math 221 Midterm 2

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ & & & & \\ 1 & 2 & 3 & & n \end{bmatrix} \quad m \times n$$

row reduce

$$A_{red} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad m \times 1 \quad \text{Basis for } \mathrm{Col} A$$

b) $n = \dim \mathrm{Nul} A + \dim \mathrm{Col} A$

$n = \dim \mathrm{Nul} A + 1$

$\dim \mathrm{Nul} A = n - 1$

c) <u>basis for null space</u>

$$A_{red} = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

$$a_1 + 2a_2 + 3a_3 + \cdots + n a_n = 0$$

$$\begin{cases} a_1 = -2a_2 - 3a_3 - 4a_4 - \cdots - n a_n \\ a_2 = 1a_2 + 0a_3 + 0a_4 + \cdots + 0a_n \\ a_3 = 0a_2 + 1a_3 + 0a_4 + \cdots + 0a_n \\ a_4 = 0a_2 + 0a_3 + 1a_4 + \cdots + 0a_n \\ \vdots \end{cases}$$

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$a_n = 0a_2 + 0a_3 + \dots + 1a_n$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = a_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_3\begin{pmatrix} -3 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + a_n\begin{pmatrix} -n \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Basis for Nul A

2) a)

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

YES! Since we have 2 pivot columns dim Col A = 2
∴ we need 2 linearly independent columns
and we can take any 2 column vectors
that are independent.

b)

$$\begin{bmatrix} 1 & 1 \\ 1 & x \\ x & 0 \end{bmatrix} \quad \begin{bmatrix} y \\ 1 \\ 0 \end{bmatrix}$$

$-x$, -1

$$\left[\begin{array}{cc|c} 1 & 1 & y \\ 1 & x & 1 \\ x & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & y \\ 0 & -1+x & -y+1 \\ 0 & -x & -xy \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & y \\ 0 & 1-x & y-1 \\ 0 & x & xy \end{array}\right]$$

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$$\left[\begin{array}{cc|c} 1 & 1 & y \\ 0 & 1-x & y-1 \\ 0 & x & xy \end{array}\right] \rightarrow$$

Consider two cases $x=0$ in which case the system is consistent regardless of y

$$x=0 \Rightarrow \left[\begin{array}{cc|c} 0 & 1 & y \\ 0 & 1 & y-1 \\ 0 & 0 & 0 \end{array}\right]$$ Consistent!

Now consider $x \neq 0$

divide row 3 by x since $x \neq 0$

$$R_3 \rightarrow \frac{R_3}{x}$$

$$\left[\begin{array}{cc|c} 0 & 1 & y \\ 0 & 1-x & y-1 \\ 0 & 1 & y \end{array}\right]$$

now swap bottom 2 rows

$$R_2 \leftrightarrow R_3$$

$$-(1-x)\left[\begin{array}{cc|c} 0 & 1 & y \\ 0 & 1 & y \\ 0 & 1-x & y-1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & y \\ 0 & 1 & y \\ 0 & 0 & -(1-x)y+y-1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 1 & y \\ 0 & 1 & y \\ 0 & 0 & xy-1 \end{array}\right]$$

$\therefore$ consistent if $xy-1=0$ or $y=\frac{1}{x}$

$\therefore$ 2 cases $x=0$ <u>OR</u> $y=\frac{1}{x}$ system consistent

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3 (a) Easy

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

not invertible since A is not square $\underline{n \times n}$

4) Calculate the inverse of the matrix

$$A = \begin{bmatrix} 0 & 4 & 3 \\ -1 & 2 & 0 \\ 3 & 2 & 5 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 0 & 4 & 3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$3 \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$-2 \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 8 & 5 & 0 & 3 & 1 \end{array}\right]$$

$$-2 \left[\begin{array}{ccc|ccc} -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & -4 & 0 & 0 & -2 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 3 & 1 \end{array}\right]$$

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$$3 \curvearrowright \left[\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & -2 & 0 \\ 0 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 3 & 1 \end{array}\right]$$

$$3 \curvearrowright \left[\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & -2 & 0 \\ 0 & 4 & 0 & -5 & 9 & 3 \\ 0 & 0 & -1 & -2 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 7 & 3 \\ 0 & 4 & 0 & -5 & 9 & 3 \\ 0 & 0 & -1 & -2 & 3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/2 & 7/2 & 3/2 \\ 0 & 1 & 0 & -5/4 & 9/4 & 3/4 \\ 0 & 0 & 1 & 2 & -3 & -1 \end{array}\right] \qquad (A \mid I) \Rightarrow (A^{-1} \mid I)$$

$\uparrow$ A^{-1}

test Solution

$A A^{-1}$

$$\begin{pmatrix} 0 & 4 & 3 \\ -1 & 2 & 0 \\ 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} -5/2 & 7/2 & 3/2 \\ -5/4 & 9/4 & 3/4 \\ 2 & -3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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5) FALSE

a) Counter Example

$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

invertible $\quad$ invertible

$C + D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ not invertible

5b) $CDC^{-1} = D$

$CDC^{-1}C = DC$

$CD = DC$ not necessarily True

5c) $Ax = b$ $\quad$ A $n \times n$

$Ax = b$ unique solution which means every $\vec{b} \in \mathbb{R}^n$ is linear combination of the columns of A, ∴ linear transformation is onto $\quad$ A invertible

TRUE

$Ax = 0$ has unique solution $x = 0$

null space of A has dimension 0

∴ A is invertible

P.270 LAY IMT

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\text{Col } A = \mathbb{R}^n$
o. $\text{rank } A = n$
p. $\text{nullity } A = 0$
q. $\text{Nul } A = \{\mathbf{0}\}$

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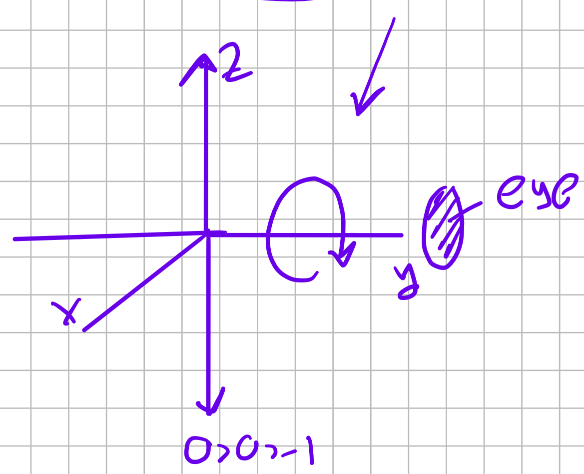
Concept: when you rotate about an axis, means ⊥ to axis.

9b) Solution

(1, 0, 0) ⟶ ⟨-1, 0, 0⟩ ⟶ ⟨0, 0, -1⟩

(0, 1, 0) ⟶ ⟨0, -1, 0⟩ ⟶ ⟨0, -1, 0⟩

(0, 0, 1) ⟶ ⟨0, 0, 1⟩ ⟶ ⟨-1, 0, 0⟩



6)

$$A = \begin{bmatrix} 1 & k-6 & 2 \\ 7 & k & k+7 \end{bmatrix}$$

$T: R^n \to R^m$

a) $n = 3 \quad m = 2$

b) one-to-one means pivot in every column which is not possible with only 2 rows

none

c) ONTO

$$-7 \begin{bmatrix} 1 & k-6 & 2 & | & 0 \\ 7 & k & k+7 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k-6 & 2 & | & 0 \\ 0 & -6k+42 & -7+k & | & 0 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|c} 1 & K-6 & 2 & 0 \\ 0 & -6(K-7) & K-7 & 0 \end{array}\right]$$

$K \neq 7$ will have pivot in every row and hence onto.

If $K \neq 7$ transformation is onto.

d) D matrix of $S: \mathbb{R}^8 \to \mathbb{R}^7$

S not onto   rank $A < 7$

dim Nul A + dim Col A = 8

A) ✓ Null space dim 6 ✓

B) ✓ Plane

C) ✓ True since S is not onto some vecter in $\mathbb{R}^7$ can not be a linear combination of the columns of D

7) Easy

a)

$$X = 4\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + -3\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

b) $[y]_B$ row reduction

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8 a) $x_1 - x_2 + 2x_3 - x_4 = 0$

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \end{bmatrix}$$

nul A = (x_1, x_2, x_3, x_4) such that $Ax = 0$

b)

$x_1 - x_2 + 2x_3 - 3x_4 = 0$

$$\begin{bmatrix} x_1 = x_2 - 2x_3 + 3x_4 \\ x_2 = x_2 + 0x_3 + 0x_4 \\ x_3 = 0x_2 + 1x_3 + 0x_4 \\ x_4 = 0x_2 + 0x_3 + 1x_4 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

∴ matrix A such that entries of its columns satisfy $x_1 - x_2 + 2x_3 - 3x_4 = 0$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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