Trigonometric Substitution 1

Integrand : $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

we can eliminate the radical sign $\sqrt{}$ by applying following trig. substitution

Integrand	Trig. Substitution		Trig. Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ $-\frac{\pi}{2}\leq\theta\leq\pi/2$		$\cos^2\theta=1-\sin^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$ $-\frac{\pi}{2}<\theta<\pi/2$		$\sec^2\theta=1+\tan^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$0<\theta<\frac{\pi}{2}$	$\tan^2\theta=\sec^2\theta-1$

Compute ∫ 1/√(4+x^2) dx , Trigonometric substitution solved example

Ex Compute $\int \frac{1}{\sqrt{4+x^2}}\,dx$

Pattern $\sqrt{a^2+x^2}$ $\quad x = a\tan\theta \quad dx = a\sec^2\theta\,d\theta$

Pattern $\sqrt{2^2+x^2}$ $\quad x = 2\tan\theta \quad dx = 2\sec^2\theta\,d\theta$

$$\sqrt{4+x^2} = \sqrt{4+(2\tan\theta)^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta}$$

$$= 2\sqrt{\sec^2\theta} = 2\,|\sec\theta| = 2\cdot\sec\theta \quad \text{since}$$

$$\sec\theta = \frac{1}{\cos\theta} > 0 \quad \text{when} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{1}{\cancel{2}\sec\theta}\cdot\cancel{2}\sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$

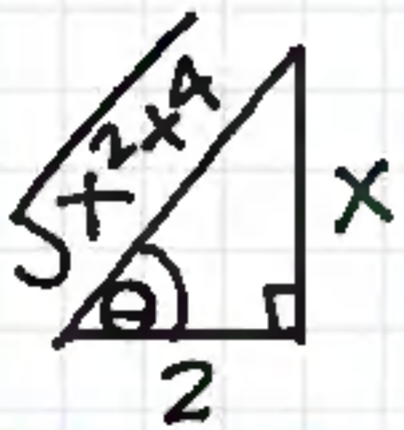
Recall: $\boxed{1+\tan^2\theta = \sec^2\theta}$

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

now we need to express answer in terms of x

$x = 2\tan\theta \qquad \tan\theta = \dfrac{x}{2} = \dfrac{\text{opp}}{\text{adj}}$

Reference Triangle



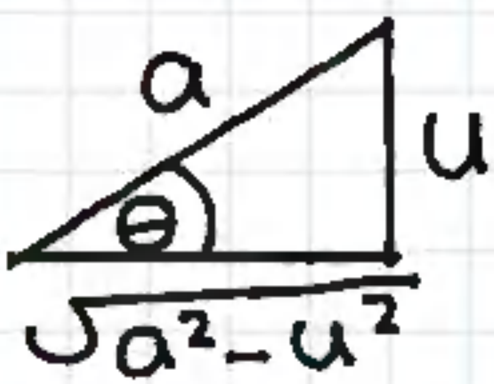
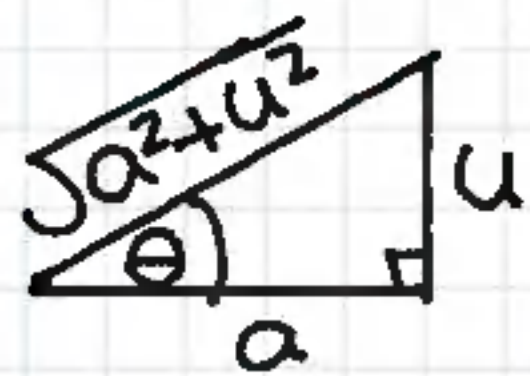
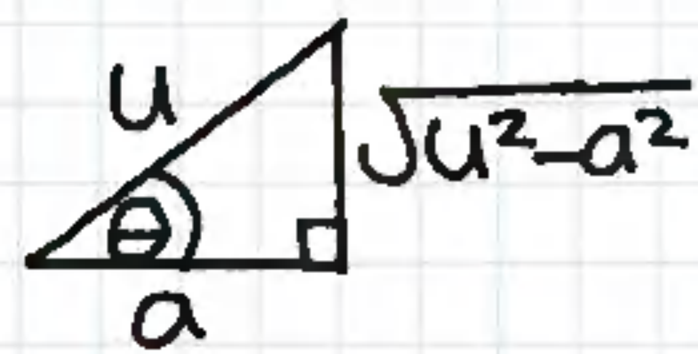
$$\tan\theta = \frac{x}{2}\ ; \ \sec\theta = \frac{\sqrt{x^2+4}}{2} = \frac{\text{Hypot.}}{\text{Adjacent}}$$

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\Big|\overbrace{\sec\theta}^{\frac{\sqrt{x^2+4}}{2}} + \overbrace{\tan\theta}^{\frac{x}{2}}\Big| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

Trigonometric Substitution Strategy review and reference triangles

Strategy Review: Trigonometric Substitution

$\sqrt{a^2-u^2}$	$\sqrt{a^2+u^2}$	$\sqrt{u^2-a^2}$
$u=a\sin\theta$	$u=a\tan\theta$	$u=a\sec\theta$
$du=a\cos\theta\, d\theta$	$du=a\sec^2\theta\, d\theta$	$du=a\sec\theta\tan\theta\, d\theta$
$1-\sin^2\theta=\cos^2\theta$	$1+\tan^2\theta=\sec^2\theta$	$\sec^2\theta-1=\tan^2\theta$
		
$\sin\theta=\frac{u}{a}$	$\tan\theta=\frac{u}{a}$	$\sec\theta=\frac{u}{a}$

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Integrate ∫ √(4-x^2) dx , Trigonometric Substitution solved example

Trigonometric Substitution 2

Ex| Integrate $\int \sqrt{4-x^2}\, dx$

Pattern: $\sqrt{a^2-x^2}$ $\quad x = a\sin\theta \quad dx = a\cos\theta\, d\theta$

Pattern: $\sqrt{2^2-x^2}$ $\quad x = 2\sin\theta \quad dx = 2\cos\theta\, d\theta$

$$\sqrt{4-x^2} = \sqrt{4-(2\sin\theta)^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta}$$

$$= 2\sqrt{\cos^2\theta} = 2|\cos\theta| = 2\cos\theta \quad \text{Since}$$

$\cos\theta > 0$ when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int \sqrt{4-x^2}\, dx = \int \underbrace{2\cos\theta}_{\sqrt{4-x^2}} \cdot \underbrace{2\cos\theta\, d\theta}_{dx}$$

$$\int \sqrt{4-x^2}\,dx = \int 4\cos^2\theta\,d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\int \sqrt{4-x^2}\,dx = 4\int \cos^2\theta\,d\theta = 4\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 2\int (1+\cos 2\theta)\,d\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= 2\theta + \sin 2\theta + C = 2\theta + \underbrace{2\sin\theta\cos\theta}_{\sin 2\theta} + C$$

$$\int \sqrt{4-x^2}\,dx = 2\theta + 2\sin\theta\cos\theta + C$$

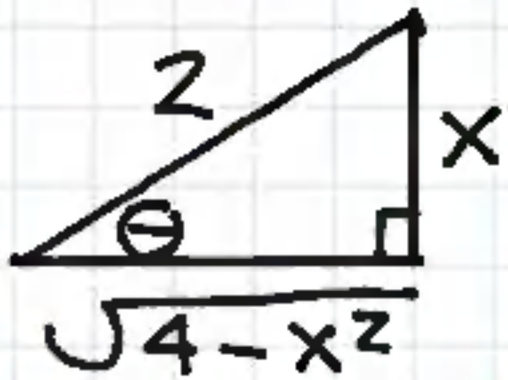
Now we need to express answer in terms of x

Note $\int \cos 2\theta\,d\theta = \frac{\sin(2\theta)}{2} + C$

$$\boxed{\begin{array}{l} u = 2\theta \\ du = 2\,d\theta \end{array}}$$

$$\int \sqrt{4-x^2}\,dx = 2\theta + 2\sin\theta\cos\theta + C$$

$x = 2\sin\theta$ $\quad$ $\sin\theta = \frac{x}{2} \quad \theta = \sin^{-1}(x/2)$



$$\cos\theta = \frac{\sqrt{4-x^2}}{2}$$

Reference triangle

$$\int \sqrt{4-x^2}\,dx = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}(x/2) + \cancel{2}\,(x/\cancel{2})\,\frac{\sqrt{4-x^2}}{2} + C$$

$$= 2\sin^{-1}(x/2) + \frac{x\sqrt{4-x^2}}{2} + C$$

Trig. identities to memorize for applying trigonometric substitution

Some notes

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\int \cos(2\theta)\,d\theta = \frac{\sin(2\theta)}{2} + C$$

$u = 2\theta \quad du = 2\,d\theta$

U - Substitution

Compute ∫1/(x^2 √(x^2-16)) dx , Trigonometric Substitution solved example

Trigonometric Substitution 3

Ex Compute $\int \frac{dx}{x^2\sqrt{x^2-16}}$

$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta \quad dx = a\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-16}$ $\quad x = 4\sec\theta \quad dx = 4\sec\theta\tan\theta\, d\theta$

$$x^2\sqrt{x^2-16} = (4\sec\theta)^2\sqrt{(4\sec\theta)^2-16}$$

$$= 16\sec^2\theta\sqrt{16\sec^2\theta-16} = 64\sec^2\theta\sqrt{\sec^2\theta-1}$$

$$= 64\sec^2\theta\sqrt{\tan^2\theta} = 64\sec^2\theta\,|\tan\theta|$$

$$= 64\sec^2\theta\tan\theta$$

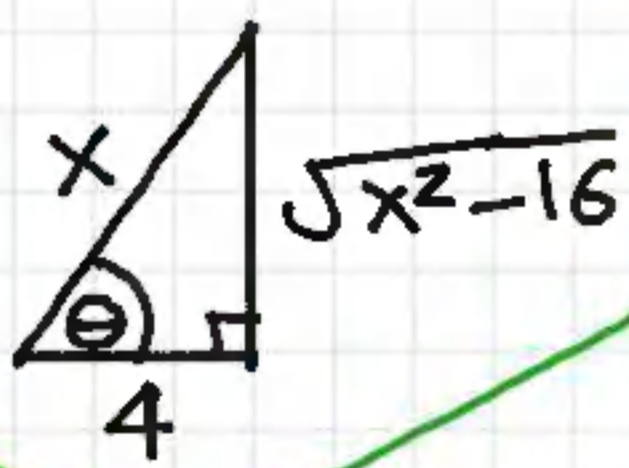
$\sec^2\theta - 1 = \tan^2\theta$
Trig. Identity

$$\int \frac{dx}{x^2\sqrt{x^2-16}} = \int \frac{4\cancel{\sec\theta}\,\cancel{\tan\theta}}{64\,\sec^{\cancel{2}}\theta\,\cancel{\tan\theta}}\,d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + C$$

Now we need to express answer in terms of x

$x = 4\sec\theta$



Reference Triangle

$$\sec\theta = \frac{x}{4} = \frac{\text{Hyp}}{\text{Adj}}$$

$$\sin\theta = \frac{\sqrt{x^2-16}}{x}$$

$$\int \frac{dx}{x^2\sqrt{x^2-16}} = \frac{1}{16}\,\frac{\sqrt{x^2-16}}{x} + C$$

Trigonometric Substitution 4

Ex Evaluate $\int_0^a x^2\sqrt{a^2-x^2}\,dx$

$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta \quad dx = a\cos\theta\,d\theta$

$x^2\sqrt{a^2-x^2} = (a\sin\theta)^2\sqrt{a^2-(a\sin\theta)^2}$

$= a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta} = a^3\sin^2\theta\sqrt{1-\sin^2\theta}$

$= a^3\sin^2\theta\sqrt{\cos^2\theta} = a^3\sin^2\theta\,|\cos\theta| = a^3\sin^2\theta\cos\theta$

Change limits:

$x = 0 \Rightarrow \theta = \sin^{-1}(x/a) \Rightarrow \theta = \sin^{-1}0 = 0$

$x = a \Rightarrow \theta = \sin^{-1}(x/a) \Rightarrow \theta = \sin^{-1}(1) = \pi/2$

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^3 \sin^2\theta \cos\theta \; a\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} a^4 \sin^2\theta \cos^2\theta \, d\theta = a^4 \int_0^{\pi/2} (\sin\theta\cos\theta)^2 d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$ Trig. Identity

$$a^4 \int_0^{\pi/2} (\sin\theta\cos\theta)^2 d\theta = a^4 \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} \sin^2(2\theta)\, d\theta$$

$\sin^2(2\theta) = \frac{1}{2}(1-\cos 4\theta)$ Trig. Identity

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \frac{a^4}{4}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} \frac{1}{2}\left(1-\cos(4\theta)\right)\,d\theta$$

$$= \frac{a^4}{8}\int_0^{\pi/2} \left(1-\cos(4\theta)\right)d\theta = \frac{a^4}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]\Bigg|_0^{\pi/2}$$

$$= \frac{a^4}{8}\left(\left[\frac{\pi}{2} - \frac{\overset{=0}{\sin(2\pi)}}{4}\right] - \left[0 - \frac{\overset{=0}{\sin 0}}{4}\right]\right)$$

$$= \frac{a^4}{8}\left(\frac{\pi}{2} - 0 + 0\right) = \frac{\pi a^4}{16}$$

Trig. Identity + U-Substitution Review

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$$

$$\sin^2(2\theta) = \frac{1}{2}(1-\cos(4\theta))$$

$$\int \cos(4\theta)\,d\theta = \frac{\sin(4\theta)}{4} + C$$

$$u = 4\theta \qquad du = 4\,d\theta \qquad \frac{du}{4} = d\theta$$

Evaluate ∫ √(9x^2-25)/x^3 dx , Trigonometric substitution solved example

Trigonometric Substitution 5

Ex| Evaluate $\int \frac{\sqrt{9x^2-25}}{x^3}\,dx$

$\sqrt{u^2-a^2}$ $\quad u=a\sec\theta \quad du=a\sec\theta\tan\theta\,d\theta$

$\sqrt{(3x)^2-(5)^2}$ $\quad 3x=5\sec\theta \quad dx=\frac{5\sec\theta\tan\theta\,d\theta}{3}$

$\sqrt{9x^2-25}=\sqrt{(5\sec\theta)^2-25}=\sqrt{25(\sec^2\theta-1)}$

$=5\sqrt{\tan^2\theta}=5|\tan\theta|=5\tan\theta$

$\sqrt{9x^2-25}=5\tan\theta$

$$\int \frac{\sqrt{9x^2-25}}{x^3}\,dx$$

$x = \frac{5}{3}\sec\theta$

$dx = \frac{5}{3}\sec\theta\tan\theta\,d\theta$

$\sqrt{9x^2-25} = 5\tan\theta$

$x^3 = \frac{125}{27}\sec^3\theta$

$$\int \frac{\sqrt{9x^2-25}}{x^3}\,dx = \int \frac{5\tan\theta}{\frac{125}{27}\sec^3\theta}\,\frac{5}{3}\sec\theta\tan\theta\,d\theta$$

$$= \frac{\cancel{25}}{\cancel{3}}\cdot\frac{\overset{9}{\cancel{27}}}{\underset{5}{\cancel{125}}}\int \frac{\tan^2\theta\,\cancel{\sec\theta}}{\sec^{\cancel{3}\,2}\theta}\,d\theta = \frac{9}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$\int \frac{\sqrt{9x^2-25}}{x^3}\,dx = \frac{9}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{9}{5}\int \frac{\sec^2\theta - 1}{\sec^2\theta}\,d\theta = \frac{9}{5}\int \left(1 - \frac{1}{\sec^2\theta}\right) d\theta$$

$$= \frac{9}{5}\int (1 - \cos^2\theta)\,d\theta = \frac{9}{5}\int \left(1 - \frac{1}{2}(1+\cos(2\theta)\right)d\theta$$

$$= \frac{9}{5}\int \left[\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right] d\theta$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{5}\left[\ \frac{1}{2}\theta - \frac{1}{2}\ \frac{\sin 2\theta}{2}\right] + C$$

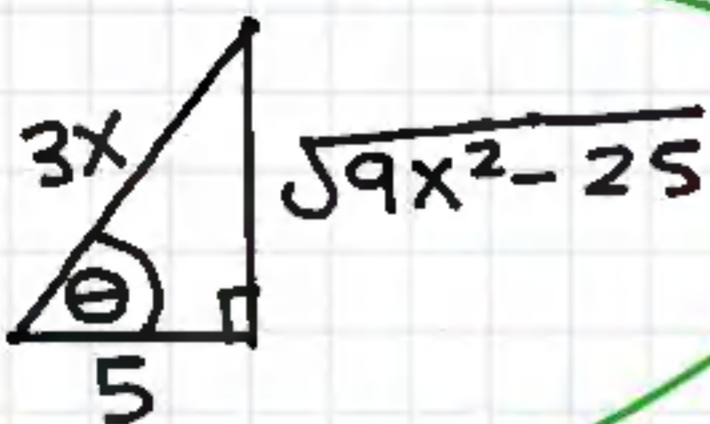
$$= \frac{9}{5}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right] + C \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$= \frac{9}{5}\left[\frac{1}{2}\theta - \frac{1}{4}\cdot 2\sin\theta\cos\theta\right] + C$$

$$= \frac{9}{10}\theta - \frac{9}{10}\sin\theta\cos\theta + C$$

Now we need to express answer in terms of x

$\sec\theta = 3x/5$



Reference triangle

$$3x = 5\sec\theta \qquad \sec\theta = \frac{3x}{5} = \frac{\text{hyp.}}{\text{adj.}}$$

$$\theta = \sec^{-1}(3x/5)$$

$$\sin\theta = \frac{\sqrt{9x^2-25}}{3x}$$

$$\cos\theta = \frac{5}{3x}$$

$$\int \frac{\sqrt{9x^2-25}}{x^3}\,dx = \frac{9}{10}\theta - \frac{9}{10}\sin\theta\cos\theta + C$$

$$\theta = \sec^{-1}(3x/5) \; ; \; \sin\theta = \frac{\sqrt{9x^2-25}}{3x} \; ; \; \cos\theta = \frac{5}{3x}$$

$$= \frac{9}{10}\sec^{-1}(3x/5) - \frac{9}{10}\,\frac{\sqrt{9x^2-25}}{3x}\cdot\frac{5}{3x}$$

$$= \frac{9}{10}\sec^{-1}(3x/5) - \frac{45}{90x^2}\sqrt{9x^2-25} + C$$

$$= \frac{9}{10}\sec^{-1}(3x/5) - \frac{1}{2x^2}\sqrt{9x^2-25} + C$$

Trigonometric Identities Review

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\int \cos(2\theta)\, d\theta = \frac{\sin(2\theta)}{2} + C$$

U-Substitution

$u = 2\theta \quad du = 2d\theta$

$d\theta = \frac{du}{2}$

Trigonometric Substitution 6

Ex| Evaluate $\displaystyle\int_0^{3/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$

$\sqrt{u^2+a^2}$ $\quad u = a\tan\theta \quad du = a\sec^2\theta\, d\theta$

$\sqrt{(2x)^2+(3)^2}$ $\quad 2x = 3\tan\theta \quad 2\,dx = 3\sec^2\theta\, d\theta$

$(4x^2+9)^{3/2} = \left(\left((2x)^2+(3)^2\right)^{1/2}\right)^3$

$(4x^2+9)^{3/2} = \left(\sqrt{4x^2+9}\right)^3$

$x = 0 \Rightarrow \theta = \tan^{-1}(2x/3) \Rightarrow \theta = \tan^{-1} 0 = 0$

$x = 3/2 \Rightarrow \theta = \tan^{-1}(2x/3) \Rightarrow \theta = \tan^{-1}(1) = \pi/4$

$$\int_0^{3/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{3/2} \frac{x^3}{[(2x)^2+(3)^2]^{3/2}}\,dx$$

$$2x = 3\tan\theta \qquad x = \frac{3}{2}\tan\theta \qquad dx = \frac{3}{2}\sec^2\theta\, d\theta$$

$$x^3 = \left(\frac{3}{2}\tan\theta\right)^3 = \frac{27}{8}\tan^3\theta$$

$$[(2x)^2+(3)^2]^{3/2} = [(3\tan\theta)^2+(3)^2]^{3/2}$$

$$\int_0^{3/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{\pi/4} \frac{27}{8}\frac{\tan^3\theta}{(9\tan^2\theta+9)^{3/2}} \cdot \frac{3}{2}\sec^2\theta\, d\theta$$

$$= \frac{27}{8}\cdot\frac{3}{2}\,\frac{1}{9^{3/2}} \int_0^{\pi/4} \frac{\tan^3\theta\sec^2\theta}{(\tan^2\theta+1)^{3/2}}\,d\theta$$

$$\int_0^{3/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \frac{3}{16}\int_0^{\pi/4} \frac{\tan^3\theta\,\sec^2\theta}{(\tan^2\theta+1)^{3/2}}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/4} \frac{\tan^3\theta\,\sec^2\theta}{(\sec^{\cancel{2}}\theta)^{3/\cancel{2}}}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/4} \frac{\tan^3\theta\,\cancel{\sec^2\theta}}{\sec^{\cancel{3}\,1}\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/4} \tan^3\theta \cdot \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/4} \frac{\sin^3\theta}{\cos^{\cancel{3}\,2}\theta} \cdot \cancel{\cos}\theta\,d\theta$$

$$\int_0^{3/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \frac{3}{16}\int_0^{\pi/4} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta}\cdot\sin\theta\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/4} \frac{(1-\cos^2\theta)}{\cos^2\theta}\cdot\sin\theta\,d\theta$$

$u = \cos\theta \quad du = -\sin\theta\,d\theta$ U-Substitution

$\theta = 0 \Rightarrow u = \cos\theta \Rightarrow u = \cos 0 = 1$

$\theta = \pi/4 \Rightarrow u = \cos\theta \Rightarrow u = \cos(\pi/4) = 1/\sqrt{2}$

$$\frac{3}{16}\int_0^{\pi/4} \frac{(1-\cos^2\theta)}{\cos^2\theta} \sin\theta \, d\theta$$

$$= -\frac{3}{16}\int_1^{1/\sqrt{2}} \frac{(1-u^2)}{u^2} \, du$$

$$= -\frac{3}{16}\int_1^{1/\sqrt{2}} \left(\frac{1}{u^2}-1\right) du = -\frac{3}{16}\int_1^{1/\sqrt{2}} (u^{-2}-1)\, du$$

$$= -\frac{3}{16}\left[\frac{u^{-1}}{-1} - u\right]\Big|_1^{1/\sqrt{2}} = -\frac{3}{16}\left[\frac{-1}{u} - u\right]\Big|_1^{1/\sqrt{2}}$$

$$= -\frac{3}{16}\left[\frac{-1}{1/\sqrt{2}} - \frac{1}{\sqrt{2}} - \left(\frac{-1}{1} - 1\right)\right]$$

$$= -\frac{3}{16}\left[-\sqrt{2} - \frac{1}{\sqrt{2}} + 2\right]$$

Evaluate ∫ √(2x-x^2) dx ,by completing the square and trig. substitution example

Trigonometric Substitution 7

Ex| Evaluate $\int \sqrt{2x-x^2}\,dx$

Complete the square:

$$-x^2+2x=-1[x^2-2x+1-1]=-[x^2-2x+1]+1$$

$$=-(x-1)^2+1$$

$$\sqrt{2x-x^2}=\sqrt{1-(x-1)^2}$$

$\sqrt{a^2-u^2}$ $\quad u=a\sin\theta \quad du=a\cos\theta\,d\theta$

$\sqrt{1^2-(x-1)^2}$ $\quad x-1=1\sin\theta \quad dx=\cos\theta\,d\theta$

$$\int \sqrt{2x-x^2}\,dx = \int \sqrt{1^2-(x-1)^2}\,dx$$

$x-1=\sin\theta \qquad dx=\cos\theta\,d\theta$

$$\sqrt{1-(x-1)^2} = \sqrt{1-(\sin\theta)^2} = \sqrt{\cos^2\theta} = |\cos\theta|$$

$$= \cos\theta$$

$$\int \sqrt{1-(x-1)^2}\,dx = \int \overbrace{\cos\theta}^{\sqrt{1-(x-1)^2}} \cdot \overbrace{\cos\theta\,d\theta}^{dx}$$

$$= \int \cos^2\theta\,d\theta = \int \frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$$

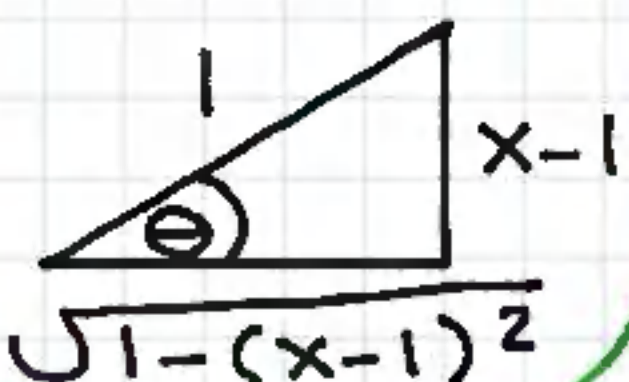
$$\int \sqrt{2x - x^2}\, dx = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{2}\left[\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

Now we need to express answer in terms of x

$x - 1 = \sin\theta$



Reference triangle

$\theta = \sin^{-1}(x-1)$

$$\sin\theta = \frac{x-1}{1} = \frac{\text{opp.}}{\text{Hypot.}}$$

$$\cos\theta = \frac{\sqrt{1-(x-1)^2}}{1} = \frac{\text{adj.}}{\text{Hypot.}}$$

$$\int \sqrt{2x - x^2}\, dx = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$\theta = \sin^{-1}(x-1)$; $\sin\theta = x - 1$; $\cos\theta = \sqrt{1-(x-1)^2}$

$$= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\cdot\sqrt{1-(x-1)^2} + C$$

Trig. identities we applied

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

Completing the square short cut by using the vertex formula x=-b/(2a)

Review notes

Completing the square SHORTCUT!

$y = ax^2 + bx + C \Rightarrow y = a(x-h)^2 + k$

$y = -x^2 + 2x$

$a = -1 \quad b = 2$

$h =$ x coordinate of Vertex

$k =$ y coordinate of Vertex

$y = -x^2 + 2x \qquad a = -1 \quad b = 2$

$y = -1[x-h]^2 + k$

$h = -\frac{b}{2a} = \frac{-2}{2(-1)} = 1$

$h = 1 \rightarrow y = -x^2 + 2x$

$y = -(1)^2 + 2(1) = 1$

$\therefore k = 1$

$y = -1[x-1]^2 + k$

$-x^2 + 2x = -(x-1)^2 + 1$

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Evaluate ∫ 1/√(-x^2-12x+64) dx ,Trig. Substitution and completing the square

Trigonometric Substitution 8

Ex| Evaluate $\int \frac{1}{\sqrt{-x^2-12x+64}}\,dx$

Complete the square:

$-x^2-12x+64 = -1[x^2+12x+36-36]+64$

$= -[x+6]^2+100$

$-x^2-12x+100 = -[x+6]^2+100$

$\sqrt{-x^2-12x+64} = \sqrt{10^2-(x+6)^2}$

$\sqrt{a^2-u^2} \quad u = a\sin\theta \quad du = a\cos\theta\, d\theta$

$\sqrt{10^2-(x+6)^2} \quad x+6 = 10\sin\theta \quad dx = 10\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{-x^2-12x+64}}\,dx = \int \frac{1}{\sqrt{10^2-(x+6)^2}}\,dx$$

$x+6 = 10\sin\theta \qquad dx = 10\cos\theta$

$$\sqrt{10^2-(x+6)^2} = \sqrt{100-(10\sin\theta)^2} = 10\sqrt{1-\sin^2\theta}$$

$$= 10\sqrt{\cos^2\theta} = 10\,|\cos\theta| = 10\cos\theta$$

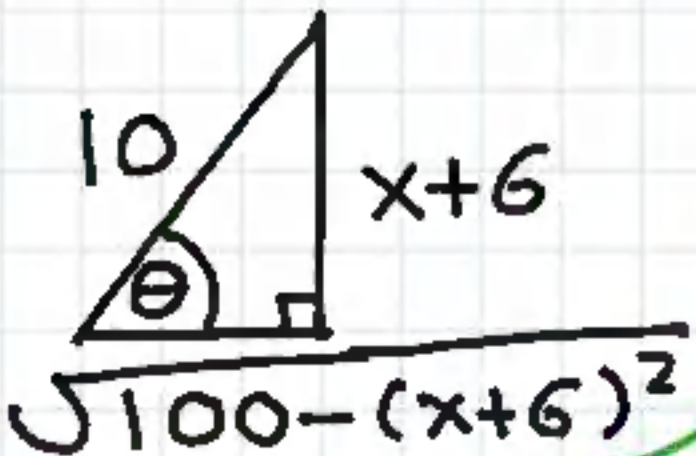
$$\int \frac{1}{\sqrt{10^2-(x+6)^2}}\,dx = \int \frac{1}{\cancel{10}\,\cancel{\cos\theta}} \cdot \cancel{10}\,\cancel{\cos\theta}\,d\theta$$

$$= \int 1\,d\theta = \theta + C$$

$$\int \frac{1}{\sqrt{-x^2-12x+64}}\,dx = \theta + C$$

now we need to express answer in terms of x

$$x+6 = 10\sin\theta$$



Reference Triangle

$$\sin\theta = \frac{x+6}{10}$$

$$\theta = \sin^{-1}\left(\frac{x+6}{10}\right)$$

$$\int \frac{1}{\sqrt{-x^2-12x+64}}\,dx = \sin^{-1}\left(\frac{x+6}{10}\right) + C$$

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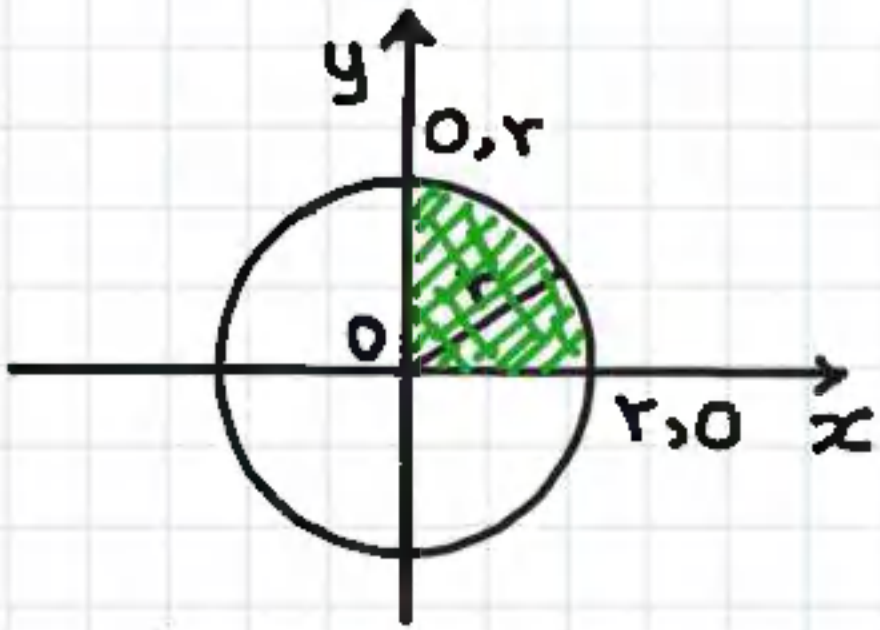
Apply Trig. Substitution to prove that the area of a circle is πr^2

Trigonometric Substitution 9

Ex] Prove that the area of a circle of radius r is πr^2.

Equation of a circle is $x^2+y^2=r^2$



$y^2=r^2-x^2$

$y=\pm\sqrt{r^2-x^2}$

$y=\sqrt{r^2-x^2}$ is the upper half of a circle centered at the origin. So the area of full circle is $4\int_0^r \sqrt{r^2-x^2}\,dx$

$$A=4\int_0^r \sqrt{r^2-x^2}\,dx$$

$$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx$$

$\sqrt{r^2 - x^2}$ $\quad x = r\sin\theta \quad dx = r\cos\theta\,d\theta$

$$\sqrt{r^2 - x^2} = \sqrt{r^2 - (r\sin\theta)^2} = \sqrt{r^2 - r^2\sin^2\theta}$$

$$= \sqrt{r^2(1 - \sin^2\theta)} = r\sqrt{\cos^2\theta} = r|\cos\theta| = r\cos\theta$$

Change limits: $\sin\theta = \frac{x}{r} \Rightarrow \theta = \sin^{-1}(x/r)$

$x = 0 \Rightarrow \theta = \sin^{-1}(x/r) \Rightarrow \theta = \sin^{-1}(0) = 0$

$x = r \Rightarrow \theta = \sin^{-1}(x/r) \Rightarrow \theta = \sin^{-1}(1) = \pi/2$

$$4\int_0^r \sqrt{r^2 - x^2}\,dx = 4\int_0^{\pi/2} \underbrace{r\cos\theta}_{\sqrt{r^2 - x^2}} \cdot \underbrace{r\cos\theta\,d\theta}_{dx}$$

$$4\int_0^r \sqrt{r^2-x^2}\,dx = 4\int_0^{\pi/2} \underbrace{r\cos\theta}_{\sqrt{r^2-x^2}}\,\underbrace{r\cos\theta\,d\theta}_{dx}$$

$$= 4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= 4r^2\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 2r^2\int_0^{\pi/2} (1+\cos 2\theta)\,d\theta$$

$$= 2r^2\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2}$$

$$= 2r^2\left[\left(\frac{\pi}{2} + \frac{\overset{=0}{\sin\pi}}{2}\right) - \left(0 + \frac{\overset{=0}{\sin 0}}{2}\right)\right] = \pi r^2$$

Area of circle of radius r

Trig. Identity + U-Substitution Review

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\int \cos(2\theta)\, d\theta = \frac{\sin(2\theta)}{2} + C$$

$u = 2\theta \qquad du = 2\, d\theta \qquad d\theta = \frac{du}{2}$

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