3] $V(t)$ volume of water in tank in m^3

4] Assume every layer of fluid free falls from a height of y metres and exits thru hole at bottom

5] $g = 9.8\ m/s^2$

Potential energy at height $y = mgy$

kinetic energy of fluid as it exits hole $= \frac{mv^2}{2}$

$PE = KE \Rightarrow \cancel{m}gy = \frac{\cancel{m}v^2}{2} \Rightarrow v = \sqrt{2gy}$

velocity of fluid exiting

Volume of water that leaves hole at bottom: $avdt$

$$dV = \underset{\text{area}}{a}\, \underbrace{v\,dt}_{\text{height of cylinder of liquid exiting tank}}$$

$dV = -avdt$ change in Volume of water in tank

but velocity of liquid $v = \sqrt{2gy}$

$dV = -a\sqrt{2gy}\,dt$

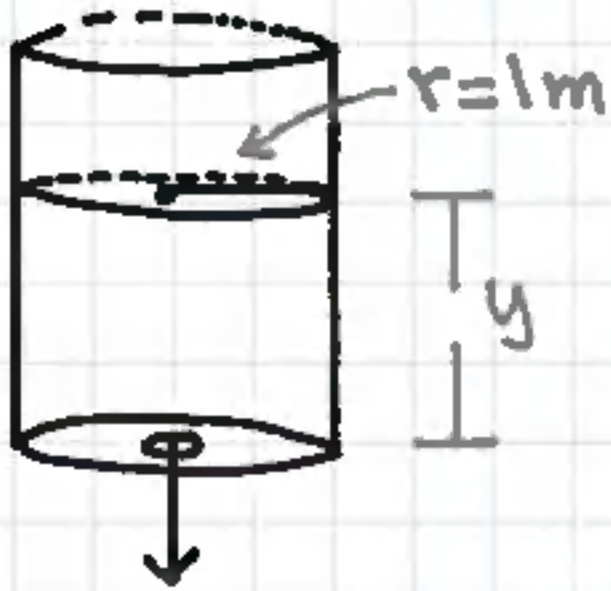
$$dV = A(y)\,dy = -av\,dt = -a\sqrt{2gy}\,dt$$

$$A(y)\,dy = -a\sqrt{2gy}\,dt \Rightarrow A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$ Torricelli's Law

$$A(y)\frac{dy}{dt} = -k\sqrt{y}$$ If area of bottom hole is not known.

Ex] Water is draining from a cylindrical tank that is filled to a height of 4 metres at $t=0$. The tank has a base radius of 1 metre. After 1 hour the level of water is 2 metres. When will the tank be empty.



$$A(y)\frac{dy}{dt} = -k\sqrt{y}$$

radius = 1m

$y(0) = 4$ m

$y(1) = 2$ m

$$A(y) = \pi(1)^2$$

$$\pi\frac{dy}{dt} = -k\sqrt{y}$$

$$\frac{dy}{\sqrt{y}} = -\frac{k}{\pi}\,dt \Rightarrow \int y^{-1/2}\,dy = \int -\frac{k}{\pi}\,dt$$

Integrate both sides

$$y^{1/2}\cdot 2 = -\frac{k}{\pi}t + C \Rightarrow y^{1/2} = \frac{-k}{2\pi}t + \frac{C}{2}$$

$$\sqrt{y} = \frac{-k}{2\pi}t + C_1 \quad \text{where } C_1 = \frac{C}{2}$$

$$\sqrt{y} = \frac{-k}{2\pi} t + C_1 \qquad y(0) = 4\,m \quad \Rightarrow t = 0 \quad y = 4$$

$$\sqrt{4} = \frac{-k}{2\pi} . 0 + C_1 \Rightarrow C_1 = 2 \qquad \text{Apply } \; y(0) = 4\,m$$

$$\sqrt{y} = \frac{-k}{2\pi} t + 2 \quad \Rightarrow \quad y(t) = \left(\frac{-k}{2\pi} t + 2\right)^2$$

$$y(t) = \left(\frac{-k}{2\pi} t + 2\right)^2 \qquad \text{Apply } \; y(1) = 2\,m \Rightarrow t = 1, y = 2$$

$$2 = \left(-\frac{k}{2\pi} + 2\right)^2 \quad \Rightarrow \sqrt{2} = -\frac{k}{2\pi} + 2$$

$$\sqrt{2} - 2 = \frac{-k}{2\pi} \Rightarrow k = (2 - \sqrt{2})\, 2\pi$$

$$y(t) = \left(\frac{-K}{2\pi} t + 2 \right)^2 \qquad K = (2-\sqrt{2})2\pi$$

$$y(t) = \left(-(2-\sqrt{2})t + 2 \right)^2$$

Now to find time t such that tank is empty set $y(t) = 0$ and solve for t.

$$y(t) = 0 \Rightarrow 0 = \left(-(2-\sqrt{2})t + 2 \right)^2$$

$$0 = \sqrt{(2-(2-\sqrt{2})t)^2} \Rightarrow 2-(2-\sqrt{2})t = 0$$

$$t = \frac{2}{(2-\sqrt{2})} \cong 3.41 \text{ hours}$$

It will take approx. <u>3.41</u> hours for the tank to empty.

Separable equations 7

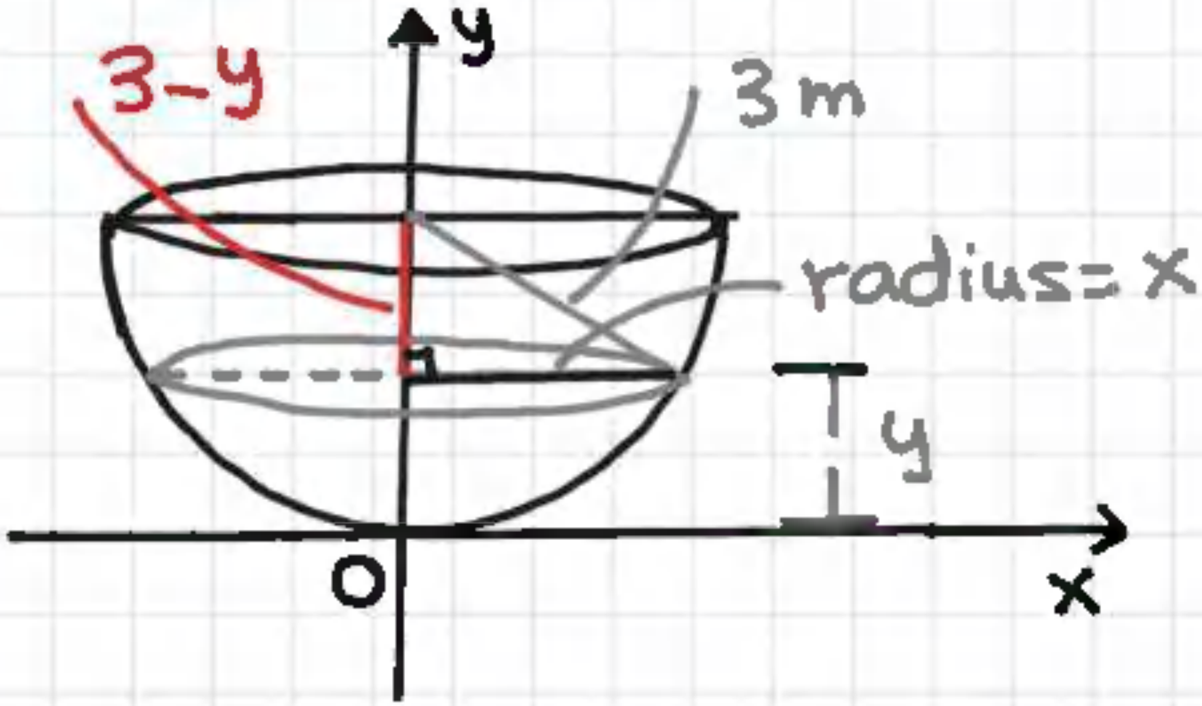
Ex] A hemispherical tank with radius 3 metres is filled with water at time $t=0$. At $t=0$, a circular hole with radius 2cm at bottom of the tank is opened and the water exits thru the hole at the bottom. How long will it take for all the water to drain out?

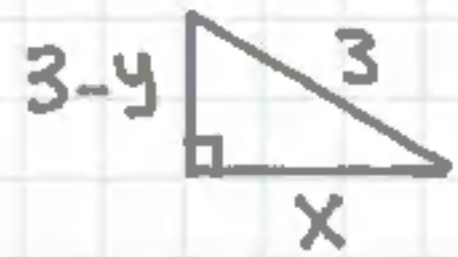
Hint: Apply Torricelli's Law

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

a → Area of hole at bottom

$g = 9.8 \ m/s^2$





$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$ Torricelli law

$$A(y) = \pi x^2$$ cross sections ⊥ to y axis are circles with radius x

$$x^2 + (3-y)^2 = 9$$ apply pythagoras to relate X and Y

$$a = \pi(0.02)^2 = 0.0004\pi$$ area of hole at bottom.

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

Solve for x^2 in terms of y

$$x^2 + (3-y)^2 = 9$$

$$A(y) = \pi x^2 = \pi(9-(3-y)^2)$$

$$a = 0.0004\pi \text{ m}^2$$

Area of hole at bottom

$$\pi(9-(3-y)^2)\frac{dy}{dt} = -0.0004\pi\sqrt{2(9.8)y}$$

$$\frac{(9-(3-y)^2)}{\sqrt{y}}\,dy = -0.0004\sqrt{2(9.8)}\,dt$$

$$\frac{(9-(9-6y+y^2))}{y^{1/2}}\,dy = -0.0004\sqrt{19.6}\,dt$$

$$\frac{6y-y^2}{y^{1/2}}\,dy = -0.0004\sqrt{19.6}\,dt$$

$$\left(6y^{1/2}-y^{3/2}\right)dy = -0.0004\sqrt{19.6}\,dt$$

$$\int \left(6y^{1/2}-y^{3/2}\right)dy = \int -0.0004\sqrt{19.6}\,dt$$

$$6y^{3/2}\cdot\frac{2}{3} - y^{5/2}\cdot\frac{2}{5} = -0.0004\sqrt{19.6}\,t + C$$

$$4y^{3/2} - \frac{2}{5}y^{5/2} = -0.0004\sqrt{19.6}\,t + C$$

Now Apply $y(0)=3\text{ m}$ to solve for C

$$4y^{3/2} - \frac{2}{5} y^{5/2} = -0.0004\sqrt{19.6}\,t + C$$

$y(0) = 3$

$t = 0\ ,\ y = 3$

$$4(3)^{3/2} - \frac{2}{5}(3)^{5/2} = 0 + C$$

$$4y^{3/2} - \frac{2}{5} y^{5/2} = -0.0004\sqrt{19.6}\,t + 4(3)^{3/2} - \frac{2}{5}(3)^{5/2}$$

Now to find time t such that tank is empty set $y(t) = 0$ and solve for t.

$$0 = -0.0004\sqrt{19.6}\,t + 4(3)^{3/2} - \frac{2}{5}(3)^{5/2}$$

$$0.0004\sqrt{19.6}\,t = 4(3)^{3/2} - \frac{2}{5}(3)^{5/2}$$

$$0.0004\sqrt{19.6}\ t = 4(3)^{3/2} - \frac{2}{5}(3)^{5/2}$$

$$t = \frac{4(3)^{3/2} - \frac{2}{5}(3)^{5/2}}{0.0004\sqrt{19.6}} \cong 8215.84 \text{ seconds}$$

$$t \cong 8215.84 \text{ seconds}$$

$$t \cong 2.28 \text{ hours} \qquad 1 \text{ hour} = 3600 \text{ sec}$$

It will take approx. 2.28 hours for all the water to drain out of the hemispherical tank.

Ex] According to the Logistic model of population growth $\frac{dP}{dt} = KP(1 - \frac{P}{M})$ where P is population at time t, k is a constant and M is the carrying capacity of the population.

Find an explicit solution to the logistic equation by applying separation of variables.

(Hint: Apply Partial Fraction Decomposition)

$$\frac{dP}{dt} = KP(1 - \frac{P}{M}) \qquad P(0) = P_0$$

differential equation

initial population

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$ differential equation

$$\frac{dP}{P(1 - P/M)} = k\,dt$$ separate the variables

$$\frac{M}{P(M-P)}\,dP = k\,dt$$ multiply by $\frac{M}{M}$ to simplify

$$\int \frac{M}{P(M-P)}\,dP = \int k\,dt$$ integrate both sides

$$\frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$ apply Partial Fractions

$$\frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

Apply Partial Fractions

$$P(M-P)\frac{M}{P(M-P)} = P(M-P)\frac{A}{P} + P(M-P)\frac{B}{M-P}$$

$$M = A(M-P) + BP$$

Clear out the fractions

subst. $P=0 \Rightarrow M = AM + 0 \Rightarrow A = 1$

subst. $P=M \Rightarrow M = A(M-M) + BM \Rightarrow B = 1$

$$\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$$

Partial Fraction Decomposition

$$\int \frac{M}{P(M-P)}\, dP = \int k\, dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int k\, dt$$

U-Substitution

$U = M-P \quad du = -dP$

$$\ln|P| - \ln|M-P| = kt + C$$

$$\ln|M-P| - \ln|P| = -kt - C$$

multiply by -1

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$$e^{\ln|(M-P)/P|} = e^{(-kt-C)} = e^{-kt} \cdot e^{-C}$$

$$\left|\frac{M-P}{P}\right| = e^{-kt}\, e^{-C}$$

$$\frac{M-P}{P} = Ae^{-kt} \quad \text{where } A = \pm e^{-C}$$

$$M-P = APe^{-kt}$$

$$APe^{-kt} + P = M$$

$$P(Ae^{-kt}+1) = M$$

$$P(t) = \frac{M}{1+Ae^{-kt}}$$

$$P(0) = P_0 \Rightarrow P_0 = \frac{M}{1+Ae^{-0}}$$

cross-multiply to solve for $P(t)$

Apply initial condition

$P(0) = P_0$

$t=0 \Rightarrow P = P_0$

$$P(t) = \frac{M}{(1+Ae^{-kt})}$$

$$P_0 = \frac{M}{1+A} \Rightarrow P_0 + P_0 A = M \Rightarrow P_0 A = M - P_0 \Rightarrow A = \frac{M-P_0}{P_0}$$

Solution to $\frac{dP}{dt} = KP\left(1 - \frac{P}{M}\right)$ $\quad P(0) = P_0$

$$P(t) = \frac{M}{(1+Ae^{-kt})} \quad \text{where } A = \frac{M-P_0}{P_0}$$

Separable equations 9 (Logistic growth cont.)

EX| Assume that the population of salmon in a fish farm follows the logistic growth model:

$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$. At $t=0$ the fish farm is populated with 100 fish. When the fish population is 500 the rate of growth of the population is 200 fish/year. Assume that the total carrying capacity of the fish farm is 1000 fish. After how many years does the fish population reach 800.

$$\frac{dP}{dt} = KP(1 - P/M)$$

$P(0) = 100$
initial population

$M = 1000$
carrying capacity

Given: $\frac{dP}{dt} = 200$ fish/year when $P = 500$ fish

Lets apply Logistic differential equation to solve for K.

$$\frac{dP}{dt} = KP(1 - P/M) \Rightarrow 200 = K(500)(1 - 500/1000)$$

$$K = 200/250 = 0.8$$

$$\frac{dP}{dt} = 0.8P(1 - P/1000) \qquad P(0) = 100$$

$$\frac{dP}{dt} = kP(1 - P/M) \qquad P(0) = P_0$$

We found the explicit solution to the above differential equation in the previous lesson.

$$P(t) = \frac{M}{(1 + Ae^{-kt})} \qquad \text{where } A = (M - P_0)/P_0$$

$M = 1000$; $P_0 = 100$; $K = 0.8$ $\quad A = (1000 - 100)/100$

$A = 9$

$$P(t) = \frac{1000}{(1 + 9e^{-0.8t})}$$

Find the time in years such that $P(t) = 800$

$$P(t) = \frac{1000}{(1+9e^{-0.8t})}$$

$P(t) = 800$ solve for t

$$800 = \frac{1000}{(1+9e^{-0.8t})} \Rightarrow \frac{1000}{800} = 1+9e^{-0.8t}$$

$$\frac{5}{4} = 1+9e^{-0.8t} \Rightarrow \frac{1}{4} = 9e^{-0.8t} \Rightarrow \frac{1}{36} = e^{-0.8t}$$

$$Ln\left(\frac{1}{36}\right) = Ln\, e^{-0.8t} \Rightarrow Ln(1/36) = -0.8t$$

$$t = Ln(1/36)/_{-0.8} = \frac{-Ln36}{-0.8} = \frac{Ln36}{0.8} \cong 4.48 \text{ years}$$

It will take approximately 4.48 years for the fish population to reach 800.

Basic Skills review

$\ln 1 = 0$; $\ln e = 1$, $e^{\ln x} = x$; $\ln e^{x} = x$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$; $\ln(xy) = \ln x + \ln y$

$\ln x^{r} = r\ln x$; $e^{x+y} = e^{x} \cdot e^{y}$; $e^{0} = 1$

Separable equations 10

Ex| Find a family of curves that intersect every cubic function of the form $y=Cx^3$ at right angles.
(This is also known as an orthogonal trajectory)

$$y=Cx^3 \Rightarrow \frac{dy}{dx}=3Cx^2$$

Now, Let's get rid of constant C.

$$C=\frac{y}{x^3} \Rightarrow \frac{dy}{dx}=3\left(\frac{y}{x^3}\right)x^2=\frac{3y}{x}$$

$\dfrac{dy}{dx}=\dfrac{3y}{x}$ differential equation for $y=Cx^3$

Now any curve that meets the cubic function $y = cx^3$ at right angles must have slope of tangent line to be equal to the negative reciprocal of the slope of tangent line to $y = cx^3$ at the point of intersection. $m_1 m_2 = -1$

since for $y = cx^3 \Rightarrow \frac{dy}{dx} = \frac{3y}{x}$

The orthoganal trajectory of family of curves must satisfy $\frac{dy}{dx} = -\frac{x}{3y}$

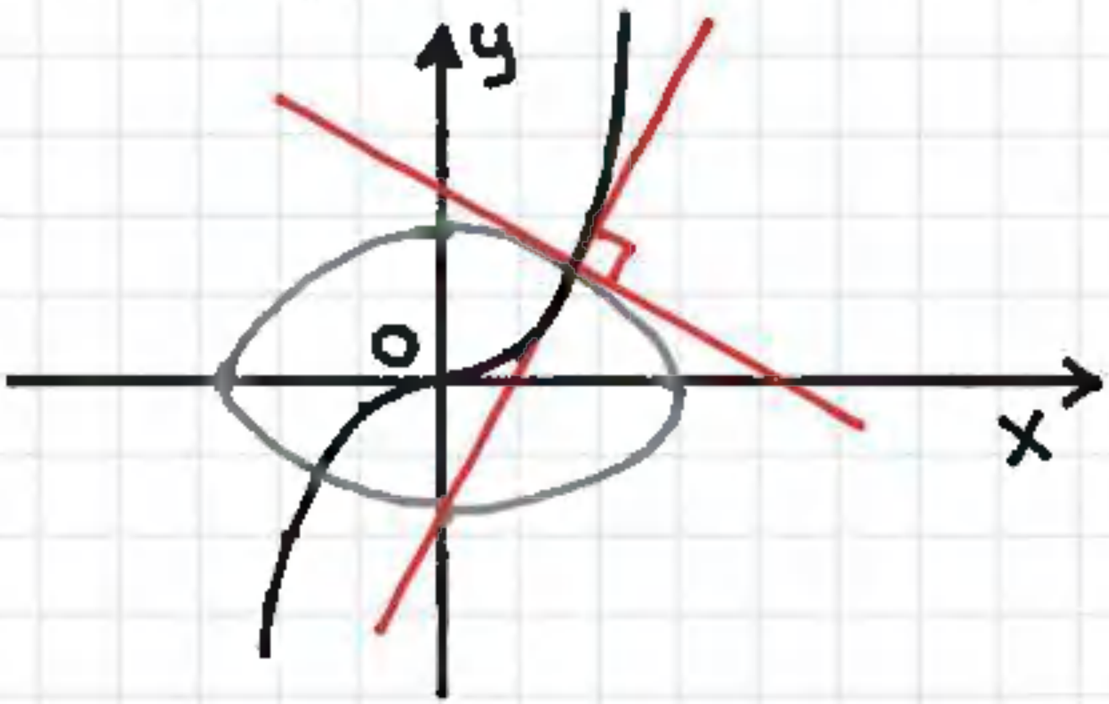
$$\frac{dy}{dx} = \frac{-x}{3y}$$

differential equation
separation of variables

$$3y\,dy = -x\,dx \Rightarrow \frac{3y^2}{2} = \frac{-x^2}{2} + K$$

$\frac{x^2}{2} + \frac{3y^2}{2} = K$ is the orthogonal trajectory of $y = Cx^3$

Let's choose $C=1$ and $K=1$ to visualize



$y = x^3$ cubic function

$\frac{x^2}{2} + \frac{3y^2}{2} = 1$ ellipse

Ex| Find the orthogonal trajectory of $y = e^{cx}$

$$y = e^{cx} \Rightarrow \frac{dy}{dx} = ce^{cx}$$

Now let's get rid of constant C and find a differential equation satisfied by $y = e^{cx}$

$$y = e^{cx} \Rightarrow \ln y = \ln e^{cx} \Rightarrow \ln y = cx \Rightarrow c = \frac{\ln y}{x}$$

$$\frac{dy}{dx} = ce^{cx} = cy \quad \text{since } y = e^{cx}$$

subst. $c = \ln y / x$ into $\frac{dy}{dx} = cy$

$$\frac{dy}{dx} = \frac{\ln y}{x} y$$ diff. equation satisfied by $y = e^{cx}$

Now to find orthogonal trajectory of $y=e^{cx}$ we apply $m_1 m_2 = -1$, since the family of curves whose tangent lines are perpendicular to $y=e^{cx}$ must have the slope of the tangent line to be negative reciprocal of the slope of the tangent line to $y=e^{cx}$ at the point of intersection.

$$y = e^{cx} \Rightarrow \frac{dy}{dx} = \frac{y}{x} \ln y$$

apply $m_1 m_2 = -1 \Rightarrow \dfrac{dy}{dx} = \dfrac{-x}{y \ln y}$

Now solve $\dfrac{dy}{dx} = \dfrac{-x}{y \ln y}$ to find orthogonal trajectory

$$\frac{dy}{dx} = \frac{-x}{y \ln y}$$ differential equation

$$y \ln y = -x\,dx$$ separation of the variables

$$\int y \ln y \, dy = \int -x \, dx$$ integrate both sides

Integration by Parts

$$\frac{y^2}{2} \ln y - \int \frac{y^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{y}} du = -\frac{x^2}{2} + K$$

$u = \ln y \qquad du = \frac{1}{y} dy$

$dv = y\,dy \qquad v = y^2/2$

$\int u\,dv = uv - \int v\,du$

$$\frac{y^2}{2} \ln y - \frac{y^2}{4} = -\frac{x^2}{2} + K$$

$$\boxed{\frac{y^2}{2} \ln y - \frac{y^2}{4} + \frac{x^2}{2} = K}$$

Orthogonal trajectory of $y = e^{cx}$

implicit solution

Separable equations II

Ex] Find an equation of the curve that has slope of the tangent line $f'(x) = 1 + \sqrt{f(x)}$ and passes thru $(0,9)$.

Solution: Lets rewrite $\frac{dy}{dx} = f'(x)$ and $y = f(x)$

$$\frac{dy}{dx} = 1 + \sqrt{y}$$

differential equation

$$\frac{dy}{(1+\sqrt{y})} = dx$$

separate the variables

$$\frac{dy}{1+\sqrt{y}} = dx$$

$$\int \frac{1}{1+\sqrt{y}}\,dy = \int 1\,dx$$

Integrate both sides

Apply U-Substitution

$u = 1+\sqrt{y}$ $\quad du = \frac{1}{2\sqrt{y}}\,dy$

$\sqrt{y} = u-1$ $\quad dy = 2\sqrt{y}\,du$

$dy = 2(u-1)\,du$

$$\int \frac{2(u-1)}{u}\,du = x+C$$

$$2\int \left(1-\frac{1}{u}\right)du = x+C$$

$$2\left[u - \ln|u|\right] = x+C$$

$$2[u - \ln|u|] = x + C \qquad \text{substitute } u = 1+\sqrt{y}$$

$$2\left[1+\sqrt{y} - \ln|1+\sqrt{y}|\right] = x + C$$

$$1+\sqrt{y} - \ln(1+\sqrt{y}) = \frac{x}{2} + \frac{C}{2} = \frac{x}{2} + C_1 \qquad C_1 = \frac{C}{2}$$

$$1+\sqrt{y} - \ln(1+\sqrt{y}) = \frac{x}{2} + C_1$$

Now apply initial condition $y(0) = 9 \Rightarrow x = 0,\ y = 9$

$$1+\sqrt{9} - \ln(1+\sqrt{9}) = 0 + C_1$$

$$4 - \ln 4 = C_1$$

$$1+\sqrt{y} - \ln(1+\sqrt{y}) = \frac{x}{2} + 4 - \ln 4$$

$$1+\sqrt{y}-\ln(1+\sqrt{y}) = \frac{x}{2}+4-\ln 4$$

Implicit solution for $\frac{dy}{dx} = 1+\sqrt{y}$; $y(0)=9$

Separable equations 12

Solve the following Integral equation:

Ex] $y(x) = 16 + \int_0^x 2t\sqrt{y(t)}\,dt$

solution strategy:

1] Differentiate both sides.

2] Apply F.T.C to get rid of the integral.

3] Solve the resulting differential equation.

4] Obtain an initial condition from the integral equation to solve for C.

$$y(x) = 16 + \int_0^x 2t\sqrt{y(t)}\, dt$$

$$\frac{dy}{dx} = \frac{d}{dx}(16) + \frac{d}{dx}\int_0^x 2t\sqrt{y(t)}\, dt$$

Apply the second Fundamental Theorem Of Calculus

$$\frac{d}{dx}\int_c^x f(t)\, dt = f(x)$$

$$\frac{dy}{dx} = 0 + 2x\sqrt{y(x)} \Rightarrow \frac{dy}{dx} = 2x\sqrt{y}$$

$$\frac{dy}{\sqrt{y}} = 2x\, dx$$

Separate the variables

$$y^{-1/2}\,dy = 2x\,dx$$

$$\int y^{-1/2}\,dy = \int 2x\,dx$$

Integrate both sides

$$y^{1/2}\cdot 2 = \frac{\cancel{2}x^2}{\cancel{2}} + C$$

$$2y^{1/2} = x^2 + C \Rightarrow y^{1/2} = \frac{x^2}{2} + \frac{C}{2} \Rightarrow \sqrt{y} = \frac{x^2}{2} + C_1$$

$C_1 = C/2$

$$(\sqrt{y})^2 = \left(\frac{x^2}{2} + C_1\right)^2$$

square both sides

$$y = \left(\frac{x^2}{2} + C_1\right)^2$$

$$y(x) = 16 + \int_0^x 2t\sqrt{y(t)}\,dt$$

Plug in $x=0$

$$y(0) = 16 + \int_0^0 2t\sqrt{y(t)}\,dt = 16 + 0$$

$$\int_a^a f(t)\,dt = 0$$

$$y(0) = 16$$

$$y(x) = \left(\frac{x^2}{2} + C_1\right)^2$$

Apply initial condition

$y(0) = 16 \Rightarrow x=0,\ y=16$

$$16 = (0 + C_1)^2 \Rightarrow C_1^2 = 16 \Rightarrow C_1 = \pm 4$$

choose $C_1 = 4$

$$y(x) = \left(\frac{x^2}{2} + 4\right)^2$$

Solution for Integral differential equation.

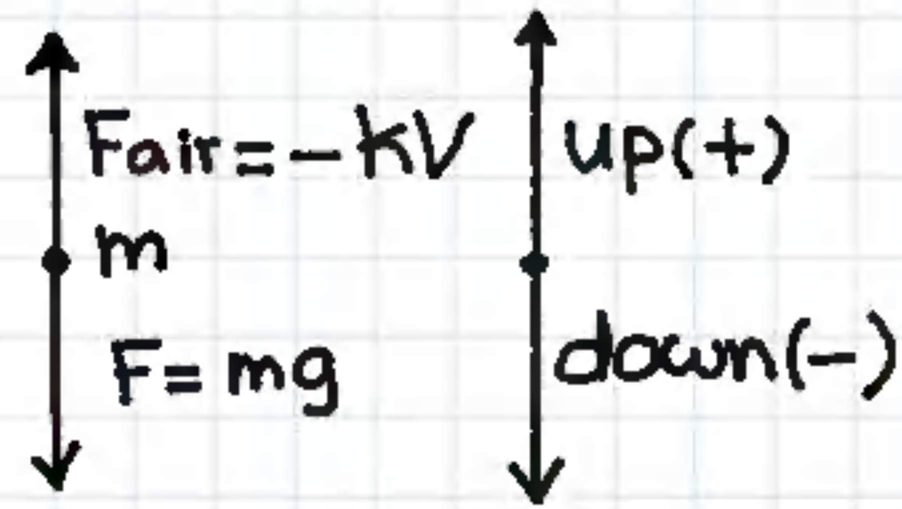
Separable equations 13

Ex] An object of mass m falls from rest at time $t=0$ from an initial height of h metres. Assume air resistance is proportional to the instant velocity of the object. Find the limiting velocity of the object as t becomes large. Also find an expression for the height $S(t)$ of the object at time t assuming $dS/dt = V(t)$; $V(0)=0$; $S(0)=h$

According to Newton's second law of motion

$$m\frac{dv}{dt} = -KV - mg \quad ; g = 9.8\, m/s^2 \; ; k > 0$$

$$m\frac{dV}{dt} = -kV - mg$$



$$\frac{dV}{-kV-mg} = \frac{1}{m}dt$$

$$\int \frac{dV}{-kV-mg} = \int \frac{1}{m}dt$$

U-substitution

$u = -kV - mg \quad du = -k\,dV$

$dV = -\frac{1}{k}du$

$$\int \frac{1}{u} \cdot -\frac{1}{k}\,du = \frac{1}{m}t + C$$

$$-\frac{1}{k}\ln|u| = \frac{1}{m}t + C \Rightarrow \ln|u| = -\frac{k}{m}t - Ck$$

$$\ln|-kV-mg| = -\frac{k}{m}t - Ck$$

$$\ln|-KV-mg| = -\frac{K}{m}t + C_2$$ where $C_2 = -CK$

$$e^{\ln|-KV-mg|} = e^{(-\frac{K}{m})t + C_2}$$

$$|-KV-mg| = e^{(-K/m)t} \cdot e^{C_2}$$

$$-KV-mg = \pm e^{C_2} e^{(-K/m)t}$$

$$-KV-mg = A\, e^{(-K/m)t}$$ $A = \pm e^{C_2}$

$$-KV = Ae^{(-K/m)t} + mg$$

$$V(t) = -\frac{A}{K} e^{(-K/m)t} - \frac{mg}{K}$$

Apply initial condition

$V(0) = 0 \Rightarrow t=0, V=0$

$$0 = -\frac{A}{K} e^{0} - \frac{mg}{K} \Rightarrow A = -mg$$

$$V(t) = \frac{mg}{K} e^{(-K/m)t} - \frac{mg}{K}$$

Now find limiting velocity of the object as $t \to \infty$

$$\lim_{t \to \infty} V(t) = \frac{-mg}{K}$$ Terminal Velocity

Note $V_{terminal} = -\frac{mg}{K} < 0$ since we chose down direction to be negative

Now find $S(t)$

$$S(t) = \int V(t)\,dt = \int \left(\frac{mg}{K} e^{(-K/m)t} - \frac{mg}{K} \right) dt$$

$$S(t) = \frac{mg}{K} e^{-(K/m)t} \cdot -\frac{m}{K} - \frac{mg}{K} t + D$$

$$S(t) = -\frac{m}{K}\left(\frac{mg}{K}e^{-(K/m)t} + gt\right) + D$$

Now apply initial condition $S(0) = h \Rightarrow t = 0\,;\ S = h$

$$h = -\frac{m}{K}\left(\frac{mg}{K}e^{-(K/m)0} + 0\right) + D$$

$$h = \frac{-m^2g}{K^2} + D \Rightarrow D = h + \frac{m^2g}{K^2}$$

$$\boxed{S(t) = -\frac{m}{K}\left(\frac{mg}{K}e^{-(K/m)t} + gt\right) + h + \frac{m^2g}{K^2}}$$

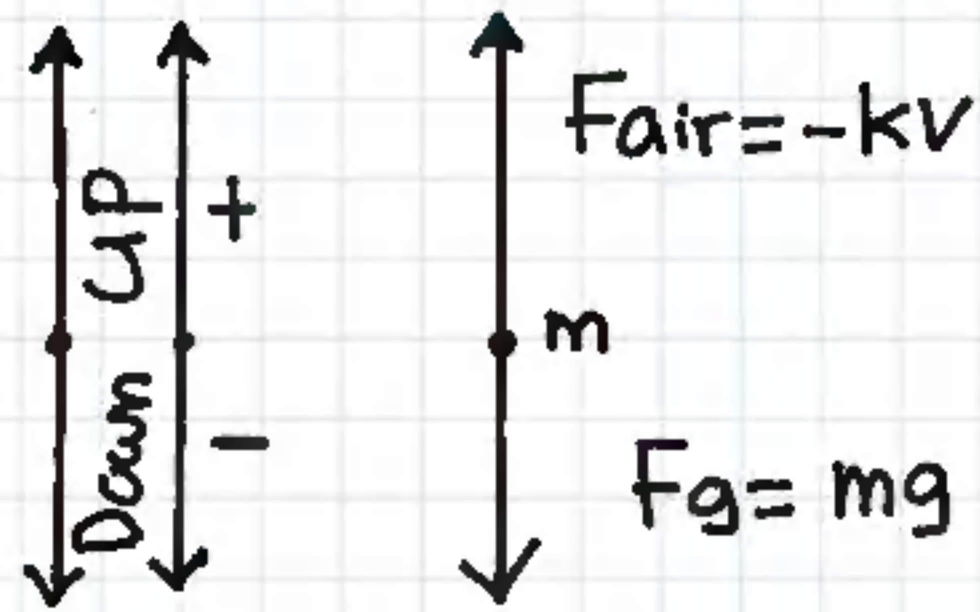
Let's check to see if $S(0) = h$

$$S(0) = \cancel{-\frac{m}{K}\left(\frac{mg}{K}\right.}e^{-0} + 0\Big) + h + \cancel{\frac{m^2g}{K^2}} = h \quad \text{YES!}$$

Some theory notes

$$m\frac{dv}{dt} = -kV - mg$$

$$\frac{dv}{dt} = -\frac{k}{m}v - g$$



since we chose up direction to be positive, as the object falls $V<0$ (since down is negative) and $F_{air} = -kV > 0$ since $V<0$ and we expect the force of air resistance to be pointing in the positive (up) direction since Fair opposes the falling motion of the object.

Furthermore when $V=0$ as the object falls from rest $\frac{dV}{dt} = -\frac{K}{m}V - g \Rightarrow \frac{dV}{dt} = -g$ as we expect since $a = -g < 0$ due to gravity. As the object falls $\frac{dV}{dt} = -g + (-\frac{K}{m}V)$ gets smaller in absolute value since the force of air resistance is a positive quantity $\Rightarrow -\frac{K}{m}V > 0$ since $V<0$

As $t \to \infty$ Force of air resistance matches or equals to $F_g = mg$ and acceleration of the mass becomes 0 and object falls down with constant velocity. $V_{terminal} = -mg/K$

Separable equations 14

A beverage containing 6% alcohol by volume and another beverage containing 4% alcohol are both pumped into a vat which initially contains 1000 L of beer at 3% alcohol by volume. The rate at which the 6% beer is pumped into the tank is 10 L/min, and the rate at which the 4% beer is pumped into the vat is 5 L/min. The resulting well mixed solution is pumped out of the vat at the rate of 15 L/min. What is the percentage of alcohol in the tank after 60 minutes.
What is the percentage of alcohol in the tank as time grows to infinity.

$y(t)$ = Volume in Litres of pure alcohol in vat.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\frac{dy}{dt} = \underbrace{(0.06)(10) + (0.04)(5)}_{\text{L/min of pure alcohol entering the tank}} - \underbrace{\frac{y}{1000}(15)}_{\text{L/min of pure alcohol leaving the tank}}$$

$$\frac{dy}{dt} = 0.8 - 0.015y$$

$$\frac{dy}{0.8 - 0.015y} = \frac{dt}{1}$$

Separate the variables

$$\int \frac{dy}{0.8-0.015y} = \int 1\, dt$$

Integrate both sides

$$\frac{\ln|0.8-0.015y|}{-0.015} = t + C$$

Apply U-Substitution

$U = 0.8 - 0.015y$

$du = -0.015\, dy$

$$\ln|0.8-0.015y| = -0.015t - 0.015C$$

$$e^{\ln|0.8-0.015y|} = e^{(-0.015t-0.015C)}$$

$$= e^{-0.015t} \cdot e^{-0.015C}$$

$$|0.8-0.015y| = A e^{-0.015t}$$

$A = e^{-0.015C}$

$$0.8 - 0.015y = \pm A e^{-0.015t}$$

$$0.8-0.015y = \pm Ae^{-0.015t} = Be^{-0.015t}$$

$$0.8-0.015y = Be^{-0.015t} \qquad \text{where } B=\pm A$$

$$y(0) = (0.03)(1000) = 30\text{ L} \Rightarrow t=0 \quad y=30$$

$$0.8-0.015(30) = Be^{-0.015(0)}$$

$$0.35 = Be^{-0} \Rightarrow B=0.35 \qquad e^0=1$$

$$0.8-0.015y = 0.35e^{-0.015t}$$

$$-0.015y = 0.35e^{-0.015t} - 0.8$$

$$y(t) = \frac{-350}{15}e^{-0.015t} + \frac{800}{15}$$

$$y(t) = \frac{-70}{3}e^{-0.015t} + \frac{160}{3}$$

$$y(t) = \frac{-70}{3} e^{-0.015t} + \frac{160}{3}$$

Litres of pure alcohol in vat at time t.

at $t = 60$ min $\quad y(60) = \frac{-70}{3} e^{-0.015(60)} + \frac{160}{3}$

$y(60) = \frac{-70}{3} e^{-0.9} + \frac{160}{3} \cong 43.85$ Litres of pure alcohol in vat after 60 minutes

$$\text{Percentage alcohol} = \frac{y(60)}{1000} \times 100\ \%$$

$$\%\ \text{alcohol} = \frac{\left(-70/3\ e^{-0.9} + \frac{160}{3}\right)}{1000} \times 100\ \%$$

$$\% \text{ alcohol} \cong \frac{43.85}{1000} \times 100\% \cong 4.39\%$$

We have found the % alcohol in vat after 60 minutes.

Let's now find % alcohol in the vat after a very long time as $t \to \infty$.

$$y(t) = -\frac{70}{3} e^{-0.015t} + \frac{160}{3} \qquad \lim_{t\to\infty} e^{-0.015t} = 0$$

$$\lim_{t\to\infty} \frac{y(t)}{1000} \times 100\% = \frac{160/3}{1000} \times 100\% \cong 5.33\%$$

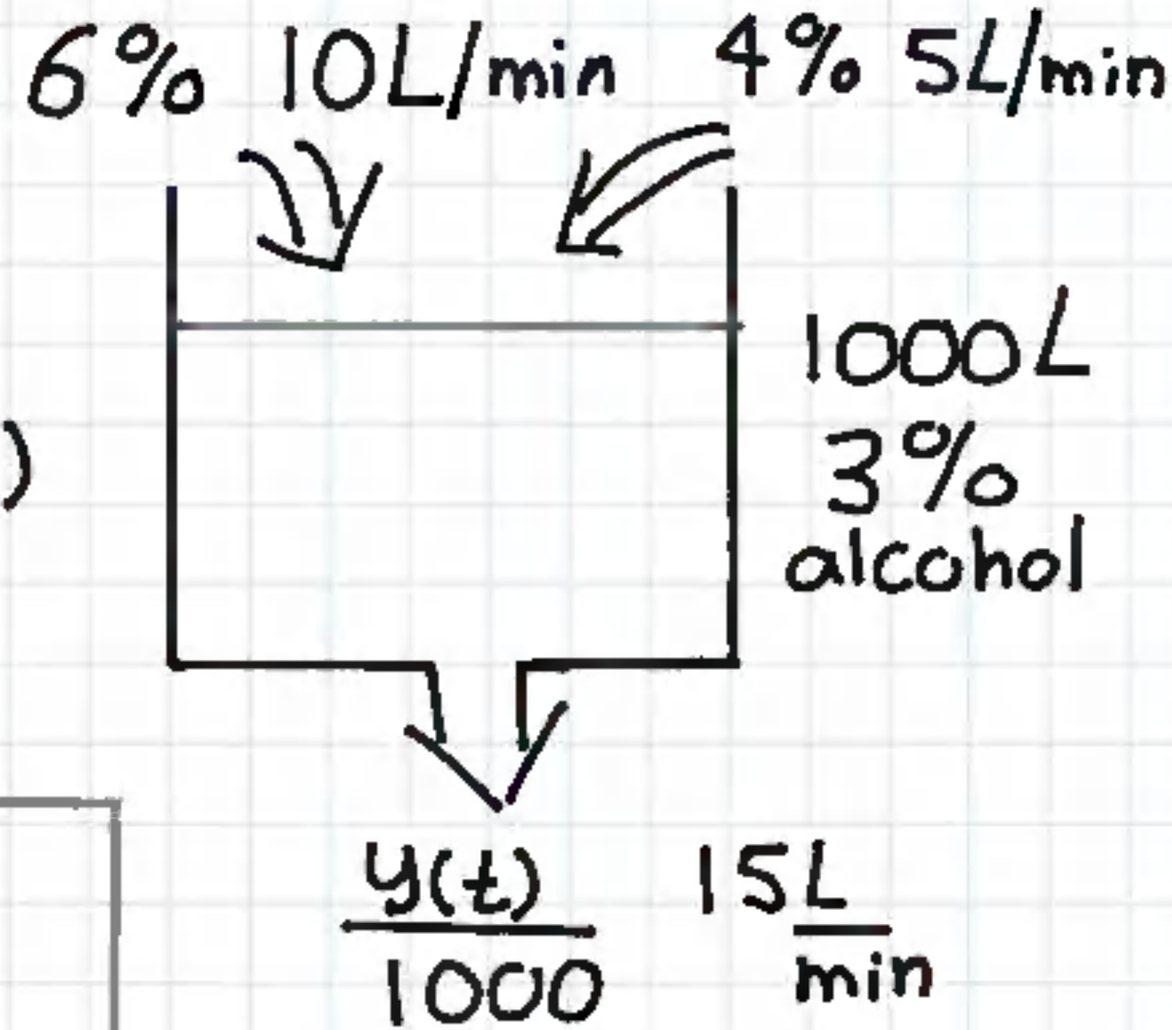
Lets now verify our answer by intuition. Let's compute a weighed average of % alcohol based on the 2 rates of alcohol entering the tank and since the volume of vat always remains the same at 1000 Litres, since inflow rate = outflow rate $\Rightarrow$ $\underbrace{10\frac{L}{min} + 5\frac{L}{min}}_{\text{Volume in}} = \underbrace{15L/min}_{\text{Volume out}}$

$$\% \text{ alcohol} = \left(\frac{10}{10+5} \times 6\%\right) + \left(\frac{5}{10+5} \times 4\%\right)$$

$$\% \text{ alcohol} \cong 4\% + 1.33\% = 5.33\%$$

SAME
Answer

Solution Diagram



$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$(\text{rate in}) = (0.06)(10) + (0.04)(5)$$

$$(\text{rate out}) = \frac{y(t)}{1000} \times 15$$

$$\boxed{\begin{array}{l} \dfrac{dy}{dt} = 0.8 - 0.015y \\ y(0) = 1000L \times 0.03 = 30L \end{array}}$$

$$\boxed{y(t) = \frac{160}{3} - \frac{70}{3} e^{-0.015t}}$$

$\frac{y(t)}{1000} \times 100\% = \%$ alcohol in vat at time t minutes

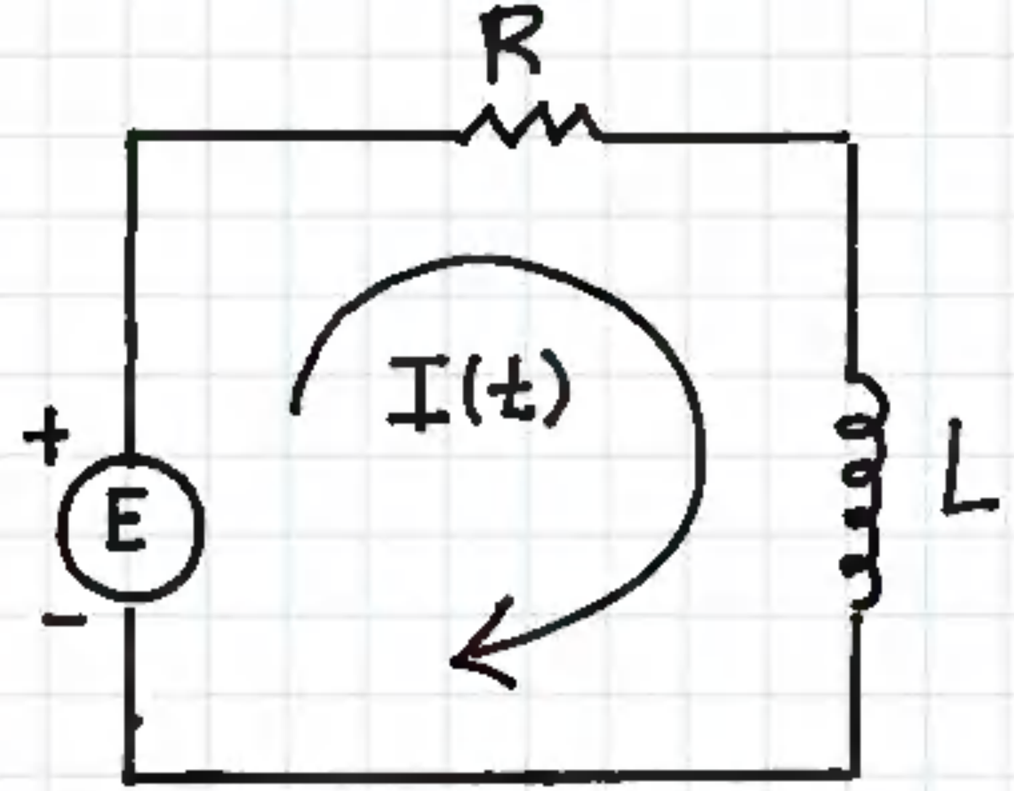
Separable equations 15

In a series circuit containing a resistor with resistance of R ohms (Ω) and an inductor with an inductance of L henries (H) and a battery that produces a voltage of E(t) volts (V), Kirchoff's second law states that the sum of the voltage drop across the resistor $V = IR$ and the voltage drop across the inductor $V = L\,dI/dt$ is equal to the supplied voltage E(t) by the battery. Thus we obtain

$$L\frac{dI}{dt} + RI = E(t) \quad ; \; I(0) = I_0 \quad \text{(initial current at } t=0\text{)}$$

This differential equation models the current I as a function of time I(t)

Ex] A 24 volt battery is connected to a simple series circuit in which the resistor $R = 12\,\Omega$ and the inductance $L = 2$ henry. Find the current $I(t)$ and also find the limiting value of the current as time grows to infinity. Assume that the initial current $I(0) = 0$



Given: $L = 2\text{ H}$; $R = 12\,\Omega$

$E(t) = 24$ Volts

Find $I(t)$

Strategy : Apply Kirchoff's voltage law to set up a differential equation for $I(t)$

Voltage drop across Resistor $= IR$

Voltage drop across Inductor $= L\,dI/dt$

Voltage supplied $= E(t)$

$IR + L\,dI/dt = E(t)$ Kirchoff's voltage law

Given: $R = 12\,\Omega$; $L = 2\,H$; $E(t) = 24\,V$; $I(0) = 0$

$12I + 2\,dI/dt = 24$; $I(0) = 0$

Above is a differential equation modelling current $I(t)$.

$12I + 2\,dI/dt = 24$ $\quad I(0) = 0$

$\frac{dI}{dt} = 12 - 6I$ $\quad$ Rearrange to solve for $\frac{dI}{dt}$

$\frac{dI}{12-6I} = dt$ $\quad$ Separate the variables

$\int \frac{1}{12-6I}\,dI = \int 1\,dt$ $\quad$ Integrate both sides

Apply U-Substitution

$U = 12 - 6I \quad du = -6\,dI$

$dI = du/-6$

$\frac{\ln|12-6I|}{-6} = t + C$

$\ln|12-6I| = -6t - 6C$

$$\ln|12-6I| = -6t + C_1 \quad \text{where } C_1 = -6C$$

$$e^{\ln|12-6I|} = e^{(-6t+C_1)} = e^{-6t} \cdot e^{C_1}$$

$$|12-6I| = Ae^{-6t} \quad \text{where } A = e^{C_1}$$

$$12-6I = \pm Ae^{-6t} = Be^{-6t} \quad \text{where } B = \pm A$$

$$-6I = Be^{-6t} - 12$$

$$I(t) = 2 - \frac{B}{6}e^{-6t}$$

Now apply initial condition $I(0)=0 \Rightarrow t=0, I=0$

$$0 = 2 - \frac{B}{6}e^{-0} \Rightarrow -2 = \frac{-B}{6} \Rightarrow B = 12$$

$$I(t) = 2 - 2e^{-6t}$$

We have found the explicit solution to the current $I(t)$

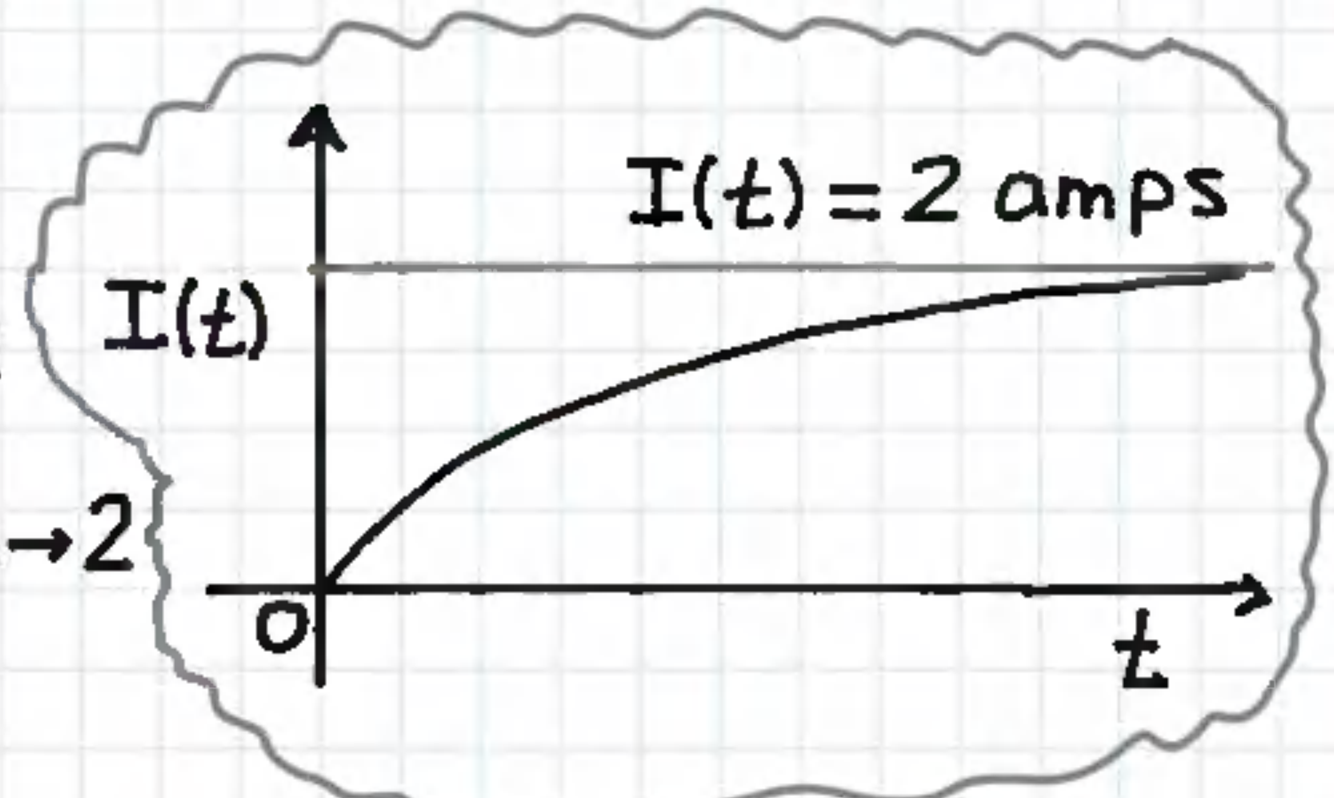
Now Let's find the limiting value of the current as $t \to \infty$

$$\lim_{t \to \infty} I(t) = \lim_{t \to \infty} (2 - 2e^{-6t}) = 2 - 0 = \underline{2 \text{ amps}}$$

since $\lim_{t \to \infty} e^{-6t} = 0$

Graph of $I(t) = 2 - 2e^{-6t}$

Notice as $t \to \infty \Rightarrow I(t) \to 2$



Do you need a Calculus tutor ?
Website: https://www.ubcmathtutor.com
Email: ubcmathtutor1@gmail.com
iMessage: ubcmathtutor1@gmail.com
Cell/Text: 604-318-1970