Calculus 2 ∫ U-Substitution method introduction notes

Substitution Method 1 (Introduction)

Differentials

$y = f(x)$ $\qquad \frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)\,dx$

$y = f(x)$	Derivative	Differential
$y = x^5$	$\frac{dy}{dx} = 5x^4$	$dy = 5x^4\,dx$
$u = \sin x$	$\frac{du}{dx} = \cos x$	$du = \cos x\,dx$
$u = e^x$	$\frac{du}{dx} = e^x$	$du = e^x\,dx$
$u = \sin^{-1} x$	$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$	$du = \frac{1}{\sqrt{1-x^2}}\,dx$

Chain Rule

$y = f(g(x))$ $\qquad$ $y' = f'(g(x)) \cdot g'(x)$

Ex $\quad y = \sin(x^2)$ $\qquad$ $\frac{dy}{dx} = \cos(x^2) \cdot 2X$

Ex $\quad y = e^{x^2}$ $\qquad$ $\frac{dy}{dx} = e^{x^2} \cdot 2X$

Ex $\quad y = \tan(x^2)$ $\qquad$ $\frac{dy}{dx} = [\sec(x^2)]^2 \cdot 2X$

Ex $\quad y = (ax+b)^n$ $\qquad$ $\frac{dy}{dx} = n[ax+b]^{n-1} \cdot a$

Ex $\quad y = Ln(x^2+1)$ $\qquad$ $\frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2X$

Integration U substitution method

Chain Rule

$$\int \underbrace{f(g(x))}_{f(u)} \cdot \underbrace{g'(x)\,dx}_{du}$$

$u = g(x) \qquad du = g'(x)\,dx$

$$\frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} F(g(x)) = f(g(x)) \cdot g'(x)$$

$$\int f(u)\,du = F(u) + C = F(g(x)) + C$$

Guidelines

Step 1: Substitute $u = g(x) \quad du = g'(x)\,dx$

Step 2: Transform $\int f(g(x)) \cdot g'(x)\,dx \Rightarrow \int f(u)\,du$

Step 3: Integrate simpler function $f(u)$ with respect to u.

Find ∫ 1/(2x+1) dx solved example (U-Substitution method)

Step 4: Substitute $u = g(x)$ and obtain result in terms of x

Ex Find $\int \frac{1}{2x+1}\,dx$

$u = 2x+1 \quad du = 2\,dx$

$dx = \frac{du}{2}$

$$\int \frac{1}{u}\,\frac{du}{2} \Rightarrow \frac{1}{2}\int \frac{1}{u}\,du \Rightarrow \frac{1}{2}\ln|u| + C$$

$$\Rightarrow \frac{1}{2}\ln|2x+1| + C$$

Ex $\int \frac{x}{x^2+1}\,dx$

$\int \frac{1}{x^2+1} \cdot x\,dx$

$u = x^2+1 \quad du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int \underbrace{\boxed{\frac{1}{x^2+1}}}_{\frac{1}{u}} \underbrace{\boxed{x\,dx}}_{\frac{du}{2}}$$

$$\int \frac{1}{u}\,\frac{du}{2} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C$$

$$= \frac{1}{2}\ln(x^2+1) + C$$

Note $|x^2+1| = x^2+1$

Since $x^2+1 > 0$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C$$

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Integrate ∫ x e^(x^2) dx solved example

Substitution Method 2

Ex | $\int x e^{x^2} dx$

$$\boxed{\begin{array}{l} u = x^2 \qquad du = 2x\,dx \\ \frac{1}{2} du = x\,dx \end{array}}$$

$$\int \underbrace{e^{x^2}}_{e^u} \cdot \underbrace{x\,dx}_{\frac{1}{2}du}$$

$$\int e^u \cdot \frac{1}{2} du = \frac{1}{2}\int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

$$\therefore \int x e^{x^2} dx = \boxed{\frac{1}{2} e^{x^2} + C}$$

Evaluate ∫ x * (5-4x^2)^(1/3) dx solved example

Ex $\int x \sqrt[3]{5-4x^2}\, dx$

$$\boxed{\begin{aligned} u &= 5-4x^2 \\ du &= -8x\, dx \\ -\tfrac{1}{8}\, du &= x\, dx \end{aligned}}$$

$$\int \underbrace{\sqrt[3]{5-4x^2}}_{\sqrt[3]{u}}\ \underbrace{x\, dx}_{-\frac{1}{8} du}$$

$$\int \sqrt[3]{u}\left(-\frac{1}{8}\right) du = -\frac{1}{8}\int u^{1/3}\, du = -\frac{1}{8}\, u^{4/3} \cdot \frac{3}{4} + C$$

$$= -\frac{3}{32}\, u^{4/3} + C = -\frac{3}{32}\left(5-4x^2\right)^{4/3} + C$$

$$\therefore \int x \sqrt[3]{5-4x^2}\, dx = -\frac{3}{32}\left(5-4x^2\right)^{4/3} + C$$

U-Substitution method involving a Trigonometric (sine) function example

Ex| $\int x \cdot \sin(x^2)\,dx$

Integrate ∫ x * sin(x^2) dx solved example
U-Substitution method

$$\boxed{\begin{array}{ll} u = x^2 & du = 2x\,dx \\ & \frac{1}{2}du = x\,dx \end{array}}$$

$$\int x \sin(x^2)\,dx = \int \underbrace{\sin(x^2)}_{\sin u}\ \underbrace{x\,dx}_{\frac{1}{2}du} = \int \sin(u) \cdot \frac{1}{2}du$$

$$= \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2) + C$$

$$\therefore \int x \sin(x^2)\,dx = \frac{-1}{2}\cos(x^2) + C$$

U-Substitution method finding an integral with algebraic root functions

Ex $\int \frac{t}{\sqrt[3]{1-3t^2}}\,dt$

$u = 1-3t^2$

$du = -6t\,dt$

$-\frac{1}{6}\,du = t\,dt$

Integrate ∫ t/(1-3t^2)^(1/3) dt solved example U-Substitution method.

$$\int \frac{1}{\sqrt[3]{1-3t^2}} \cdot t\,dt$$

$$\int \frac{1}{\sqrt[3]{u}} \cdot \left(-\frac{1}{6}\right) du = -\frac{1}{6}\int u^{-1/3}\,du = -\frac{1}{6}u^{2/3} \cdot \frac{3}{2} + C$$

$$= -\frac{3}{12}u^{2/3} + C = -\frac{1}{4}u^{2/3} + C = -\frac{1}{4}(1-3t^2)^{2/3} + C$$

$$\therefore \int \frac{t}{\sqrt[3]{1-3t^2}}\,dt = \frac{-1}{4}(1-3t^2)^{2/3} + C$$

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U-Substitution method outline of the proof for definite integrals notes

Substitution method 3 (Definite Integrals)

Theory: If $g'(x)$ is continuous on $[a,b]$ and $f(x)$ is continuous on the range of $u=g(x)$ then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof (not formal!)

let $u=g(x)$ $du=g'(x)\,dx$

Don't forget to change the limits

$$x=a \xrightarrow{u=g(x)} u=g(a)$$

$$x=b \xrightarrow{u=g(x)} u=g(b)$$

$$\int_a^b f(g(x)) \cdot g'(x)\, dx$$

$$\boxed{\begin{array}{ll} u = g(x) & du = g'(x)\,dx \\ x = a & u = g(a) \\ x = b & u = g(b) \end{array}}$$

$$\int_{x=a}^{x=b} \overbrace{f(g(x))}^{f(u)} \cdot \overbrace{g'(x)\, dx}^{du} = \int_{u=g(a)}^{u=g(b)} f(u) \cdot du$$

$$= F(u) \Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

$$\therefore \int_a^b f(g(x)) \cdot g'(x)\, dx = F(g(b)) - F(g(a))$$

U-Substitution method applied to a definite integral by changing x limits to u limits

Ex| Evaluate $\int_0^1 x e^{x^2} \sin(e^{x^2})\,dx$

Change limits

$x=0 \xrightarrow{u=e^{x^2}} u=e^0=1$

$x=1 \xrightarrow{u=e^{x^2}} u=e^1=e$

$$\boxed{\begin{aligned} u &= e^{x^2} \\ du &= e^{x^2}\cdot 2x\,dx \\ \tfrac{1}{2}\,du &= x e^{x^2}\,dx \end{aligned}}$$

$$\int_0^1 \sin(e^{x^2})\cdot x e^{x^2}\,dx = \int_1^e \sin(u)\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_1^e \sin u\,du = -\frac{1}{2}\cos u\Big|_1^e = -\frac{1}{2}\left[\cos e - \cos 1\right]$$

Ex $\int_1^e \frac{(\ln x)^3}{x}\,dx$

$$\boxed{u = \ln x \qquad du = \frac{1}{x}\,dx}$$

Change Limits x Limits to u limits

$x = 1 \xrightarrow{u=\ln x} u = \ln 1 = 0$

$x = e \xrightarrow{u=\ln x} u = \ln e = 1$

$$\int_1^e \underbrace{(\ln x)^3}_{u^3} \cdot \underbrace{\frac{1}{x}\,dx}_{du} = \int_0^1 u^3\,du = \frac{u^4}{4}\Big|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

$$\therefore \int_1^e \frac{(\ln x)^3}{x}\,dx = \frac{1}{4}$$

Ex Theory question

If $\int_1^9 f(x)\,dx = 10$ find $\int_0^2 x^2 f(x^3+1)\,dx$

Let's start with $\int_0^2 x^2 f(x^3+1)\,dx$

$u = x^3+1 \qquad du = 3x^2\,dx \qquad \frac{1}{3}\,du = x^2\,dx$

$x=0 \xRightarrow{u=x^3+1} u=1$

$x=2 \xRightarrow{u=x^3+1} u=9$

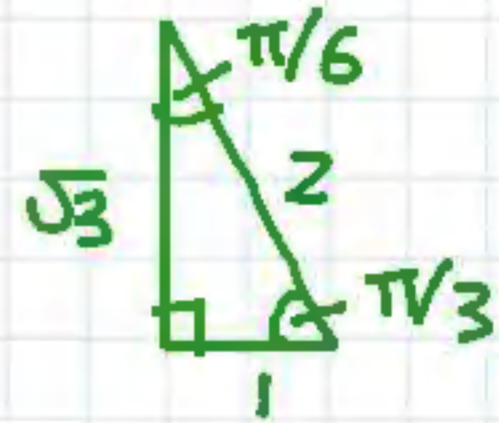
$\therefore \int_0^2 x^2 f(x^3+1)\,dx = \int_1^9 f(u)\frac{1}{3}\,du = \frac{1}{3}\int_1^9 f(u)\,du$

$= \frac{1}{3}\int_1^9 f(u)\,du = \frac{1}{3}\cdot 10 = \boxed{10/3}$

Applying U-Substitution method to evaluate definite integral of tan x

Substitution method 4

Special triangles



Ex| Find $\int_{\pi/6}^{\pi/4} \tan x \, dx$

Let's rewrite $\tan x = \frac{\sin x}{\cos x}$

$$\int_{\pi/6}^{\pi/4} \tan x \, dx = \int_{\pi/6}^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x$ $\quad du = -\sin x \, dx$

Change x limits to u limits

$x = \pi/6 \qquad u = \cos x \qquad u = \cos(\pi/6) = \sqrt{3}/2$

$x = \pi/4 \qquad u = \cos x \qquad u = \cos(\pi/4) = 1/\sqrt{2}$

$$\int_{\pi/6}^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = \int_{\pi/6}^{\pi/4} \frac{1}{\cos x}\cdot \sin x\,dx$$

$u = \cos x \quad du = -\sin x\,dx \Rightarrow -1\,du = \sin x\,dx$

$x = \pi/6 \rightarrow u = \sqrt{3}/2$

$x = \pi/4 \rightarrow u = 1/\sqrt{2}$

$$\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{u}(-1)\,du = -\int_{\sqrt{3}/2}^{1/\sqrt{2}} \frac{1}{u}\,du = -\left[\ln u \Big|_{\sqrt{3}/2}^{1/\sqrt{2}}\right]$$

$$= -\left[\ln\frac{1}{\sqrt{2}} - \ln\frac{\sqrt{3}}{2}\right] = \boxed{-\ln\frac{1}{\sqrt{2}} + \ln\frac{\sqrt{3}}{2}}$$

$$\int_{\pi/6}^{\pi/4} \tan x\,dx = -\ln\frac{1}{\sqrt{2}} + \ln\frac{\sqrt{3}}{2}$$

U-Substitution method for evaluating an algebraic function with square root

Ex Compute $\int x^5 \sqrt[3]{x^3+1}\,dx$

$\boxed{\text{Let } u = x^3+1 \qquad du = 3x^2\,dx \qquad \frac{1}{3}\,du = x^2\,dx \qquad x^3 = u-1}$

$$\int \underbrace{\sqrt[3]{x^3+1}}_{\sqrt[3]{u}}\ \underbrace{x^3}_{u-1}\ \underbrace{x^2\,dx}_{\frac{1}{3}du}$$

$$\frac{1}{3}\int (u-1)\,u^{1/3}\,du = \frac{1}{3}\int \left(u^{4/3} - u^{1/3}\right) du$$

$$= \frac{1}{3}\left[\ u^{7/3}\cdot\frac{3}{7} - u^{4/3}\cdot\frac{3}{4}\ \right] + C$$

$$= \frac{1}{7}\left(x^3+1\right)^{7/3} - \frac{1}{4}\left(x^3+1\right)^{4/3} + C$$

Applying U-Substitution method to an integral with Arcsine function

Ex| Evaluate $\int \frac{2x \sin^{-1}(x^2)}{\sqrt{1-x^4}}\, dx$

Let $u = \sin^{-1}(x^2)$ $\quad \frac{du}{dx} = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$ Chain Rule

$du = \frac{2x}{\sqrt{1-x^4}}\, dx$

$$\int \underbrace{\sin^{-1}(x^2)}_{u} \cdot \underbrace{\frac{2x}{\sqrt{1-x^4}}\, dx}_{du} = \int u\, du = \frac{u^2}{2} + C$$

$$= \frac{\left[\sin^{-1}(x^2)\right]^2}{2} + C$$

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U-Substitution method shortcuts for f(a+bx) ,let u=a+bx , linear substitution

Substitution method 5 (Linear Subst. $u=ax+b$)

$$\int f(a+bx)\,dx$$

Let $u=a+bx$ $\quad du=b\,dx$ $\quad \frac{1}{b}du=dx$

$$\int f(a+bx)\,dx=\int f(u)\cdot\frac{1}{b}\,du=\frac{1}{b}\int f(u)\,du$$

$$=\frac{1}{b}F(u)+C=\frac{1}{b}F(a+bx)+C$$

where $F'(x)=f(x)$

$$\therefore \int f(a+bx)\,dx=\frac{1}{b}F(a+bx)+C$$

where $F(x)$ is antiderivative of $f(x)$

Ex $\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + C$ $\quad u = ax+b$, $du = a\,dx$

Ex $\int \sin(kx)\,dx = -\frac{\cos(kx)}{k} + C$ $\quad u = kx$, $du = k\,dx$

Ex $\int \sqrt{ax+b}\,dx = \frac{(ax+b)^{3/2}}{(3/2)(a)} + C$ $\quad u = ax+b$, $du = a\,dx$

$$= \frac{2}{3a}(ax+b)^{3/2} + C$$

Ex $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C$ $\quad u = ax+b$, $du = a\,dx$

$n \neq -1$

$$\int (ax+b)^{m/n}\,dx = \frac{(ax+b)^{m/n+1}}{\left(\frac{m}{n}+1\right)(a)} + C$$

$u = ax+b$

$du = a\,dx$

$\frac{m}{n} \neq -1$

$$\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$$

$u = ax+b$

$du = a\,dx$

$$\int \sec^2(kx)\,dx = \frac{\tan(kx)}{k} + C$$

$u = kx$

$du = k\,dx$

$$\int\left[e^{2x}+\sin(\pi x)+\frac{1}{2x+1}+(3x+5)^2+\sqrt{4x+7}+\sec^2(10x)\right]dx$$

Solution: split up integrals and apply linear Subst.

$$\frac{e^{2x}}{2}+-\frac{\cos(\pi x)}{\pi}+\frac{\ln|2x+1|}{2}+\frac{(3x+5)^3}{3(3)}+\frac{(4x+7)^{3/2}}{(3/2)(4)}$$

$$+\frac{\tan(10x)}{10}+C$$

Key Concept: $\int f(\underbrace{ax+b}_{\text{linear}})\,dx = \frac{F(ax+b)}{a}+C$

Ex | $\int e^{3x^2}dx \neq \frac{e^{3x^2}}{6x}+C$

$\int e^{3x^2}dx$ is not integrable!

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U-Substitution method applied to an exponential function e^(x+e^x) example

Substitution method 6 (Exponential functions)

Ex) Evaluate $\int e^{x+e^x}\,dx$

Recall
$e^{a+b} = e^a \cdot e^b$

$$\int e^{x+e^x}\,dx = \int e^{e^x} \cdot e^x\,dx$$

Let $u = e^x$ $\qquad du = e^x\,dx$

$$\int e^{e^x} \cdot e^x\,dx = \int e^u \cdot du = e^u + C = e^{e^x} + C$$

$$\therefore \int e^{x+e^x}\,dx = e^{e^x} + C$$

U-Substitution method applied to exponential function ∫3^x dx

Ex| Evaluate $\int 3^x\,dx$

hint:

$a^x = e^{\ln a^x} = e^{x\ln a}$

$\int 3^x\,dx = \int e^{\ln 3^x}\,dx = \int e^{x\ln 3}\,dx$

$\int 3^x\,dx = \int e^{x\ln 3}\,dx$

Let $u = x\ln 3$ $\quad du = \ln 3\,dx \quad \frac{1}{\ln 3}\,du = dx$

$\int e^{x\ln 3}\,dx = \int e^u \cdot \frac{1}{\ln 3}\,du = \frac{1}{\ln 3}\int e^u\,du$

$= \frac{1}{\ln 3}e^u + C = \frac{1}{\ln 3}e^{x\ln 3} + C = \frac{1}{\ln 3}\cdot 3^x + C$

$\therefore \int 3^x\,dx = \frac{1}{\ln 3}\cdot 3^x + C = \frac{3^x}{\ln 3} + C$

U substitution method applied to ∫ e^sqrt(x)/sqrt(x) dx

$\boxed{\text{Ex}}$ $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Let $t = x^{1/2}$ $\quad dt = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$

$dt = \frac{1}{2\sqrt{x}}\,dx \quad 2\,dt = \frac{1}{\sqrt{x}}\,dx$

$\int e^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\,dx = \int e^{t} \cdot 2\,dt = 2\int e^{t}\,dt$

$= 2e^{t} + C = 2e^{\sqrt{x}} + C$

$\therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C$

Recall

$\sqrt{x} = x^{1/2}$

U-substitution method applied to ∫((e^x-e^(-x)/(e^x+e^(-x)) dx

Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

$U = e^x + e^{-x} \quad du = (e^x - e^{-x})\,dx$

$$\int \frac{1}{e^x + e^{-x}} (e^x - e^{-x})\,dx$$

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|e^x + e^{-x}| + C$$

$$= \ln(e^x + e^{-x}) + C$$

since $e^x + e^{-x} > 0$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln(e^x + e^{-x}) + C$$

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Substitution method 7

Ex| Evaluate $\int \frac{2x+3}{(3x^2+9x)^2}\,dx$

Let $u = 3x^2+9x$ $\quad du = (6x+9)dx = 3(2x+3)dx$

$$du = 3(2x+3)dx$$

$$\frac{1}{3}du = (2x+3)\,dx$$

$$\int \frac{1}{(3x^2+9x)^2}(2x+3)\,dx = \int \frac{1}{u^2}\cdot\frac{1}{3}du = \frac{1}{3}\int\frac{1}{u^2}du$$

$$= \frac{1}{3}\int u^{-2}\,du = \frac{1}{3}\,\frac{u^{-1}}{-1} + C = \frac{-1}{3u} + C = \frac{-1}{3(3x^2+9x)} + C$$

$$\int \frac{2x+3}{(3x^2+9x)^2}\,dx = \frac{-1}{3(3x^2+9x)} + C$$

Ex| Compute $\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx$

Let $u = x^2$ $\quad du = 2x\,dx$ $\quad \frac{1}{2}du = x\,dx$

Change x limits to U limits

$x = 0 \quad u = x^2 \quad u = 0$

$x = 1/\sqrt{2} \quad u = x^2 \quad u = 1/2$

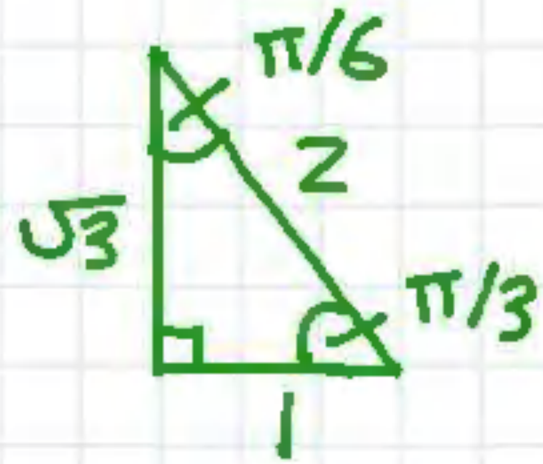
$$\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-(x^2)^2}}\,x\,dx = \int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\,\sin^{-1}u\,\Big|_0^{1/2}$$

$$= \frac{1}{2} \sin^{-1} u \Big|_0^{1/2} = \frac{1}{2}\left[\sin^{-1}(1/2) - \sin^{-1} 0 \right]$$

$$= \frac{1}{2}\left[\pi/6 - 0 \right] = \frac{\pi}{12}$$

$$\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx = \frac{\pi}{12}$$



Calculus 2 ∫ U-substitution method of function involving Arctanx (Tan^(-1)X)

Ex $\displaystyle \int \frac{1}{(1+x^2)\sqrt{\tan^{-1}x}}\,dx$

Recall

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

Let $u = \tan^{-1}x$

$$du = \frac{1}{1+x^2}\,dx$$

$$\int \overbrace{\frac{1}{\sqrt{\tan^{-1}x}}}^{1/\sqrt{u}} \cdot \overbrace{\frac{1}{1+x^2}\,dx}^{du}$$

$$\int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2u^{1/2} + C$$

$$= \underline{2(\tan^{-1}x)^{1/2} + C}$$

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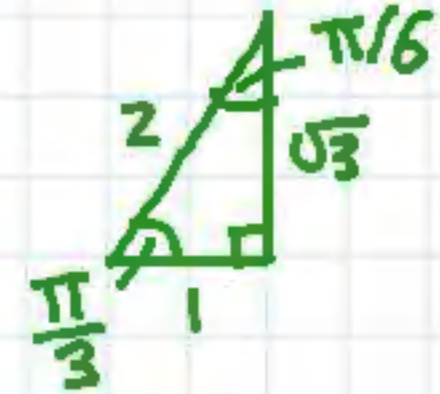
Substitution method 8

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Ex $\displaystyle\int_{\frac{\pi^2}{16}}^{\frac{\pi^2}{9}} \frac{\sin\sqrt{x}\,\cos(\sqrt{x})}{\sqrt{x}}\,dx$



Let $u = \sin\sqrt{x}$ $\qquad du = \cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx$

Change x limits to u limits

$$x = \frac{\pi^2}{16} \longrightarrow u = \sin\sqrt{x} \rightarrow u = \sin\sqrt{\frac{\pi^2}{16}} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi^2}{9} \longrightarrow u = \sin\sqrt{x} \rightarrow u = \sin\sqrt{\frac{\pi^2}{9}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\int_{\pi^2/16}^{\pi^2/9} \sin\sqrt{x}\,\frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

$u = \sin\sqrt{x}$

$du = \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,dx$

$2du = \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

$x = \frac{\pi^2}{16} \rightarrow u = 1/\sqrt{2}$

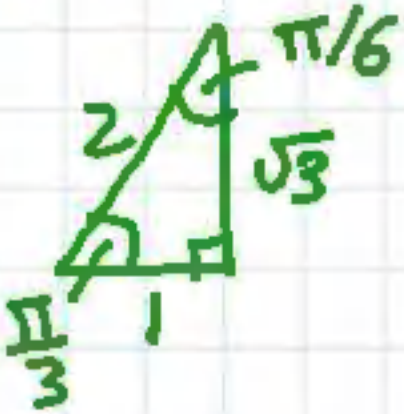
$x = \frac{\pi^2}{9} \rightarrow u = \sqrt{3}/2$

$$\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} u \cdot 2\,du = 2\int_{1/\sqrt{2}}^{\sqrt{3}/2} u\,du$$

$$= \frac{\cancel{2}u^2}{\cancel{2}}\Big|_{1/\sqrt{2}}^{\sqrt{3}/2} = u^2\Big|_{1/\sqrt{2}}^{\sqrt{3}/2}$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{4} - \frac{1}{2} = \boxed{\frac{1}{4}}$$

Special Δ's



Ex| $\displaystyle\int_{1/4}^{1/3} \sin(\pi t)\sin(\cos(\pi t))\,dt$

Let $u=\cos(\pi t)$ $\qquad du=-\sin(\pi t)\cdot\pi\,dt$

$$-\frac{1}{\pi}\,du=\sin(\pi t)\,dt$$

Change t limits to u limits

$t=\frac{1}{4}$ $\quad u=\cos(\pi t)$ $\quad u=\cos(\pi/4)=1/\sqrt{2}$

$t=\frac{1}{3}$ $\quad u=\cos(\pi t)$ $\quad u=\cos(\pi/3)=1/2$

$$\int_{\frac{1}{\sqrt{2}}}^{1/2}\sin(u)\left(-\frac{1}{\pi}\right)du=-\frac{1}{\pi}\int_{1/\sqrt{2}}^{1/2}\sin u\,du$$

$$\int_{t=1/4}^{t=1/3} \underbrace{\sin(\cos(\pi t))}_{\sin(u)} \cdot \underbrace{\sin(\pi t)\,dt}_{-\frac{1}{\pi}du} = -\frac{1}{\pi}\int_{u=1/\sqrt{2}}^{u=1/2} \sin u\, du$$

$$= -\frac{1}{\pi}\left[-\cos u \Big|_{1/\sqrt{2}}^{1/2}\right] = \frac{1}{\pi}\left[\cos u \Big|_{1/\sqrt{2}}^{1/2}\right]$$

$$= \frac{1}{\pi}\left[\cos\frac{1}{2} - \cos\frac{1}{\sqrt{2}}\right]$$

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U-Substitution method applied to integral ∫ tanx ln(cosx) dx example

Substitution method 9

Ex] Evaluate $\int \tan x \, \ln(\cos x) \, dx$

$u = \ln(\cos x) \qquad du = \frac{1}{\cos x} \cdot -\sin x \, dx$ chain rule

$du = -\tan x \, dx$

$-du = \tan x \, dx$

$$\int \underbrace{\ln(\cos x)}_{u} \, \underbrace{\tan x \, dx}_{-du} = \int -u \, du = -\int u \, du$$

$$= -\frac{u^2}{2} + C = -\frac{1}{2}\left[\ln(\cos x)\right]^2 + C$$

$$\int \tan x \, \ln(\cos x) \, dx = -\frac{1}{2}\left[\ln(\cos x)\right]^2 + C$$

Ex Compute $\int_0^a x\sqrt{x^2+a^2}\,dx \qquad a>0$

Let $u = x^2+a^2$

$du=(2x+0)\,dx \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

Change x limits to u limits

$x=0 \qquad u = x^2+a^2 \qquad u=a^2$

$x=a \qquad u = x^2+a^2 \qquad u=2a^2$

$$\int_0^a \sqrt{x^2+a^2}\,x\,dx = \int_{a^2}^{2a^2} \sqrt{u}\cdot\frac{1}{2}du = \frac{1}{2}\int_{a^2}^{2a^2} u^{1/2}\,du$$

$$= \frac{1}{\cancel{2}}\,u^{3/2}\,\frac{\cancel{2}}{3}\Big|_{a^2}^{2a^2} = \frac{1}{3}u^{3/2}\Big|_{a^2}^{2a^2} = \frac{1}{3}\left[(2a^2)^{3/2}-(a^2)^{3/2}\right]$$

$$= \frac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right]$$

$$= \frac{1}{3}\left[2^{3/2}\, a^3 - a^3\right] = \frac{a^3}{3}\left[2^{3/2} - 1\right]$$

$$\therefore \int_0^a x\sqrt{x^2+a^2}\, dx = \frac{a^3}{3}\left[2^{3/2} - 1\right]$$

U-Substitution method applied to integral ∫ sin(π/x)/x^2 dx

$\boxed{\text{Ex}}$ Evaluate $\int \frac{\sin(\pi/x)}{x^2}\,dx$

$u = \frac{\pi}{x} \qquad du = -\frac{\pi}{x^2}\,dx \quad \Rightarrow \quad -\frac{1}{\pi}\,du = \frac{1}{x^2}\,dx$

$$\int \underbrace{\sin\left(\frac{\pi}{x}\right)}_{\sin u} \cdot \underbrace{\frac{1}{x^2}\,dx}_{-1/\pi\, du}$$

$$\int \sin(\pi/x) \cdot \frac{1}{x^2}\,dx = \int -\frac{1}{\pi} \sin u\,du = -\frac{1}{\pi} \int \sin u\,du$$

$$= -\frac{1}{\pi}\left[-\cos u\right] + C = \frac{1}{\pi}\cos u + C$$

$$= \frac{1}{\pi}\cos(\pi/x) + C$$

Substitution method 10

Ex) Show that $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(x/a)+C$

Solution: $\int \frac{1}{x^2+a^2}\,dx = \int \frac{1/a^2}{\frac{x^2}{a^2}+\frac{a^2}{a^2}}\,dx = \int \frac{1/a^2}{x^2/a^2+1}\,dx$

$$= \int \frac{1/a^2}{\left(\frac{x}{a}\right)^2+1}\,dx = \frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,dx$$

Let $u = x/a \quad du = \frac{1}{a}dx \qquad a\,du = dx$

$$\frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,dx = \frac{1}{a^2}\int \frac{1}{u^2+1}\,a\,du = \frac{1}{a}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{a}\tan^{-1}(u)+C = \frac{1}{a}\tan^{-1}(x/a)+C$$

$$\frac{1}{a}\int \frac{1}{u^2+1}\,du = \frac{1}{a}\tan^{-1}u + C = \frac{1}{a}\tan^{-1}(x/a) + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(x/a) + C$$

Example: $\displaystyle\int \frac{10}{x^2+4}\,dx = 10\int \frac{1}{x^2+2^2}\,dx$

$$10\int \frac{1}{x^2+2^2}\,dx = 10\cdot\frac{1}{2}\tan^{-1}(x/2) + C$$

$$= 5\tan^{-1}(x/2) + C$$

Note: Apply Formula with $a=2$

U-Substitution method to find ∫ (1+√x)^20 dx example

Ex Evaluate $\int (1+\sqrt{x})^{20}\,dx$

Solution: Let $u = 1+\sqrt{x}$ $\quad du = \dfrac{1}{2\sqrt{x}}\,dx$ $\quad dx = 2(u-1)\,du$

$dx = 2\sqrt{x}\,du = 2(u-1)\,du$

Note
$u = 1+\sqrt{x}$
$\sqrt{x} = u-1$

$$\int \overbrace{(1+\sqrt{x})^{20}}^{u^{20}} \overbrace{dx}^{2(u-1)du} = \int u^{20}\cdot 2(u-1)\,du$$

$$= 2\int u^{20}(u-1)\,du = 2\int (u^{21}-u^{20})\,du$$

$$= 2\left[\frac{u^{22}}{22} - \frac{u^{21}}{21}\right] + C$$

$$= 2\left[\frac{(1+\sqrt{x})^{22}}{22} - \frac{(1+\sqrt{x})^{21}}{21}\right] + C$$

Ex Evaluate $\int \frac{x}{(x+1)^{10}}\,dx$

Solution: Let $u = x+1$ $du = dx$ $x = u-1$

$$\int \frac{x}{(x+1)^{10}}\,dx = \int \frac{u-1}{u^{10}}\,du = \int \left(\frac{u}{u^{10}} - \frac{1}{u^{10}}\right) du$$

$$= \int \left(u^{-9} - u^{-10}\right) du = \frac{u^{-8}}{-8} - \frac{u^{-9}}{-9} + C$$

$$= \frac{-1}{8u^{8}} + \frac{1}{9u^{9}} + C$$

$$= \frac{-1}{8(x+1)^{8}} + \frac{1}{9(x+1)^{9}} + C$$

Substitution method II

Ex Evaluate $\int_0^1 \frac{2}{e^{-x}+1}\,dx$

$$\int_0^1 \frac{2}{e^{-x}+1}\cdot\frac{e^x}{e^x}\,dx = \int_0^1 \frac{2e^x}{e^x \cdot e^{-x}+e^x}\,dx = \int_0^1 \frac{2e^x}{e^{x-x}+e^x}\,dx$$

$$= \int_0^1 \frac{2e^x}{e^0+e^x}\,dx = \int_0^1 \frac{2e^x}{1+e^x}\,dx = 2\int_0^1 \frac{e^x}{1+e^x}\,dx$$

Let $u = 1+e^x$ $\quad du = e^x dx$

Change x limits to u limits

$x=0 \qquad u = 1+e^x \qquad u=2$

$x=1 \qquad u = 1+e^x \qquad u = 1+e$

$$2\int_0^1 \frac{e^x}{1+e^x}\,dx = 2\int_2^{1+e} \frac{du}{u} = 2\ln|u|\Big|_2^{1+e}$$

$$= 2\ln u\Big|_2^{1+e} = 2\left[\ln(1+e) - \ln 2\right] = 2\ln\left(\frac{1+e}{2}\right)$$

$$\int_0^1 \frac{2}{e^{-x}+1}\,dx = 2\ln\left(\frac{1+e}{2}\right)$$

U-Substitution method applied to ∫ x . 3^(x^2+1) dx solved example

Ex| Evaluate $\int x \cdot 3^{x^2+1}\,dx$

Recall

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int x \cdot 3^{x^2+1}\,dx$$

$u = x^2+1 \quad du = 2x\,dx \quad \frac{1}{2}\,du = x\,dx$

$$\int 3^{x^2+1} \cdot x\,dx = \int 3^u \cdot \frac{1}{2}\,du = \frac{1}{2}\int 3^u\,du$$

$$= \frac{1}{2}\,\frac{3^u}{\ln 3} + C = \frac{1}{2}\,\frac{3^{x^2+1}}{\ln 3} + C$$

$$\int x \cdot 3^{x^2+1}\,dx = \frac{3^{x^2+1}}{2\ln 3} + C$$

U-Substitution method theoretical example solved

Ex| If $f'(x)$ is continuous on $[a,b]$, show

that $\int_a^b \frac{f'(x)}{f(x)}\,dx = Ln|f(b)| - Ln|f(a)|$

Assume $f(x) \neq 0$ for x in $[a,b]$

Solution:

Let $u = f(x)$ $\quad du = f'(x)\,dx$

Change x limits to u limits

$x = a \qquad u = f(x) \qquad u = f(a)$

$x = b \qquad u = f(x) \qquad u = f(b)$

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \int_{f(a)}^{f(b)} \frac{du}{u} = Ln|u| \Big|_{f(a)}^{f(b)}$$

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \int_{f(a)}^{f(b)} \frac{1}{u}\,du = \ln|u|\Big|_{f(a)}^{f(b)}$$

$$= \ln|f(b)| - \ln|f(a)|$$

$$\therefore \int_a^b \frac{f'(x)}{f(x)}\,dx = \ln|f(b)| - \ln|f(a)|$$

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Substitution method 12 (Double Substitution)

Ex Evaluate $\int \frac{x}{4-x^2+\sqrt{4-x^2}}\,dx$

Solution: Let $u=4-x^2$ $\quad du=-2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$\int \overbrace{\frac{1}{4-x^2+\sqrt{4-x^2}}}^{\frac{1}{u+\sqrt{u}}}\ \overbrace{x\,dx}^{-\frac{1}{2}du} = \int \frac{1}{u+\sqrt{u}}\ -\frac{1}{2}du$$

$$= -\frac{1}{2}\int \frac{1}{u+\sqrt{u}}\,du$$

$$= -\frac{1}{2}\int \frac{1}{u+\sqrt{u}}\,du$$

We need to do another Substitution

Let $t = \sqrt{u}$ $\quad dt = \frac{1}{2\sqrt{u}}\,du \quad du = 2\sqrt{u}\,dt$

$t^2 = u \qquad du = 2t\,dt$

$$-\frac{1}{2}\int \frac{1}{\underbrace{u}_{t^2}+\underbrace{\sqrt{u}}_{t}}\,\overbrace{du}^{2t\,dt} = -\frac{1}{2}\int \frac{1}{t^2+t}\,2t\,dt$$

$$-\frac{1}{\cancel{2}}\cdot\cancel{2}\int \frac{t}{t^2+t}\,dt = -\int \frac{t/t}{\frac{t^2}{t}+\frac{t}{t}}\,dt$$

$$= -\int \frac{1}{t+1}\,dt$$

$$-\int \frac{1}{t+1}\,dt = -\ln|t+1| + C$$

$$-\int \frac{1}{t+1}\,dt = -\ln|t+1| + C$$

Recall

$t = \sqrt{u}$

$u = 4 - x^2$

$$= -\ln|\sqrt{u}+1| + C$$

$$= -\ln|\sqrt{4-x^2}+1| + C$$

$$\therefore \int \frac{x}{4-x^2+\sqrt{4-x^2}}\,dx = -\ln|\sqrt{4-x^2}+1| + C$$

$$= -\ln(\sqrt{4-x^2}+1) + C$$

Apply U-Substitution to ∫ 2x/(e^(-x^2)+e^(x^2)) dx solved example

Ex) Evaluate $\int \frac{2x}{e^{-x^2}+e^{x^2}}\,dx$

Solution:

Let $u = x^2$ $\quad du = 2x\,dx$

$$\int \frac{1}{e^{-x^2}+e^{x^2}} \cdot 2x\,dx = \int \frac{1}{e^{-u}+e^{u}}\,du$$

$$\int \frac{1}{e^{-u}+e^{u}} \cdot \frac{e^u}{e^u}\,du = \int \frac{e^u}{e^{-u}e^{u}+e^{2u}}\,du = \int \frac{e^u}{1+e^{2u}}\,du$$

$$\int \frac{e^u}{1+e^{2u}}\,du$$

We need to do another Substitution!

$$\int \frac{e^u}{1+e^{2u}}\,du \qquad t=e^u \quad dt=e^u\,du$$

$$\int \frac{1}{1+(e^u)^2}\,e^u\,du = \int \frac{1}{1+t^2}\cdot dt = \tan^{-1}(t)+C$$

Recall $t=e^u$ and $u=x^2$

$$= \tan^{-1}(e^u)+C = \tan^{-1}(e^{x^2})+C$$

$$\therefore \int \frac{2x}{e^{-x^2}+e^{x^2}}\,dx = \tan^{-1}(e^{x^2})+C$$

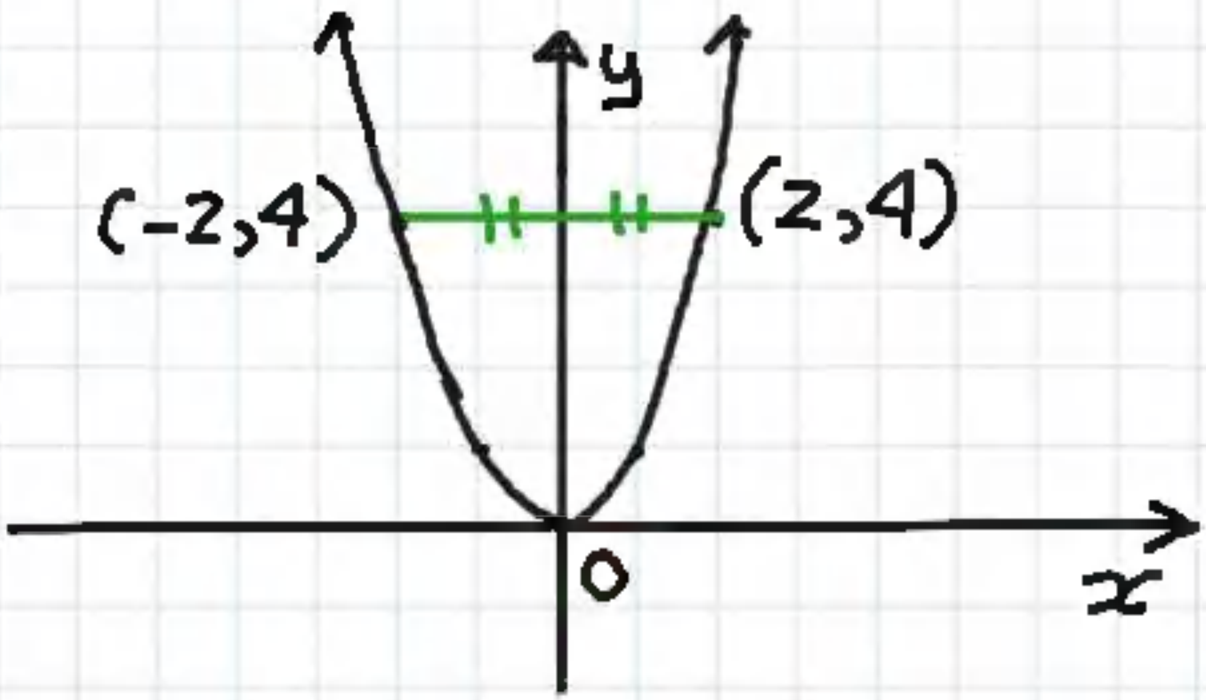
U-Substitution method (Even and odd functions theory)

Substitution method 13 (Even + Odd functions)

$f(x)$ is even if $f(-x) = f(x)$

Symmetric with respect to y axis

Ex. $f(x) = x^2$ $\quad f(x) = x^4$ $\quad f(x) = \cos x$



$f(x) = x^2$

$f(2) = f(-2) = 4$

$f(-x) = (-x)^2 = x^2$

$f(x) = x^2$

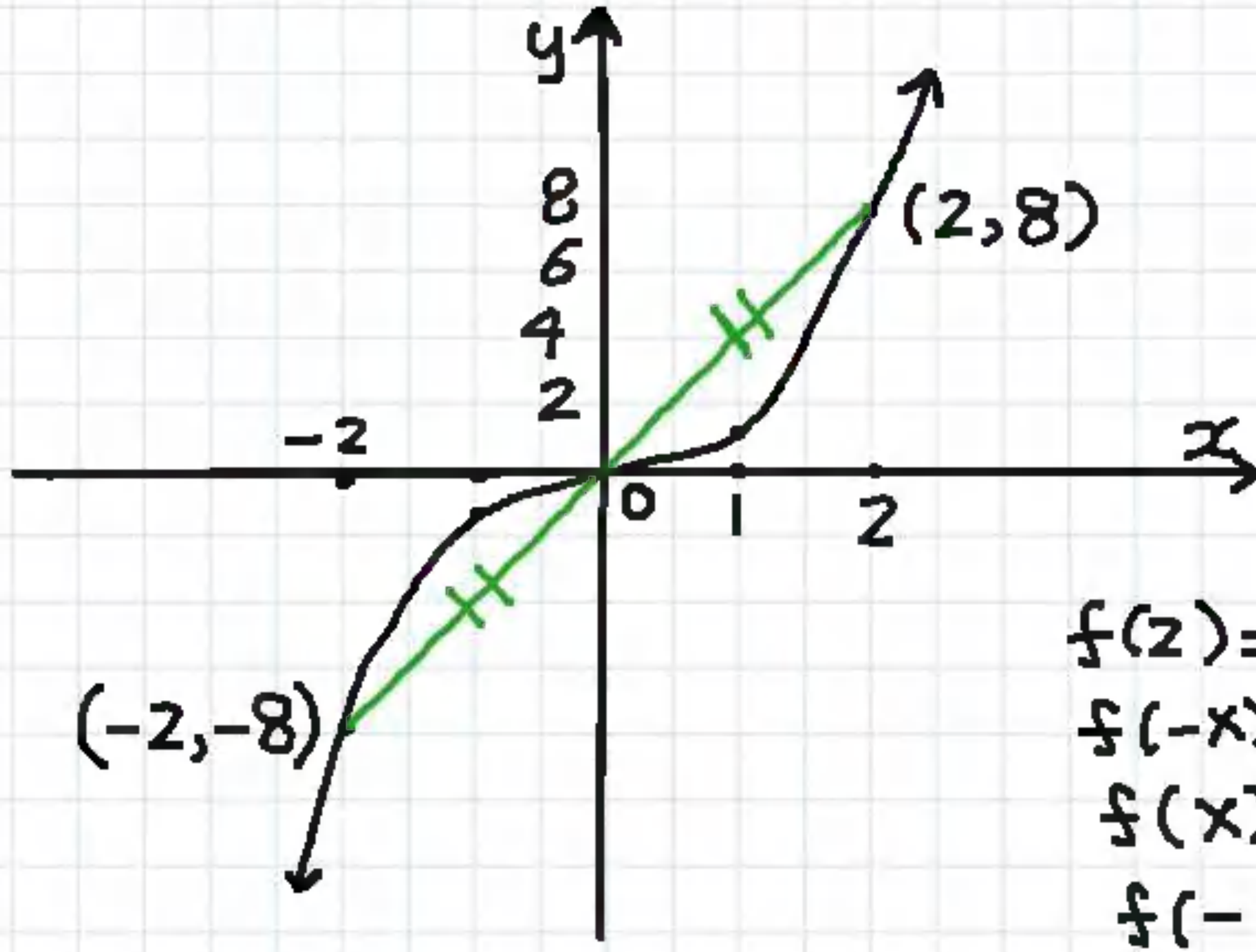
SAME

SAME

$f(x)$ is odd if $f(-x) = -f(x)$

Symmetric with respect to the origin

Ex. $f(x) = x^3$ $\quad f(x) = x^5$ $\quad f(x) = \sin x$



$f(x) = x^3$

$f(2) = 8 \quad f(-2) = -8$

$f(-x) = (-x)^3 = -x^3$

$f(x) = x^3$

$f(-x) = -f(x)$

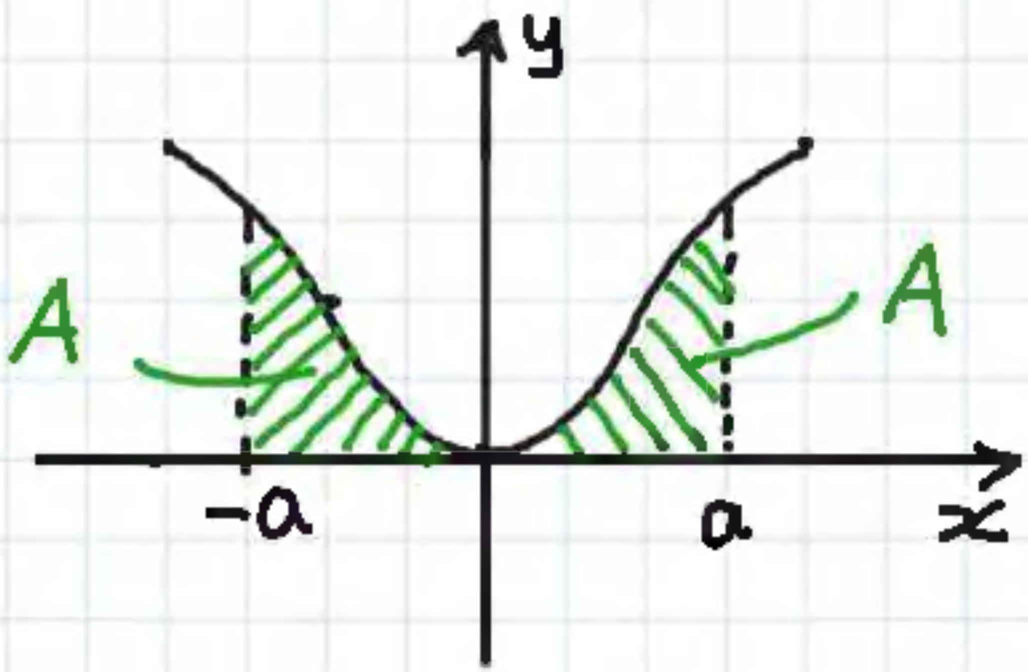
Applying U-Substitution method to find integral of even and odd functions

Integrals of Even and Odd functions

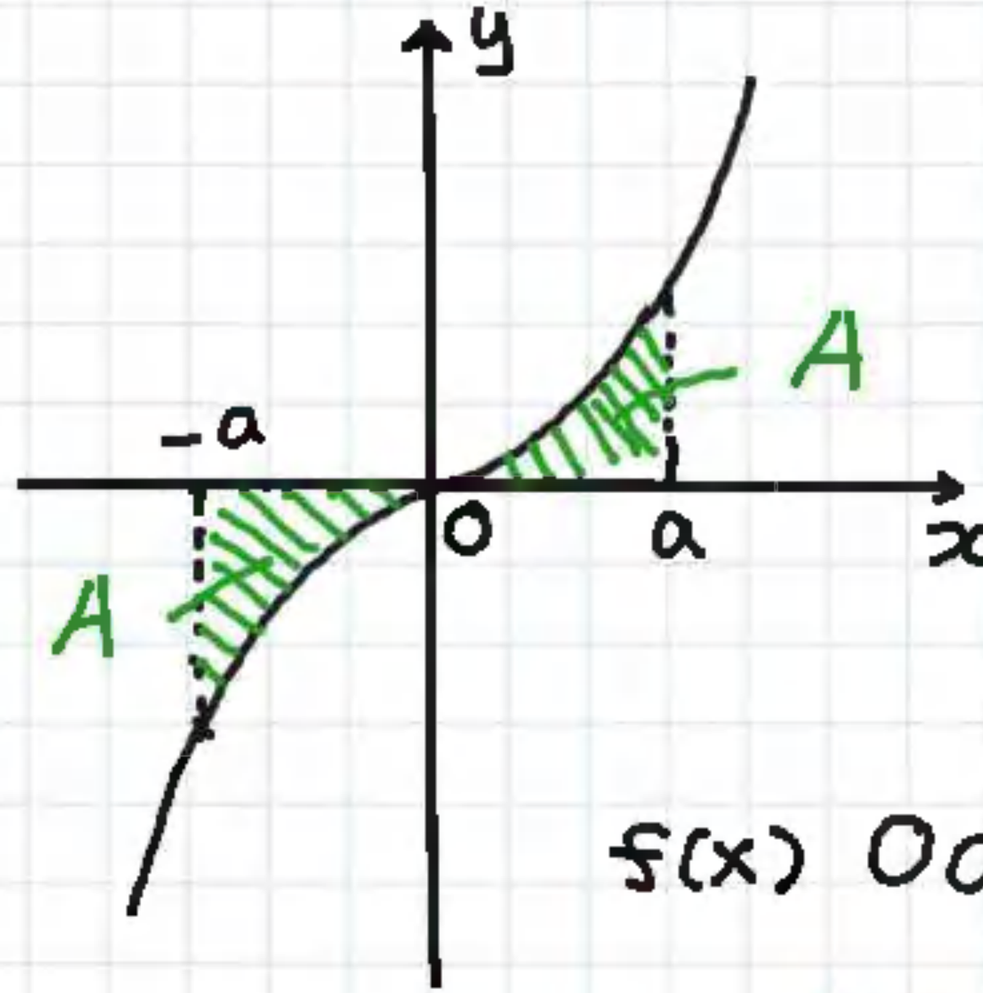
Assume $f(x)$ is continuous on $[-a, a]$

1) If $f(x)$ is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

2) If $f(x)$ is odd $\int_{-a}^{a} f(x)\,dx = 0$



f(x) Even



f(x) Odd

Proof: If $f(x)$ is odd $\int_{-a}^{a} f(x)\,dx = 0$

Split integral in two

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$\int_{-a}^{0} f(x)\,dx$ Let $u=-x$ $du=-dx$ $x=-u$

$x=-a$ $u=a$

$x=0$ $u=0$

$$\int_{a}^{0} f(-u)\,(-du) = -\int_{0}^{a} f(-u)\,(-du) = \int_{0}^{a} \overset{-f(u)}{\overset{\shortparallel}{f(-u)}}\,du$$

$$= \int_{0}^{a} -f(u)\,du = -\int_{0}^{a} f(u)\,du$$

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx$$

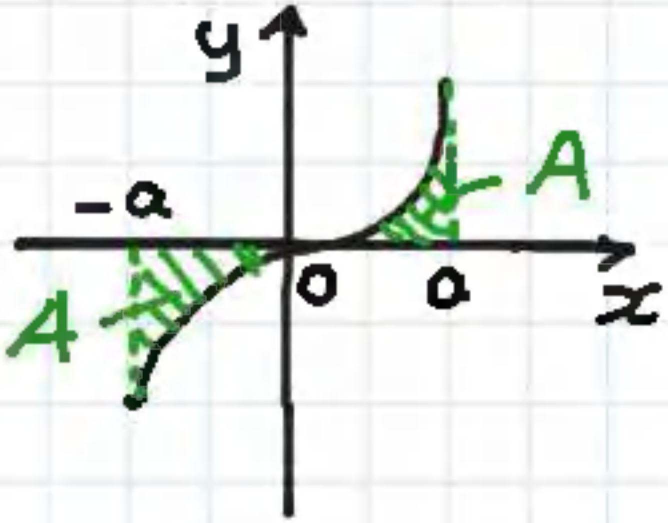
$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

$\therefore$ If $f(x)$ is odd $\int_{-a}^{a} f(x)\,dx = 0$

It can be proven that if $f(x)$ is even

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

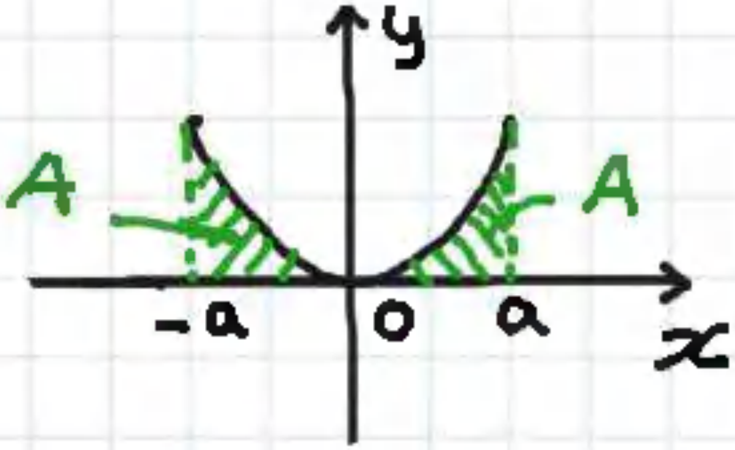
If $f(x)$ is odd Areas Cancel out due to symmetry $\int_{-a}^{a} f(x)\,dx = 0$



$$\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -A + A = 0$$

If $f(x)$ is Even Area doubles due to symmetry

$$\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = A + A = 2A$$



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U-Substitution method applied to integral of odd function over symmetric interval

Substitution method 14 (Even + Odd functions)

If $f(x)$ is odd $\int_{-a}^{a} f(x)\,dx = 0$

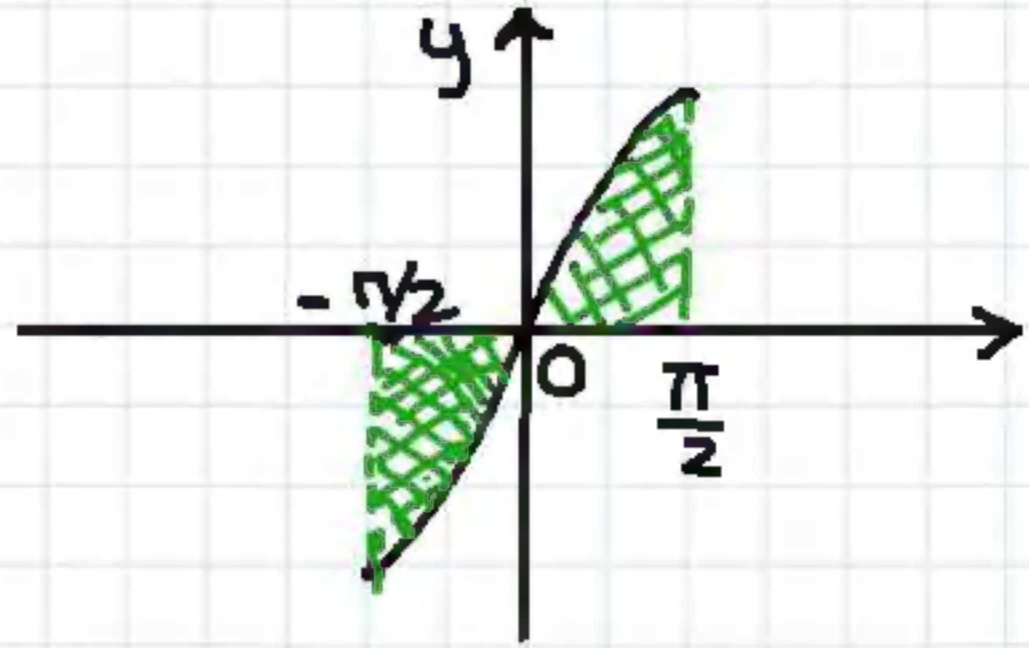
Ex| Evaluate $\int_{-\pi/2}^{\frac{\pi}{2}} \frac{\sin x}{1+x^4}\,dx$

Solution: $f(x) = \frac{\sin x}{1+x^4}$ is an odd function

$$f(-x) = \frac{\sin(-x)}{1+(-x)^4} \xrightarrow{\text{odd}} = \frac{-\sin x}{1+x^4} = -f(x)$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\frac{\sin x}{1+x^4}} dx = 0$$

odd integrand
Symmetric interval



Key Concept

odd integrand
Symmetric interval

$$\int_{-a}^{a} f(x)\, dx = 0$$

Ex Evaluate $\int_{-2}^{2}(1+x^2)\,dx$

Solution: $f(x)=1+x^2$ is an even function

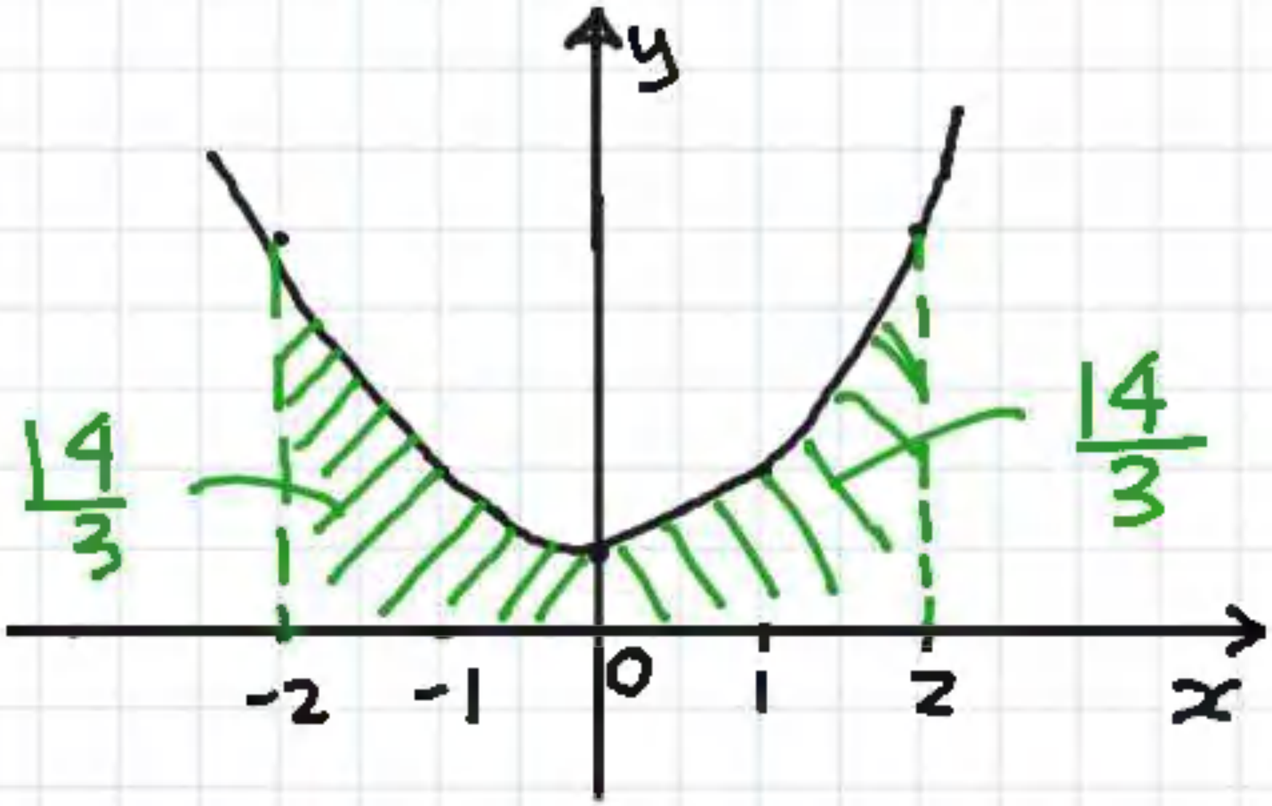
$f(-x)=1+(-x)^2=1+x^2=f(x)$

$$\int_{-2}^{2}(1+x^2)\,dx = 2\int_{0}^{2}(1+x^2)\,dx = 2\left[x+\frac{x^3}{3}\right]\Big|_0^2$$

$$= 2\left[2+\frac{8}{3}\right]-2[0+0] = 2\left[\frac{14}{3}\right] = \frac{28}{3}$$

Note: If $f(x)$ is even $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$$\int_{-2}^{2} (1+x^2)\,dx = 2\left(\frac{14}{3}\right) = \frac{28}{3}$$



$f(x) = x^2+1$

Even function

$$\int_{-2}^{2} (1+x^2)\,dx = 2\int_{0}^{2} (1+x^2)\,dx = 2\left(\frac{14}{3}\right) = \frac{28}{3}$$

U-substitution example solved (integrand is sum of an even and odd function)

Ex| Evaluate $\int_{-2}^{2} (x+1)\sqrt{4-x^2}\,dx$

Solution:

$$\int_{-2}^{2} (x+1)\sqrt{4-x^2}\,dx = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$\int_{-2}^{2} \underbrace{x\sqrt{4-x^2}}\,dx = 0$$

odd integrand + symmetric interval
so definite integral is 0

$f(x) = x\sqrt{4-x^2}$

odd function

$$f(-x) = -x\sqrt{4-(-x)^2}$$
$$= -x\sqrt{4-x^2}$$
$$f(-x) = -f(x)$$

Definite integral where integrand is semicircle of radius 2 solved example

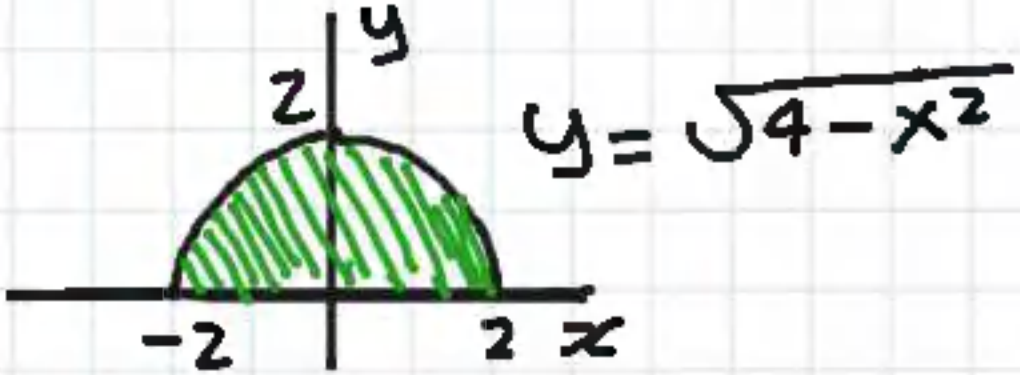
$$\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$f(x) = \sqrt{4-x^2}$ is a semi-circle of radius 2

$y = \sqrt{4-x^2}$ $y^2 = 4 - x^2$ $x^2 + y^2 = 4$

circle of radius 2

$$\int_{-2}^{2} \sqrt{4-x^2}\, dx = \frac{\pi(2)^2}{2} = 2\pi$$ (Area of semicircle of radius 2)



$$\int_{-2}^{2} (x+1)\sqrt{4-x^2}\,dx = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 0 + 2\pi = 2\pi$$

$$\int_{-2}^{2} (x+1)\sqrt{4-x^2}\,dx = 2\pi$$

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U- Substitution method word problem theoretical example solved

Substitution method 15 (word problems)

Ex) Find the function $f(x)$ such that the slope of the tangent line to the function is $x\sqrt{x^2+4}$ for all x values and graph of the function $f(x)$ passes through the point $(0,3)$

Solution:

$$f'(x) = x\sqrt{x^2+4}$$

$$f(x) = \int f'(x)\,dx$$

Strategy

Integrate $f'(x)$ to recover $f(x)$

U-Substitution applied to solve f(x)=∫x√(x^2+4) dx with initial condition f(0)=3

$$f(x) = \int x\sqrt{x^2+4}\;dx$$

Let $u = x^2+4$ $\quad du = 2x\,dx$ $\quad \frac{1}{2}du = x\,dx$

$$\int \underbrace{\sqrt{x^2+4}}_{\sqrt{u}}\;\underbrace{x\,dx}_{\frac{1}{2}du} = \int \sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}du$$

$$= \frac{1}{\cancel{2}}u^{3/2}\cdot\frac{\cancel{2}}{3} + C = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+4)^{3/2} + C$$

$$f(x) = \frac{1}{3}(x^2+4)^{3/2} + C$$

$f(0) = 3$
since $f(x)$
passes thru $(0,3)$

$$3 = \frac{1}{3}(0+4)^{3/2} + C$$

$$3 = \frac{1}{3}(4)^{3/2} + C \Rightarrow 3 = \frac{1}{3}\left[(4)^{1/2}\right]^3 + C \Rightarrow 3 = \frac{1}{3}\cdot 8 + C$$

$$3 = \frac{8}{3} + C \Rightarrow C = 3 - \frac{8}{3} = \frac{1}{3}$$

$$f(x) = \frac{1}{3}(x^2+4)^{3/2} + C$$

$$f(x) = \frac{1}{3}(x^2+4)^{3/2} + \frac{1}{3}$$

$\therefore$ Solution of $f'(x) = x\sqrt{x^2+4}$, $f(0) = 3$

is given by $f(x) = \frac{1}{3}(x^2+4)^{3/2} + \frac{1}{3}$

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Apply U-Substitution and Fundamental theorem of Calculus word problem solved

Substitution method 16 (word Problems)

Ex| A small town has a population of 27000. The population $P(t)$ will grow at the instantaneous rate of $P'(t)=\frac{dP}{dt}=\frac{36000t}{(t^2+2)^2}$

for $t \geq 0$, t is in years and $t=0$ represents the present, when will the population reach 27000 people?

Solution: Apply F. T. C (Fund. Thm of calc.)

$$F(b)=F(a)+\int_a^b F'(x)\,dx$$

$$F(b) = F(a) + \boxed{\int_a^b F'(x)\,dx}$$

$\Leftarrow$ integrate instant rate of change to get total change

$$\frac{dP}{dt} = \frac{36000t}{(t^2+2)^2} \qquad P(0) = 19000 \qquad P(T) = 27000$$

$\uparrow$ Find T

$$P(T) = P(0) + \int_0^T \frac{36000t}{(t^2+2)^2}\,dt$$

$$27000 = 19000 + \int_0^T \frac{36000t}{(t^2+2)^2}\,dt$$

$$8000 = 36000 \int_0^T \frac{t}{(t^2+2)^2}\,dt$$

$$8000 = 36000 \int_0^T \frac{t}{(t^2+2)^2}\,dt \Rightarrow \frac{8}{36} = \int_0^T \frac{t}{(t^2+2)^2}\,dt$$

$$\frac{8}{36} = \int_0^T \frac{t}{(t^2+2)^2}\,dt$$ Apply Substitution Rule

Let $u = t^2+2$ $\quad du = 2t\,dt$ $\quad \frac{1}{2}du = t\,dt$

$t=0$ $\quad u = t^2+2$ $\quad u=2$

$t=T$ $\quad u=t^2+2$ $\quad u=T^2+2$

$$\int_2^{T^2+2} \frac{1}{u^2}\frac{1}{2}\,du \Rightarrow \frac{1}{2}\int_2^{T^2+2} u^{-2}\,du = \frac{1}{2}\left(-\frac{1}{u}\right)\Big|_2^{T^2+2}$$

$$\frac{8}{36} = -\frac{1}{2}\cdot\frac{1}{u}\Big|_2^{T^2+2}$$

$$\frac{8}{36} = -\frac{1}{2} \cdot \frac{1}{u} \Big|_{2}^{T^2+2} \Rightarrow -\frac{16}{36} = \frac{1}{u}\Big|_{2}^{T^2+2}$$

$$-\frac{16}{36} = \frac{1}{T^2+2} - \frac{1}{2} \Rightarrow \frac{1}{T^2+2} = -\frac{16}{36} + \frac{1}{2}$$

$$\frac{1}{T^2+2} = \frac{-16+18}{36} \Rightarrow \frac{1}{T^2+2} = \frac{2}{36} \Rightarrow \frac{1}{T^2+2} = \frac{1}{18}$$

$$T^2 + 2 = 18 \Rightarrow T^2 = 16 \Rightarrow T = \pm 4$$

take only $T = 4$

$\therefore$ 4 years from now the population of the town will reach 27000.

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