Taylor and Maclaurin Series I

Goal: To represent a function with a power series.

Let's assume that $f(x)$ can be represented by a power series of the form $\sum_{n=0}^{\infty} C_n (x-a)^n$

$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$ for $|x-a|<R$

Let's derive the coefficients $C_0, C_1, C_2, C_3, \ldots$ in terms of $f(x)$ and its derivatives.

Plug $x=a$ into $f(x) \Rightarrow f(a) = C_0 \Rightarrow C_0 = f(a)$

$f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \cdots$

plug $x=a$ into $f'(x) \Rightarrow f'(a) = C_1 \Rightarrow C_1 = f'(a)$

$$f''(x) = 2C_2 + 3.2\,C_3(x-a) + 4.3\,C_4(x-a)^2$$

plug $x=a$ into $f''(x) \Rightarrow f''(a) = 2C_2 \Rightarrow C_2 = f''(a)/2$

$$f'''(x) = 3.2\,C_3 + 4.3.2\,C_4(x-a) + \cdots$$

plug $x=a$ into $f'''(x) \Rightarrow f'''(a) = 3.2\,C_3 = 3!\,C_3$

Now we can see the pattern developing:

$f(a) = C_0$; $f'(a) = C_1$, $f''(a) = 2!\,C_2$; $f'''(a) = 3!\,C_3$

$f^{(n)}(a) = n!\,C_n \Rightarrow C_n = f^{(n)}(a)/n!$

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

Notation: $0! = 1$, $1! = 1$

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

for $|x-a| < R$

Taylor series expansion of $f(x)$ about $x = a$

If center $a = 0$, Taylor Series $\Rightarrow$ Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \quad \text{for } |x| < R$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} \quad \text{for } |x| < R$$

Intuitive idea: We are reconstructing $f(x)$ as an infinite polynomial by evaluating its derivatives at $x = a$ (center).

Ex] Find the Maclaurin series of $f(x) = e^{2x}$ and also find it's radius of convergence.

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

Key concept: Find derivatives of $f(x)$ and find pattern

$f(x) = e^{2x}$; $f'(x) = 2e^{2x}$; $f''(x) = 2^2 e^{2x}$; $f'''(x) = 2^3 e^{2x}$

In general $f^{(n)}(x) = 2^n e^{2x}$

$f(0) = e^0 = 1$; $f'(0) = 2$; $f''(0) = 2^2$; $f'''(0) = 2^3$; ... ; $f^{(n)}(0) = 2^n$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots$$

$$f(x) = 1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \cdots + \frac{2^n x^n}{n!} + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

Maclaurin Series of $f(x) = e^{2x}$ about center $a=0$

Now let's apply the Ratio Test to find the Radius of Convergence.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}}{(n+1)!} \Big/ \frac{2^n x^n}{n!} \right|$$

$$\lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} 2 \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} |x|$$

$$= \lim_{n\to\infty} \frac{2|x|}{n+1} = 0 < 1 \Rightarrow R = \infty \quad (\text{Radius} = \infty)$$

So by the Ratio Test $R=\infty$ and this series converges for all x in $(-\infty, \infty)$.

$f(x) = e^{2x}$ can be represented by the series

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \quad \text{on } (-\infty, \infty)$$

$R = \infty$, Interval of convergence $(-\infty, \infty)$

Key steps:

1] Find $f(x), f'(x), f''(x), f'''(x), f^4(x), \ldots$

2] Plug in $x = a \Rightarrow f(a), f'(a), f''(a), f'''(a), \ldots$

3] look for pattern $f^{(n)}(x)$ or $f^{(n)}(a)$

4] Apply Taylor formula: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots$$

Taylor and Maclaurin Series 2

Ex| Find the Taylor series for $f(x) = \ln x$ at $a = 1$ and also find it's interval of convergence.

key concept: Find derivatives of $f(x)$ and look for a pattern.

$f(x) = \ln x$; $f'(x) = 1/x = x^{-1}$; $f''(x) = -x^{-2}$; $f'''(x) = 2x^{-3}$

$f^4(x) = -3.2.1\, x^{-4}$; $f^5(x) = 4.3.2.1\, x^{-5}$

$f^{(n)}(x) = (-1)^{n+1}(n-1)!\, x^{-n} \Rightarrow f^{(n)}(1) = (-1)^{n+1}(n-1)!$

$f(1) = \ln 1 = 0$; $f'(1) = 1$; $f''(1) = -1$; $f'''(1) = 2$; $f^4(1) = -3!$

Note: when differentiating $f(x)$ keep unsimplified such as $3 \times 2 \times 1$ to find pattern.

$f(1)=0$; $f'(1)=1$; $f''(1)=-1$; $f'''(1)=2\cdot 1$; $f^{4}(1)=-3!$; ...

$f^{(n)}(1) = (-1)^{n+1}(n-1)!$ Pattern of $f^{(n)}(1)$

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \cdots$$

$$f(x) = 0 + 1(x-1) + \frac{(-1)(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{-3!\ (x-1)^4}{4!}$$

$$+ \cdots + \frac{(-1)^{n+1}(n-1)!\,(x-1)^n}{n!}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!\,(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

Note: $n! = n(n-1)! \Rightarrow \frac{(n-1)!}{n!} = \frac{\cancel{(n-1)!}}{n\cancel{(n-1)!}} = \frac{1}{n}$

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} \qquad f^{(n)}(1) = (-1)^{n+1}(n-1)!$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

Taylor series of $f(x) = \ln x$ about center $a=1$

Now that we have developed a pattern for the general term of the taylor series we can apply the Ratio test to find the interval of convergence.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \Big/ \frac{(-1)^{n+1}(x-1)^n}{n} \right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(-1)^{n+1}}\cdot\frac{n}{n+1}\cdot\frac{(x-1)^{n+1}}{(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|-1\cdot\frac{n}{n+1}\cdot(x-1)\right| = \lim_{n\to\infty}\frac{n}{n+1}\left|x-1\right| = \left|x-1\right| < 1$$

$R=1$, now let's find the interval of convergence.

$$|x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$$

lets check endpoints $x=0$ and $x=2$

$$f(x) = \ln x = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^n / n$$

plug $x=0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1}(-1)^n / n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$

plug $x=0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{2n+1}/n \Rightarrow \sum_{n=1}^{\infty} \frac{-1}{n} \Rightarrow -\sum_{n=1}^{\infty} \frac{1}{n}$

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent P-series with $P=1$

Note: $(-1)^{2n+1} = -1$ for all n

$\therefore$ Dont include $x=0$ for the interval of convergence.

Plug $x=2$ into $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^n}{n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ Converges by AST (Alternating series Test)

$f(n) = \frac{1}{n}$ is a decreasing function

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ Include $x=2$ for the interval of convergence

$f(x) = \ln x$ can be represented by the taylor series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} \text{ on the interval } (0,2]$$

Key concept: The Ratio Test does not tell us anything about the convergence at endpoints so we had to plug in the endpoints $x=0$ and $x=2$ into the taylor series for $f(x) = \ln x$ to determine the interval of convergence.

Taylor Series and Maclaurin Series 3

Ex A] Find the Maclaurin series for $f(x) = \sin x$ and also find the radius of convergence.

Ex B] By differentiating the maclaurin series for $\sin x$ obtained in Ex A] find the maclaurin series for $g(x) = \cos x$

$$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$$

$$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$$

$f^4(x) = \sin x \Rightarrow f^4(0) = \sin 0 = 0$

$f^5(x) = \cos x \Rightarrow f^5(0) = \cos 0 = 1$

$f(0) = 0$; $f'(0) = 1$; $f''(0) = 0$; $f'''(0) = -1$; $f^4(0) = 0$, $f^5(0) = 1$

Notice even derivatives are always zero, odd derivatives alternate between 1 and -1.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^4(0)x^4}{4!} + \cdots$$

$$f(x) = 0 + 1x + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} + \frac{x^5}{5!} - + \cdots$$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$$

Maclaurin Series of $f(x) = \sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$$

We need to generate odd powers of x, Let $n = 2k+1$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$ Pattern for general term

Maclaurin series of $f(x) = \sin x$ with center $a=0$

Now that we have developed the pattern for the general term of the Maclaurin series, let's apply the Ratio test to find the radius of convergence.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}\,x^{2(k+1)+1}}{(2(k+1)+1)!} \Bigg/ \frac{(-1)^{k}\,x^{2k+1}}{(2k+1)!}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}\,x^{2k+3}}{(2k+3)!}\cdot\frac{(2k+1)!}{(-1)^{k}\cdot x^{2k+1}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{(-1)^{k}}\cdot\frac{(2k+1)!}{(2k+3)!}\cdot\frac{x^{2k+3}}{x^{2k+1}}\right|$$

$$= \lim_{k\to\infty}\left|(-1)\cdot\frac{\cancel{(2k+1)!}}{(2k+3)(2k+2)\cancel{(2k+1)!}}\cdot x^{2}\right|$$

$$= \lim_{k\to\infty}\frac{x^2}{(2k+3)(2k+2)} = 0 < 1 \quad \text{for all } x \Rightarrow R = \infty$$

Since the Radius of convergence is $R=\infty$, the Interval of convergence is $(-\infty, \infty)$

Ex B] Find the Maclaurin series of $g(x) = \cos x$.

In Ex A] we found the Maclaurin series of

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots$$

let's differentiate both sides.

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots$$

Or differentiate compact series for $\sin x$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$ Differentiate both sides

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$g(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Maclaurin series for $g(x) = \cos x$

What about the interval of convergence?

Interval of convergence is not affected by differentiation, only convergence at endpoints can change, but here with $(-\infty, \infty)$ we don't have any endpoints.

$g(x) = \cos x$ can be represented by the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ on the interval $(-\infty, \infty)$

Taylor and Maclaurin Series 4

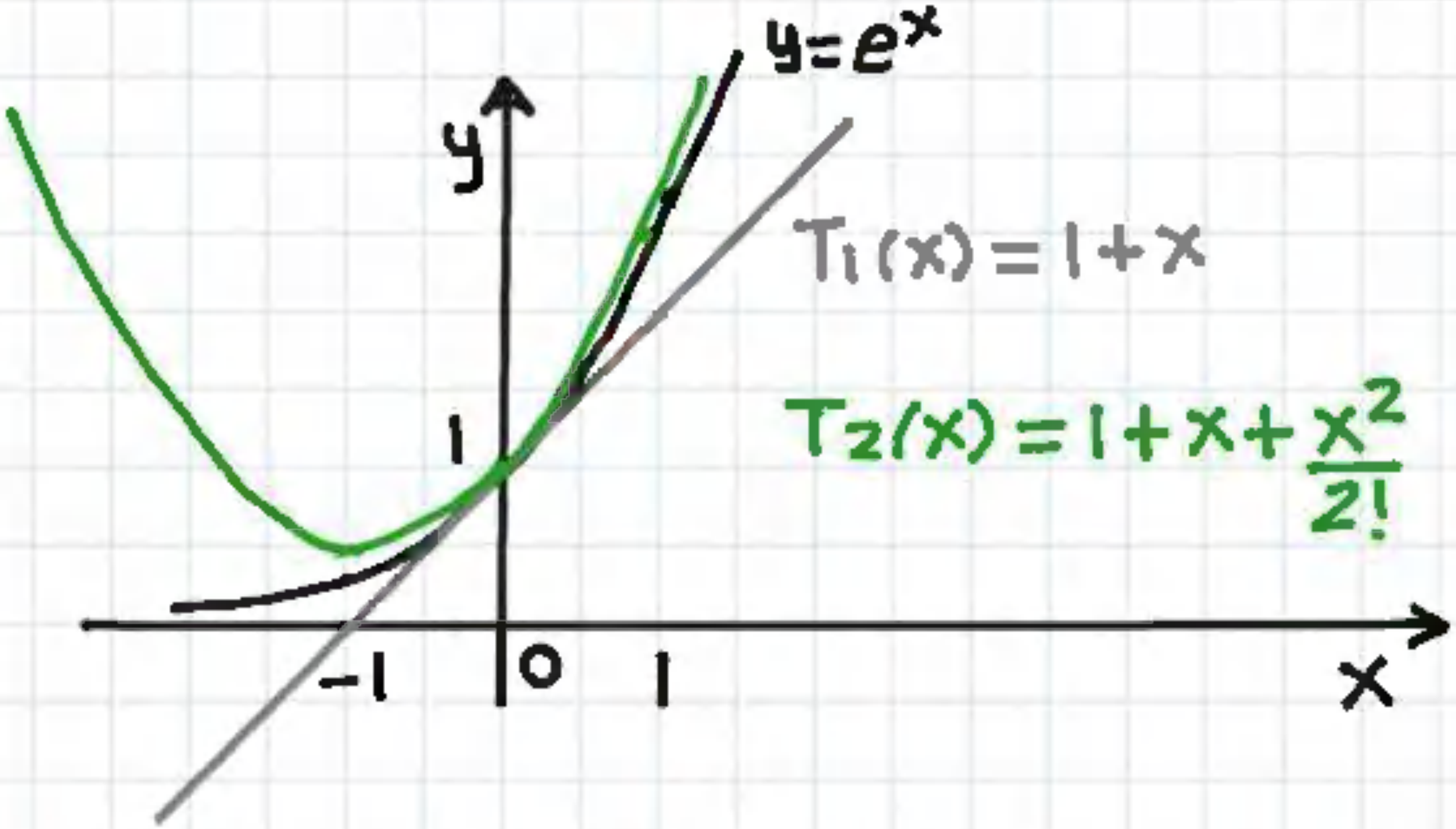
Given the Maclaurin series of $f(x) = e^x$ as $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$T_1(x) = 1 + x$; $T_2(x) = 1 + x + \frac{x^2}{2!}$; $T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$; $T_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$T_1(x), T_2(x), T_3(x), T_4(x), T_5(x)$ are approximating polynomials, the higher the degree of $T_n(x)$ the closer the approximation.



$T_1(x)$, $T_2(x)$ are approximating Polynomials to $f(x) = e^x$

We say that $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

$R_n(x) = f(x) - T_n(x)$ measures the error or difference of $f(x)$ and the approximating Polynomial $T_n(x)$

If we can show that $\lim\limits_{n\to\infty} R_n(x) = 0$ then

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} f(x) - T_n(x) = 0 \Rightarrow f(x) - \lim_{n\to\infty} T_n(x) = 0$$

$$f(x) = \lim_{n\to\infty} T_n(x)$$

Theorem: Assume $f(x)$ is infinitely differentiable on an open interval containing a, $|x-a| < R$, and $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x in $(a-R, a+R)$ then $f(x)$ is equal to the sum of its Taylor Series.

We need to show that $\lim\limits_{n\to\infty} R_n(x) = 0$

Taylor's Inequality : If $|f^{(n+1)}(x)| \leq M$ for $|x-a|<d$ then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \leq \dfrac{M\,|x-a|^{n+1}}{(n+1)!}$ for $|x-a| \leq d$

M is the maximum value of $|f^{(n+1)}(x)|$ on $(a-d, a+d)$

Key concept: If we can show that $\lim_{n\to\infty} R_n(x) = 0$ then we have proven that f(x) is equal to the sum of its Taylor Series.

Ex] Show that the Maclaurin Series for sinx ;

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

converges to $f(x) = \sin x$ for all x.

$f'(x) = \cos x$; $f''(x) = -\sin x$; $f'''(x) = -\cos x$; $f^4(x) = \sin x$;...

Clearly all the derivatives alternate between $\pm \sin x$, and $\pm \cos x$. We have $f^{(n+1)}(x) = \pm \cos x$ or $f^{(n+1)}(x) = \pm \sin x$. For sure we have $|f^{(n+1)}(x)| \leq 1$ for all x. So $M = \max |f^{(n+1)}(x)| = 1$

Let's apply Taylor's Inequality:

$$|R_n(x)| \leq \frac{M|x^{n+1}|}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$$

Holding x fixed and taking $n \to \infty$ we see that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \Rightarrow \lim_{n\to\infty} |R_n(x)| = 0$$

Since $-|R_n(x)| \le R_n(x) \le |R_n(x)|$

we can take $\lim\limits_{n\to\infty}$ and apply Squeeze theorem

$$-\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} R_n(x) \le \lim_{n\to\infty} |R_n(x)|$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

By Squeeze theorem $\lim\limits_{n\to\infty} R_n(x) = 0$, so we have proven that $\sin x$ is equal to the sum of its Maclaurin Series.

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Taylor and Maclaurin Series 5

Ex] Given the Maclaurin Series of $f(x) = \cos x$ as

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$; Find the Maclaurin Series

of $g(x) = \sin^2 x$. Hint: Apply Trignometric Identity

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

Strategy: Let's start with the Maclaurin Series of $f(x) = \cos x$ and build up the Maclaurin Series of $g(x) = \sin^2 x$.

$$\sin^2 x = \frac{1}{2}\left(1-\cos(2x)\right) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$ Plug in 2X for X

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$(-1)^{n+1}$ $\quad 2^{2n-1}$

$$\frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n \, 2^{2n}/2 \; x^{2n}}{(2n)!}$$

Notes: $(-1)(-1)^n = (-1)^{n+1}$

$$\frac{2^{2n}}{2^1} = 2^{2n-1}$$

$$\frac{1}{2}-\frac{1}{2}\cos(2x)=\frac{1}{2}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$$

let's expand summation:

$$\frac{1}{2}+\frac{(-1)^{1}2^{-1}x^{0}}{0!}+\frac{2^{1}x^{2}}{2!}-\frac{2^{3}x^{4}}{4!}+-\cdots$$

$$\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}+\frac{2^{1}x^{2}}{2!}-\frac{2^{3}x^{4}}{4!}+-\cdots+\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$$

let's start index at $n=1$ since first term of summation which is $-1/2$ cancels out.

$$\sin^{2}x=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$$

$$\sin^2 x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2^{2n-1}\, x^{2n}}{(2n)!}$$

We have found the Maclaurin Series of $\sin^2 x$ by applying Trignometric Identity $\sin^2 x = \frac{1}{2}(1-\cos(2x))$ and using Maclaurin Series of $\cos x$ as a building block.

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Taylor and Maclaurin Series 6

Ex] Find the Taylor Series of $f(x) = \sin x$ centered at $x = \pi/6$

let's start by taking some derivatives:

$f(x) = \sin x \Rightarrow f(\pi/6) = \sin(\pi/6) = 1/2$

$f'(x) = \cos x \Rightarrow f'(\pi/6) = \cos(\pi/6) = \sqrt{3}/2$

$f''(x) = -\sin x \Rightarrow f''(\pi/6) = -\sin(\pi/6) = -1/2$

$f'''(x) = -\cos x \Rightarrow f'''(\pi/6) = -\cos(\pi/6) = -\sqrt{3}/2$

$f^4(x) = \sin x \Rightarrow f^4(\pi/6) = \sin(\pi/6) = 1/2$

$f(\pi/6) = 1/2$; $f'(\pi/6) = \sqrt{3}/2$; $f''(\pi/6) = -1/2$; $f'''(\pi/6) = -\sqrt{3}/2$

$f^4(\pi/6) = 1/2$,...

The pattern is clear, even derivatives alternate between $1/2$ & $-1/2$ and odd derivatives alternate between $\sqrt{3}/2$ and $-\sqrt{3}/2$.

Taylor Series of $f(x) = \sin x$ centered at $x = \pi/6$ is:

$$\sin x = f(\pi/6) + \frac{f'(\pi/6)}{1!}(x-\pi/6) + \frac{f''(\pi/6)}{2!}(x-\pi/6)^2 + \frac{f'''(\pi/6)}{3!}(x-\pi/6)^3 + \frac{f^4(\pi/6)}{4!}(x-\pi/6)^4 + \ldots$$

$f(\pi/6) = 1/2$; $f'(\pi/6) = \sqrt{3}/2$; $f''(\pi/6) = -1/2$; $f'''(\pi/6) = -\sqrt{3}/2$

$f^4(\pi/6) = 1/2$

$$\sin x = 1/2 + \frac{\sqrt{3}}{2(1!)}(x-\pi/6) - \frac{1}{2(2!)}(x-\pi/6)^2 - \frac{\sqrt{3}}{2(3!)}(x-\pi/6)^3$$

$$+ \frac{1}{2(4!)}(x-\pi/6)^4 + \cdots$$

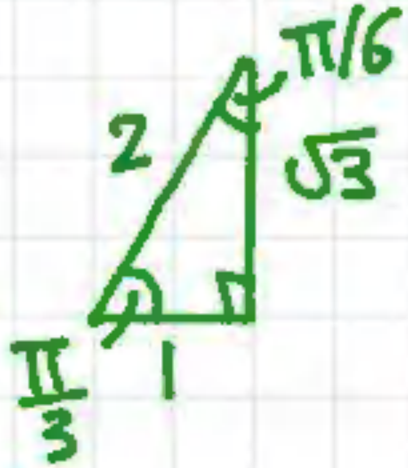
let's now separate the even and odd powers of x

$$\sin x = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!}(x-\pi/6)^{2n}}_{\text{Even powers of x}} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}}{2(2n+1)!}(x-\pi/6)^{2n+1}}_{\text{odd powers of x}}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!} (x - \pi/6)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n+1)!} \left(x - \frac{\pi}{6}\right)^{2n+1}$$

We have found the Taylor Series of $\sin x$ centered at $x = \pi/6$.

Notes: To find the exact values of $\sin(\pi/6)$ and $\cos(\pi/6)$ look at the 30-60-90 triangle.



$\sin(\pi/6) = \frac{1}{2}$ and $\cos(\pi/6) = \sqrt{3}/2$

Taylor and Maclaurin Series 7

Ex] Suppose that $g(x)$ is a function that has continuous derivatives and that $g(2)=1$, $g'(2)=3$, $g''(2)=-4$, $g'''(2)=2$. Find the Taylor Series of $g(x)$ centered at $x=2$, up to and including the term containing x^3. Find $T_3(x)$ and use this part of the Taylor Series to approximate $g(1.9)$.

Solution: Let's apply Taylor Series formula up to $n=3$

$$T_3(x) = g(2) + \frac{g'(2)}{1!}(x-2) + \frac{g''(2)}{2!}(x-2)^2 + \frac{g'''(2)}{3!}(x-2)^3$$

$$g(2)=1\ ,\ g'(2)=3\ ,\ g''(2)=-4\ ,\ g'''(2)=2$$

$$T_3(x)=1+\frac{3}{1!}(x-2)-\frac{4}{2!}(x-2)^2+\frac{2}{3!}(x-2)^3$$

$$T_3(1.9)=1+3(1.9-2)-\frac{4}{2!}(1.9-2)^2+\frac{2}{3!}(1.9-2)^3$$

$$T_3(1.9)=1-0.3-0.02-0.000\overline{333}\cong 0.6797$$

$$\therefore\ g(1.9)\approx T_3(1.9)=0.6797$$

We used the Taylor Series of $g(x)$ centered at $x=2$ to approximate $g(1.9)$.

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Taylor and Maclaurin Series 8

Ex] Write the Maclaurin Series for $f(x) = x\sin(2x^2)$ as $\sum_{n=0}^{\infty} C_n x^n$, Find the following coefficients.

Find $C_0, C_1, C_3, C_5, C_7, C_{11}, C_{15}$

Let's start with Maclaurin Series of

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and use it as a building block for developing the Maclaurin Series for

$f(x) = x \sin(2x^2)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Plug in $2x^2$ for x

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x^2)^{2n+1}$$

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \quad 2^{2n+1}}{(2n+1)!} x^{4n+2}$$

multiply both sides by x.

$$x \cdot \sin(2x^2) = x \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{4n+2}$$

Simplify

$$x \cdot \sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{4n+3}$$

$$x\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{4n+3}$$

let's expand series term by term

$$x\sin(2x^2) = \underbrace{2^1}_{C_3} x^3 - \underbrace{\frac{2^3}{3!}}_{C_7} x^7 + \underbrace{\frac{2^5}{5!}}_{C_{11}} x^{11} - \underbrace{\frac{2^7}{7!}}_{C_{15}} x^{15} + \cdots$$

$C_0 = 0$, since series does not have constant term $C_0 x^0$

$C_1 = 0$, since series does not have term $C_1 x^1$

$C_3 = 2$, since series has term $C_3 x^3 = 2x^3$

$C_5 = 0$, since series does not have term $C_5 x^5$

$$x\sin(2x^2) = 2x^3 - \frac{2^3}{3!}x^7 + \frac{2^5}{5!}x^{11} - \frac{2^7}{7!}x^{15} + \cdots$$

$C_7 = \frac{-2^3}{3!}$ since the series has term $C_7 x^7 = \frac{-2^3}{3!}x^7$

$C_{11} = \frac{2^5}{5!}$ since series has term $C_{11}x^{11} = \frac{2^5}{5!}x^{11}$

$C_{15} = \frac{-2^7}{7!}$ since series has term $C_{15}x^{15} = \frac{-2^7}{7!}x^{15}$

Key concept: Match up coefficients C_n with powers of x^n for the series $\sum_{n=0}^{\infty} C_n x^n$

Taylor and Maclaurin Series 9

Ex] let $f(x) = \dfrac{\cos(2x^2) - 1}{x^2}$; Find the 10 th derivative of $f(x)$ at $x = 0$; $f^{(10)}(0)$. Hint: Use the series for $\cos x$ as a building block for developing the Maclaurin series for $f(x)$.

Let's start with the Maclaurin series for $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Plug in $2x^2$ for x

$$\cos(2x^2) = 1 - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!} - \frac{(2x^2)^6}{6!} + \frac{(2x^2)^8}{8!} - + \cdots$$

$$\cos(2x^2) = 1 - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!} - \frac{(2x^2)^6}{6!} + \frac{(2x^2)^8}{8!} - + \cdots$$

Now subtract 1 and divide by x^2 from both sides.

$$\frac{\cos(2x^2) - 1}{x^2} = \frac{\cancel{1} - \cancel{1} - \frac{(2x^2)^2}{2!} + \frac{(2x^2)^4}{4!} - \frac{(2x^2)^6}{6!} + \frac{(2x^2)^8}{8!} \cdots}{x^2}$$

$$\frac{\cos(2x^2) - 1}{x^2} = \frac{-\frac{2^2}{2!}x^4 + \frac{2^4}{4!}x^8 - \frac{2^6}{6!}x^{12} + \frac{2^8}{8!}x^{16} - \cdots}{x^2}$$

$$\frac{\cos(2x^2)-1}{x^2} = \frac{\cancel{x^2}\left(\frac{-2^2}{2!}x^2 + \frac{2^4}{4!}x^6 - \frac{2^6}{6!}x^{10} + \frac{2^8}{8!}x^{14} \ldots\right)}{\cancel{x^2}}$$

$$\frac{\cos(2x^2)-1}{x^2} = \frac{-2^2}{2!}x^2 + \frac{2^4}{4!}x^6 - \frac{2^6}{6!}x^{10} + \frac{2^8}{8!}x^{14} - \ldots$$

We have now developed the Maclaurin series for $f(x)$, we are looking for $f^{10}(0)$, so let's match up the term involving x^{10} with the 10th general term of the Maclaurin series for $f(x)$.

$$\frac{\cos(2x^2)-1}{x^2} = \frac{-2^2}{2!}x^2 + \frac{2^4}{4!}x^6 - \frac{2^6}{6!}x^{10} + \frac{2^8}{8!}x^{14} - \cdots$$

$$\frac{f^{(10)}(0)}{10!}\cancel{x^{10}} = \frac{-2^6}{6!}\cancel{x^{10}}$$

$$f^{(10)}(0) = \frac{-2^6\, 10!}{6!}$$

Key concept: match up the term with x^{10} with $\frac{f^{(10)}(0)}{10!}x^{10} = \frac{-2^6}{6!}x^{10}$ and solve for $f^{(10)}(0)$

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Taylor and Maclaurin Series 10

Ex] Evaluate $\lim_{x \to 0} \dfrac{\cos x - 1 + x^2/2}{2x^4}$

Hint: Apply the Maclaurin series for cosx

Because the limit involves values of x near 0 we can substitute the Maclaurin series of cosx and evaluate the limit.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - + \cdots$$

$$\lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{2x^4} = \lim_{x\to 0}\frac{1 - x^2/2! + x^4/4! - x^6/6! + \cdots - 1 + x^2/2}{2x^4}$$

$$= \lim_{x\to 0}\frac{\cancel{1} - \cancel{x^2/2!} + x^4/4! - x^6/6! + \cdots \cancel{-1} + \cancel{\frac{x^2}{2}}}{2x^4}$$

$$= \lim_{x\to 0}\frac{x^4/4! - x^6/6! + \cdots}{2x^4}$$

$$= \lim_{x\to 0}\frac{\cancel{x^4}(1/4! - x^2/6! + \cdots)}{2\cancel{x^4}}$$

$$= \lim_{x\to 0}\frac{(1/4! - x^2/6! + \cdots)}{2}$$

$$\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{2x^4} = \lim_{x\to 0} \frac{(1/4! - x^2/6! + \cdots)}{2}$$

$$= \frac{1/4!}{2} = \frac{1/24}{2} = \frac{1}{48}$$

Remark: It may not be obvious how many terms of the Maclaurin series to use, general rule of thumb: Use the first 4 to 6 terms of Taylor series. In this example we used the first 4 terms of the Maclaurin Series for cos x.

Taylor and Maclaurin Series II

Ex] Evaluate $\int \frac{\sin x}{2x}\,dx$ as an infinite series, Find the first five non-zero terms of the Maclaurin series and also find the radius of convergence.

Solution: Let's start with the Maclaurin series of $\sin x$ and use it as a building block for developing the Maclaurin series for $\int \frac{\sin x}{2x}\,dx$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{2x} = \frac{x}{2x} - \frac{x^3}{3!(2x)} + \frac{x^5}{5!(2x)} - \frac{x^7}{7!(2x)} + \cdots$$

$$\frac{\sin x}{2x} = \frac{1}{2} - \frac{x^2}{3!(2)} + \frac{x^4}{5!(2)} - \frac{x^6}{7!(2)} + \cdots$$

$$\int \frac{\sin x}{2x}\,dx = \int \left(\frac{1}{2} - \frac{x^2}{3!(2)} + \frac{x^4}{5!(2)} - \frac{x^6}{7!(2)} + \cdots \right) dx$$

$$\int \frac{\sin x}{2x}\,dx = C + \frac{1}{2}x - \frac{x^3}{3(3!)(2)} + \frac{x^5}{5(5!)(2)} - \frac{x^7}{7(7!)(2)}$$

First 5 terms of Maclaurin Series including C.

$$\int \frac{\sin x}{2x}\,dx = C + \frac{x}{2} - \frac{x^3}{6(3!)} + \frac{x^5}{10(5!)} - \frac{x^7}{14(7!)} + \cdots$$

For the Radius of convergence, $\sin x$ has radius of convergence $R=\infty$ and interval of convergence $(-\infty,\infty)$, dividing $\sin x$ by x and integrating does not affect the Radius of convergence, so the radius of convergence of the resulting series remains the same $R=\infty$ and converges for all values of x.

Taylor and Maclaurin Series 12

EX] Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ as a function of x.

Let's start with the Maclaurin series for e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty$$

rewrite $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$

Replace x by $-x^2$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Replace x by $-x^2$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

Summary:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = e^{-x^2}$$

The replacement of x with $-x^2$ is permitted since $-x^2$ is inside the interval of convergence of the series for e^x that is $-\infty < -x^2 \leq 0 < \infty$

Ex] Find the sum of the series given by recognizing it as a Taylor Series evaluated at a particular value of x.

$$2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots+\frac{2^n}{n!}+\cdots$$

Above series almost fits the pattern of Maclaurin series for e^x evaluated at $x=2$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots$$

Now evaluate series at $x=2$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$ Plug in $x=2$

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + \frac{2^n}{n!} + \cdots$$

Now subtract 1 from both sides to find the sum of the given series

$$e^2 - 1 = 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + \frac{2^n}{n!} + \cdots$$

Key concept: Try to fit the pattern of a given series with one of the known Maclaurin series such as e^x, $\sin x$, $\cos x$, $\tan^{-1} x$, ...

Some common Maclaurin Series to remember:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$-\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad -\infty < x < \infty$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \; ; -1 \le x \le 1$$

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Taylor and Maclaurin Series 13

Let $F(x) = \int_0^x \sin(2t^2)\,dt$; Find the Maclaurin series for $F(x)$ up to and including the x^7 term, Use this truncated part of the Maclaurin series to estimate the value of $\int_0^{0.1} \sin(2x^2)\,dx$, Estimate the error of this approximation.

Solution: Let's start with the Maclaurin series of $\sin t$ and use it as a building block for developing the Maclaurin Series for $\int_0^x \sin(2t^2)\,dt$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

$$\sin(2t^2) = 2t^2 - \frac{(2t^2)^3}{3!} + \frac{(2t^2)^5}{5!} - \frac{(2t^2)^7}{7!} + \cdots$$

$$\sin(2t^2) = 2t^2 - \frac{2^3 t^6}{3!} + \frac{2^5 t^{10}}{5!} - \frac{2^7 t^{14}}{7!} + \cdots$$

$$\int_0^x \sin(2t^2)\, dt = \int_0^x \left[2t^2 - \frac{2^3 t^6}{3!} + \frac{2^5 t^{10}}{5!} - \frac{2^7 t^{14}}{7!} + \cdots \right] dt$$

$$= \frac{2t^3}{3} - \frac{2^3 t^7}{7(3!)} + \frac{2^5 t^{11}}{11(5!)} - \frac{2^7 t^{15}}{15(7!)} \Bigg|_0^x$$

Apply F.T.C

$$\int_0^x \sin(2t^2)\,dt = \frac{2t^3}{3} - \frac{2^3t^7}{7(3!)} + \frac{2^5t^{11}}{11(5!)} \Big|_0^x$$ Apply F.T.C

$$\int_0^x \sin(2t^2)\,dt = \frac{2x^3}{3} - \frac{2^3x^7}{7(3!)} + \frac{2^5x^{11}}{11(5!)} - \cdots$$

$$F(x) = \int_0^x \sin(2t^2)\,dt = \frac{2x^3}{3} - \frac{2^3x^7}{7(3!)}$$

Now we can use this truncated part of the series up to the x^7 term to estimate:

$$\int_0^{0.1} \sin(2t^2)\,dt$$

$$F(x) = \int_0^x \sin(2t^2)\,dt = \frac{2x^3}{3} - \frac{2^3 x^7}{7(3!)}$$

$$F(0.1) = \int_0^{0.1} \sin(2t^2)\,dt = \frac{2(0.1)^3}{3} - \frac{2^3(0.1)^7}{7(3!)}$$

$$\approx 0.000666648$$

We can estimate the error of this approximation by applying the Alternating series estimation theorem.

$|S - S_n| \leq |a_{n+1}|$; If we add the first n terms the error is less than $|a_{n+1}|$.

$$F(x) = \int_0^x \sin(2t^2)\,dt = \frac{2x^3}{3} - \frac{2^3 x^7}{7(3!)} + \frac{2^5 x^{11}}{11(5!)} - \cdots$$

$$F(0.1) = \int_0^{0.1} \sin(2t^2)\,dt = \frac{2(0.1)^3}{3} - \frac{2^3(0.1)^7}{7(3!)} + \frac{2^5(0.1)^{11}}{11(5!)} - \cdots$$

If we add the first 2 terms the error is less than the $|\text{third term}|$, that is $|S - S_2| \leq |a_3|$

$$|\text{Error}| \leq \frac{2^5(0.1)^{11}}{11(5!)} \approx 2.424 \times 10^{-13}$$

Taylor and Maclaurin Series 14

Ex] Find the first three nonzero terms in the Maclaurin Series for $f(x) = e^{2x}\sin x$

Solution: We can apply the Taylor Series formula directly by differentiating $f(x)$ many times and evaluating derivatives at $x=0$, but since we know the Maclaurin Series of e^{2x} and $\sin x$ we can simply multiply them and collect the first 3 terms.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$ Replace x by 2x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^{2x}\sin x = \left(1 + \frac{2x}{1!} + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{2x^2}{1!} - \frac{2x^4}{3!} + \frac{2x^6}{5!} + \frac{2^2x^3}{2!} - \frac{2^2x^5}{2!\,3!} + \frac{2^2x^7}{2!\,5!}$$

$$+ \frac{2^3x^4}{3!} - \frac{2^3x^6}{3!\,3!} + \frac{2^3x^8}{3!\,5!} - \cdots$$

Collecting the first 3 terms in order of increasing powers of x we obtain:

$$e^{2x}\sin x = x + 2x^2 + \left(\frac{-1}{3!} + \frac{2^2}{2!}\right)x^3 + \cdots$$

$$e^{2x}\sin x = x + 2x^2 + \frac{11}{6}x^3 + \cdots$$

Taylor and Maclaurin Series 15

Ex] Use series to evaluate $\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x}$

Hint: Apply the Maclaurin series for $\cos x$ and e^x.
Because the limit involves values of x near 0 we can substitute the Maclaurin series of $\cos x$ and e^x and evaluate the limit.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots\right)}{1+x-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{1}-\cancel{1}+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\cdots}{\cancel{1}+\cancel{x}-\cancel{1}-\cancel{x}-\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}-\cdots}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\cdots}{\frac{-x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}-\cdots}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots}{\frac{-x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \cdots}$$

Factor out x^2 from Numerator and Denominator

$$= \lim_{x\to 0} \frac{\cancel{x^2}\left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots\right)}{\cancel{x^2}\left(\frac{-1}{2!} - \frac{x}{3!} - \frac{x^2}{4!} - \cdots\right)}$$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots\right)}{\frac{-1}{2!} - \frac{x}{3!} - \frac{x^2}{4!} - \cdots}$$

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{\left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots\right)}{\frac{-1}{2!} - \frac{x}{3!} - \frac{x^2}{4!}}$$

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \frac{\frac{1}{2!} - 0 + 0 - 0 + \cdots}{\frac{-1}{2!} - 0 - 0 - \cdots}$$

Plug in $x=0$

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \frac{1}{\cancel{2}} \cdot \frac{-\cancel{2}}{1} = -1$$

Remark: It may not be obvious how many terms of the Maclaurin Series to use, general rule of thumb: use the first 4 to 6 terms of Taylor series. In this example we used the first 5 terms of the Taylor series for e^x and $\cos x$.

Summary: Given $\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x}$ plug in Maclaurin series of e^x and $\cos x$, factor and simplify and evaluate the limit.

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