Volumes by slicing I

Motivation: We now consider finding the volumes of solids that are not solids of revolution.

Volumes by Slicing: Consider a solid that lies between a plane perpendicular to the x axis at $x=a$ and a plane perpendicular to the x axis at $x=b$ and assume that the plane perpendicular to the x axis at a typical x value has a cross-sectional slice given by $A(x)$. The volume of the solid is the integral of the cross-sectional area $\Rightarrow V=\int_a^b A(x)\,dx$

Ex] A solid's base is a circle of radius 1 in the xy plane. Cross-sections perpendicular to the x axis are squares. Find the volume of the solid.

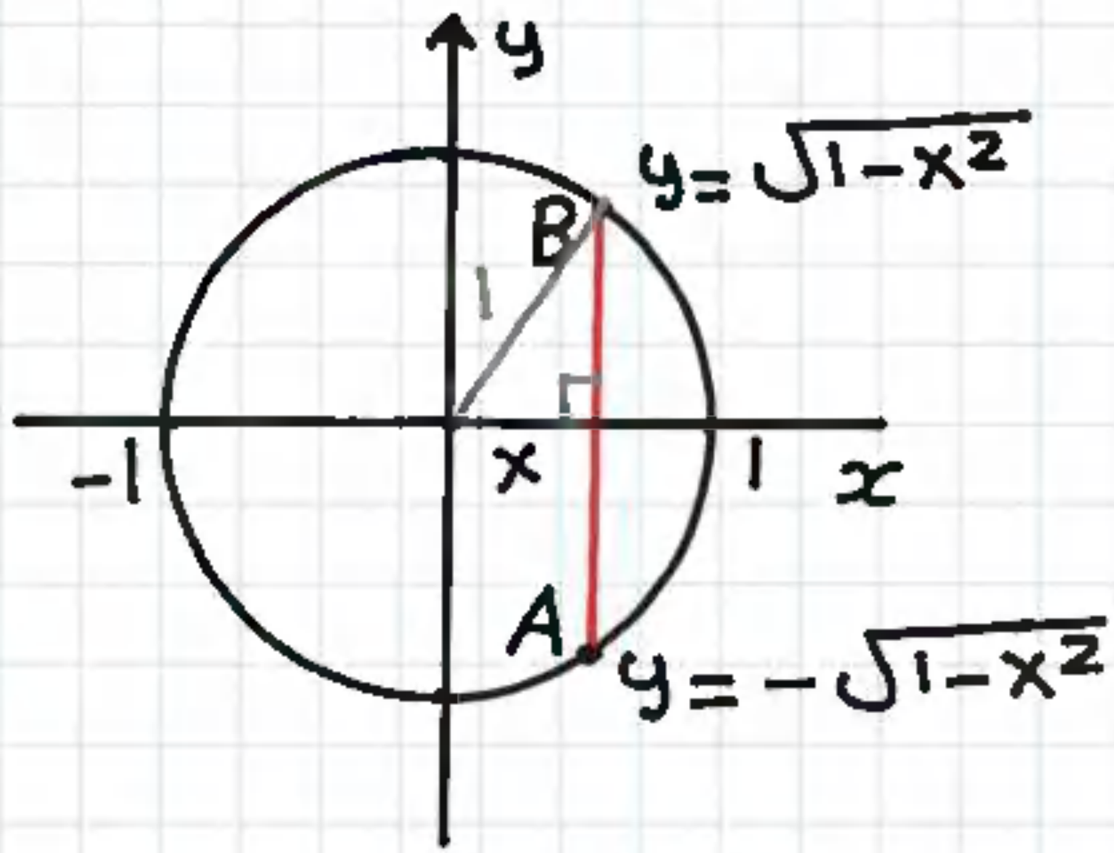
Step 1] Draw a picture of the cross sectional slice and the region in the xy plane that describes the solid.

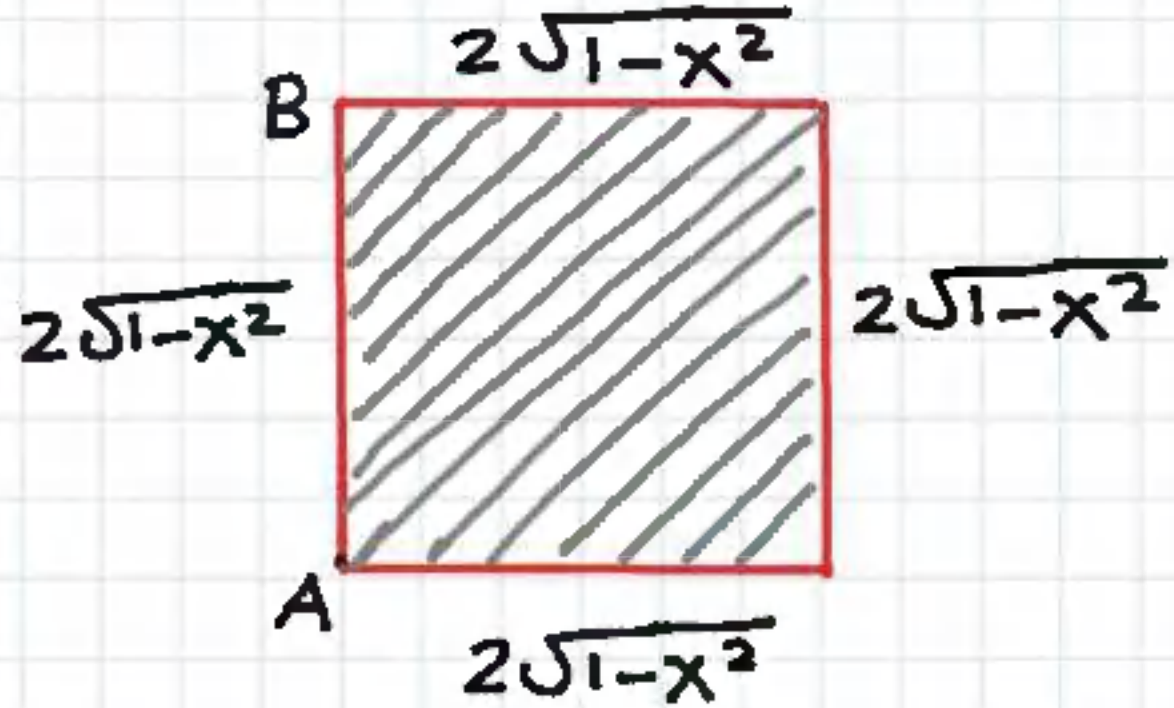
Step 2] Find a formula of cross-sectional slice perpendicular to the x axis such as $A(x)$

Step 3] Integrate Area slice $A(x)$ to find volume

$$V = \int_a^b A(x)\,dx$$

Step 1 Draw picture of base and area slice





Point B has y coord. $y=\sqrt{1-x^2}$ for typical $-1\leq x\leq 1$

Point A has y coord. $y=-\sqrt{1-x^2}$ for typical $-1\leq x\leq 1$

$\overline{AB}$ has length $\sqrt{1-x^2}--\sqrt{1-x^2}=2\sqrt{1-x^2}$

Step 2| Find a formula for $A(x)$

Since Area slice perpendicular to x axis are squares length of base $\overline{AB}$ is $2\sqrt{1-x^2}$ and height of Area slice is also $2\sqrt{1-x^2}$ since Area slices perpendicular to x axis are squares.

$\therefore\ A(x) = (2\sqrt{1-x^2})(2\sqrt{1-x^2}) = 4(1-x^2)$

Now that we have determined area slice $A(x)$ this will form the integrand of the volume formula $\Rightarrow V = \int_a^b A(x)\,dx$ where $a=-1$, $b=1$

and $A(x) = 4(1-x^2)$

step 3 | Integrate Area slice $A(x)$ to find Volume

$V = \int_a^b A(x)\,dx$ where $a=-1$, $b=1$ and $A(x)=4(1-x^2)$

$V = \int_{-1}^{1} 4(1-x^2)\,dx = 2\int_0^1 4(1-x^2)\,dx$

Symmetry
$A(x)=4(1-x^2)$
is even function

$V = 8\int_0^1 (1-x^2)\,dx = 8\left[x-\frac{x^3}{3}\right]_0^1$

$V = 8\left[1-\frac{1}{3}-(0-0)\right] = 8\left(\frac{2}{3}\right) = \frac{16}{3}$

Volume of solid $= \frac{16}{3}$

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A solid has a base in the xy plane bounded by the curves y=2x^2 and y=3-x^2. Cross sections perpendicular to the x axis are semicircular disks with diameter on the base. Find the volume of the solid solved example

Volumes by slicing 2

Ex] A solid has a base in the xy plane bounded by the curves $y=2x^2$ and $y=3-x^2$. Cross-sections perpendicular to the x axis are semicircular disks with diameter of the semicircle on the base. Find the volume of the solid.

Solution: Step 1] Find the points of intersection

$y=2x^2$, $y=3-x^2$ $\Rightarrow$ $Y=Y$ $\Rightarrow$ $2x^2=3-x^2$ $\Rightarrow$ $3x^2=3$

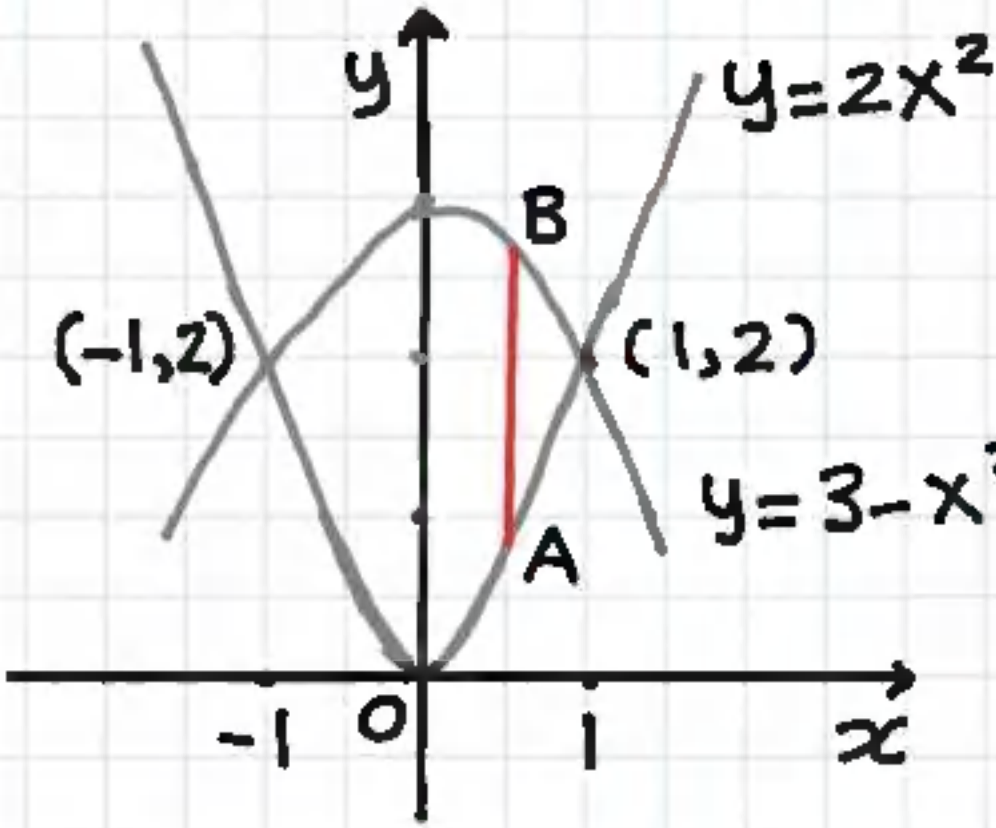
$x^2=1 \Rightarrow \sqrt{x^2}=\sqrt{1} \Rightarrow |x|=1 \Rightarrow x=\pm 1$

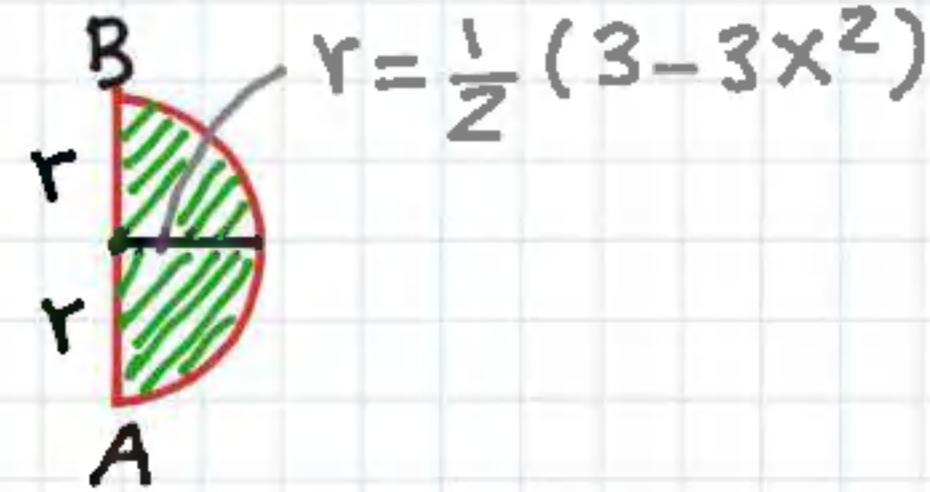
$x=1$ plug into $y=2x^2 \Rightarrow y=2 \Rightarrow (1,2)$

$x=-1$ plug into $y=3-x^2 \Rightarrow y=2 \Rightarrow (-1,2)$

$\therefore$ Points of intersection are $(1,2)$ and $(-1,2)$

step 2| Draw picture of base and area slice





Area Slice

Point B has y coord. $y = 3 - x^2$ for $-1 \leq x \leq 1$

Point A has y coord. $y = 2x^2$ for $-1 \leq x \leq 1$

$\overline{AB}$ forms diameter of semicircular Area slice and has length $|\overline{AB}| = 3 - x^2 - 2x^2 = 3 - 3x^2$

Step 3| Find a formula for $A(x)$

Since Area slice perpendicular to x axis are semicircles with Diameter $= |\overline{AB}| = 3-3x^2 \Rightarrow$ Radius $= \frac{1}{2}(3-3x^2)$

and since Area slices perpendicular to x axis are semicircles with radius $r = \frac{1}{2}(3-3x^2)$ and hence

$$A(x) = \frac{1}{2}\pi r^2 = \frac{1}{2}\cdot\pi\left(\frac{1}{2}(3-3x^2)\right)^2 = \frac{\pi}{8}(3-3x^2)^2$$

Now that we have determined area slice $A(x)$ this will form the integrand of the volume formula

$V = \int_a^b A(x)\,dx$ where $a=-1$, $b=1$ and

$$A(x) = \frac{\pi}{8}(3-3x^2)^2$$

step 4| Integrate Area slice $A(x)$ to find volume

$$V = \int_a^b A(x)\,dx \quad \text{where } a=-1,\ b=1,\ A(x) = \frac{\pi}{8}(3-3x^2)^2$$

$$V = \int_{-1}^{1} \frac{\pi}{8}(3-3x^2)^2\,dx = 2\int_0^1 \frac{\pi}{8}(3-3x^2)^2\,dx$$

symmetry
integrand $A(x)$
is even function

$$V = \frac{\pi}{4}\int_0^1 (3-3x^2)^2\,dx = \frac{\pi}{4}\int_0^1 (9-18x^2+9x^4)\,dx$$

$$V = \frac{\pi}{4}\left[9x - \frac{18x^3}{3} + \frac{9x^5}{5}\right]_0^1 = \frac{\pi}{4}\left[9-6+\frac{9}{5} - (0-0+0)\right]$$

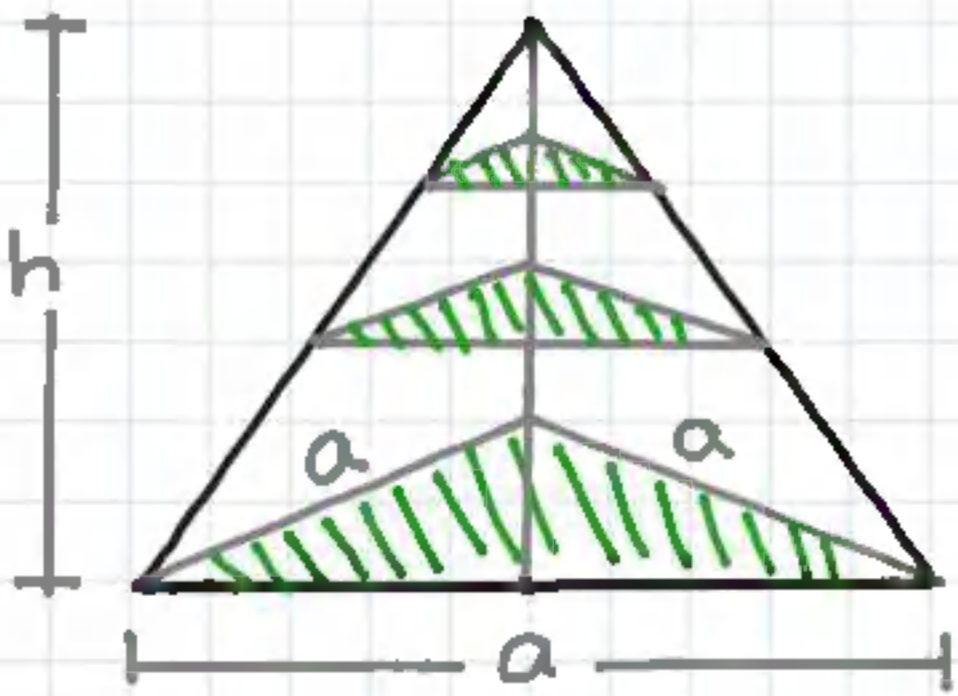
$$V = \frac{\pi}{4}\left[3+\frac{9}{5}\right] = \frac{\pi}{4}\left[\frac{15+9}{5}\right] = \frac{\pi}{4}\left[\frac{24}{5}\right] = \frac{6\pi}{5} \cong 3.77$$

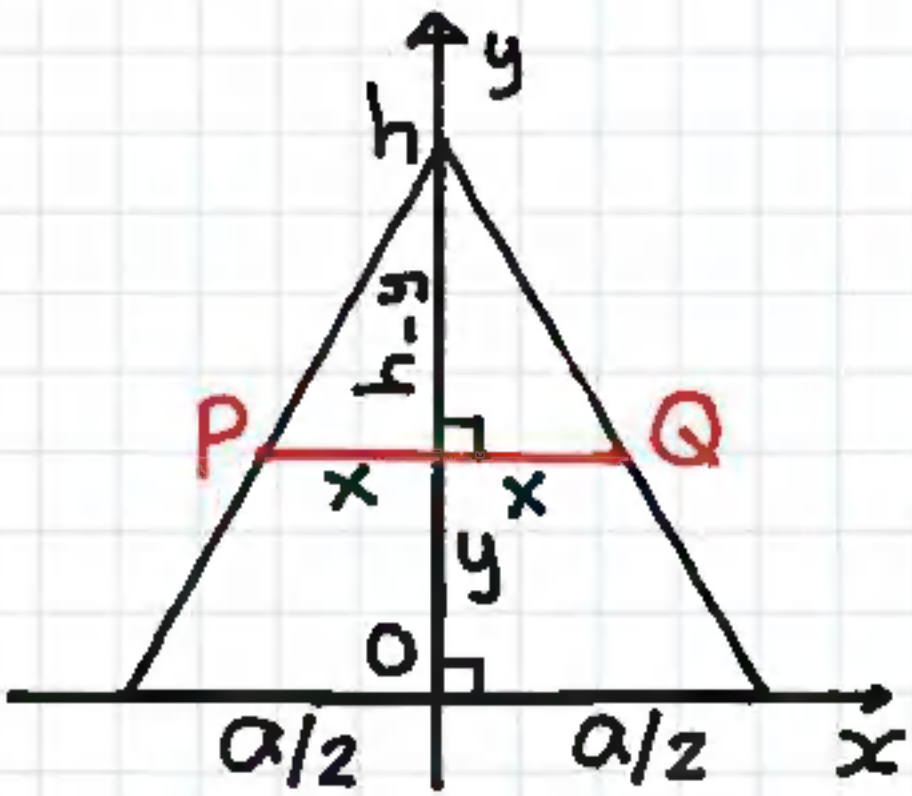
Find the volume of a pyramid with height h whose base is an equilateral triangle with side a solved Calculus 2 ∫ question

Volume by slicing 3

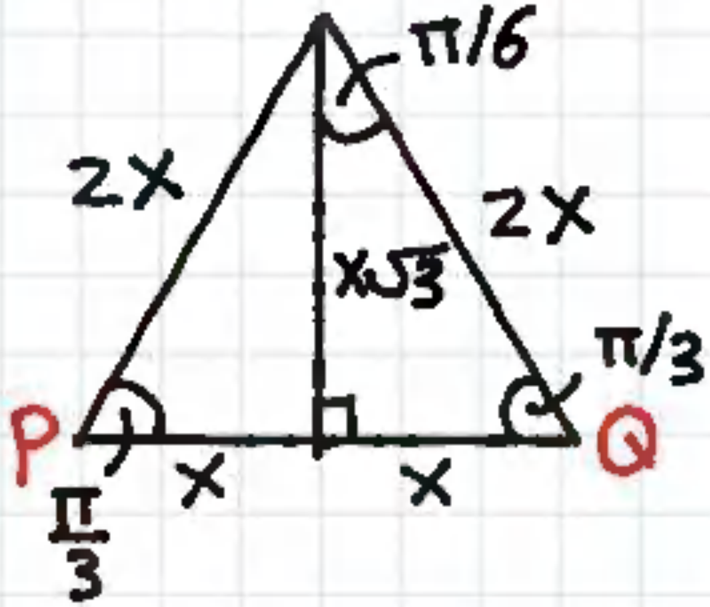
Find the volume of a pyramid with height h whose base is an equilateral triangle with side a.

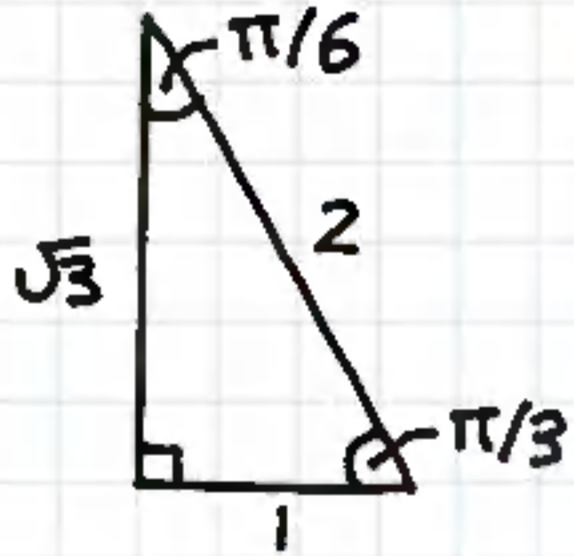
Solution: Step 1 Draw picture of solid and area slice.





Cross-sectional slice at height y is an equilateral triangle with side length $|\overline{PQ}| = 2x$

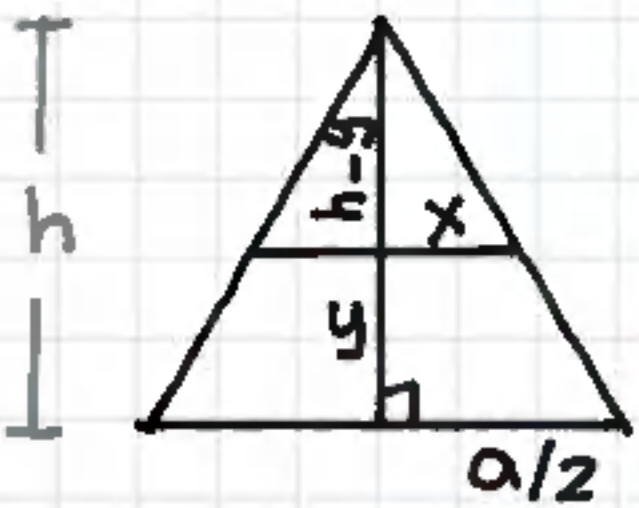


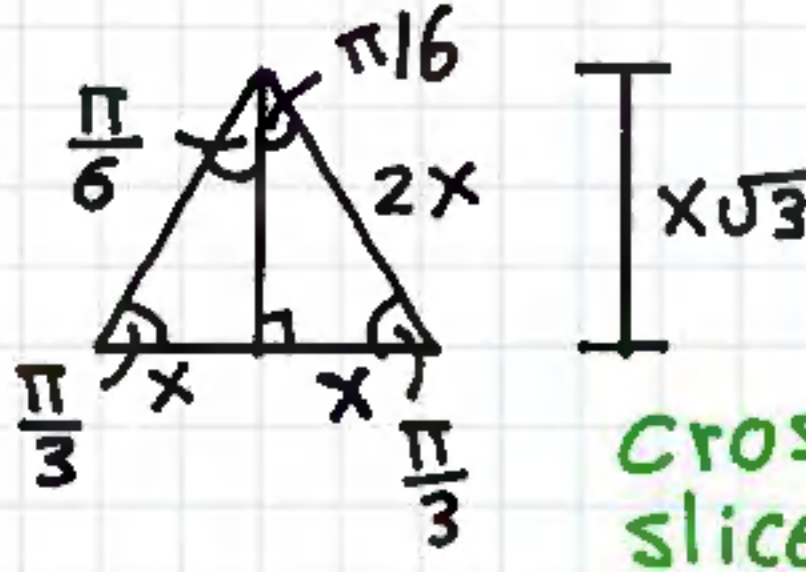


30-60-90 triangle

Key concept: Area slice at height y is Perp. ⊥ to the y axis so we need to find $A(y)$ and "add" up horizontal equilateral area slices in the direction of y axis to obtain volume.

step 2 Find a formula for $A(y)$





cross-sectional slice at height y

$$\text{Slice Area} = \frac{1}{2}(2x)(x\sqrt{3}) = \sqrt{3}\,x^2$$

but we nead slice area to be a function of y NOT x so let's apply similiar triangles to express x as a function of y

By Similiar triangles $\dfrac{h-y}{x} = \dfrac{h}{a/2}$

Slice Area $= \sqrt{3}\, x^2$

Similiar triangles $\frac{h-y}{x} = \frac{h}{a/2}$

Solving for x in terms of y $\Rightarrow$ $x = \frac{a}{2} \frac{(h-y)}{h}$

hence $A(y) = \sqrt{3}\, x^2 = \sqrt{3}\left(\frac{a}{2}\, \frac{h-y}{h}\right)^2$

$A(y) = \frac{\sqrt{3}\, a^2}{4h^2} (h-y)^2$

Now that we have found area slice A(y) this will form the integrand of the volume formula

$V = \int_c^d A(y)\, dy$ where $c = 0,\ d = h$ and $A(y)$ as

$A(y) = \frac{\sqrt{3}\, a^2}{4h^2} (h-y)^2$

step 3| Integrate Area slice $A(y)$ to find volume

$$V = \int_c^d A(y)\,dy \quad \text{where } c=0,\ d=h,\ A(y)=\frac{\sqrt{3}a^2}{4h^2}(h-y)^2$$

$$V = \int_0^h \frac{\sqrt{3}a^2}{4h^2}(h-y)^2\,dy = \frac{\sqrt{3}a^2}{4h^2}\int_0^h (h-y)^2\,dy$$

$$V = \frac{\sqrt{3}a^2}{4h^2}\left[\frac{(h-y)^3}{-3}\right]_0^h$$

Apply U-Subst.
$U = h-y \quad du = -dy$

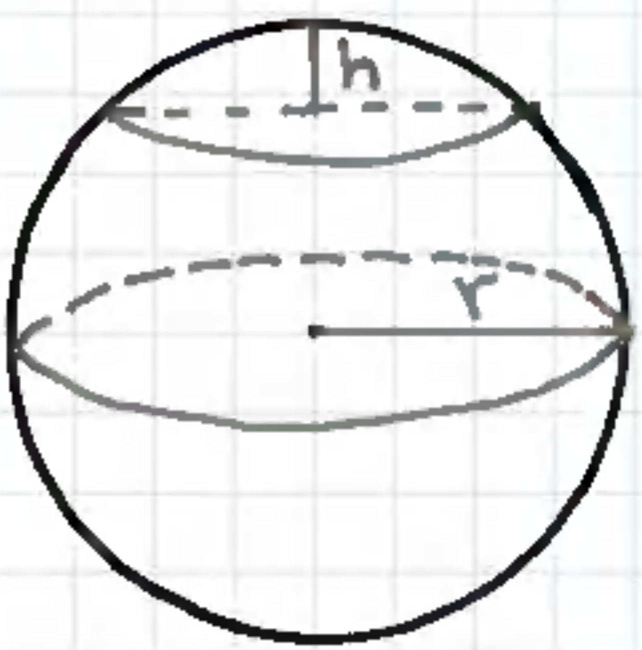
$$V = -\frac{\sqrt{3}a^2}{12h^2}\left[(h-y)^3\right]_0^h = -\frac{\sqrt{3}a^2}{12h^2}\left[0^3 - h^3\right]$$

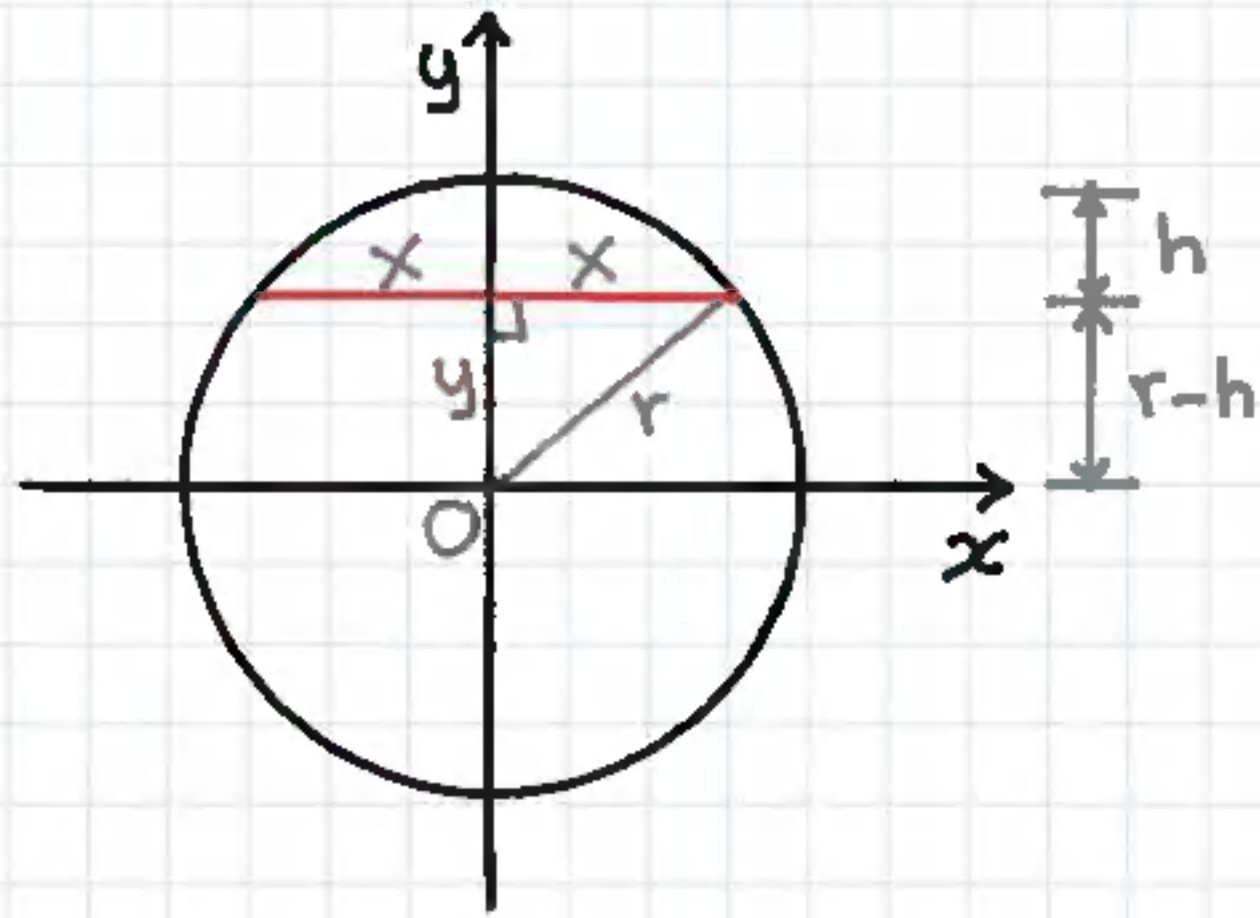
$$V = \frac{\sqrt{3}a^2 h}{12}$$ Volume of Tetrahedron

Volumes by Slicing 4

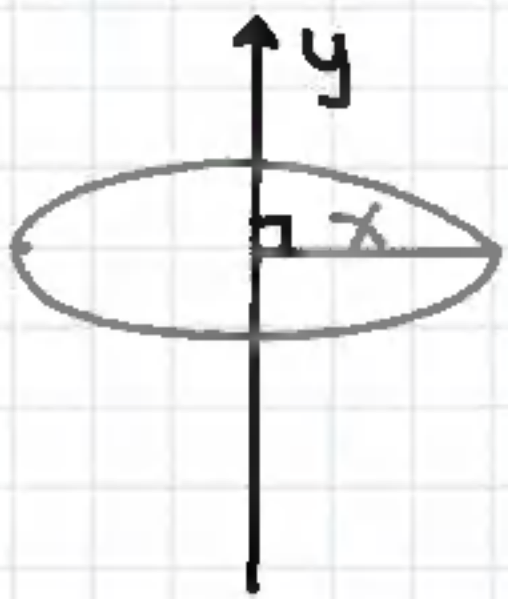
Ex] Find the volume of a cap of a sphere whose radius is r and the height of the spherical cap is h

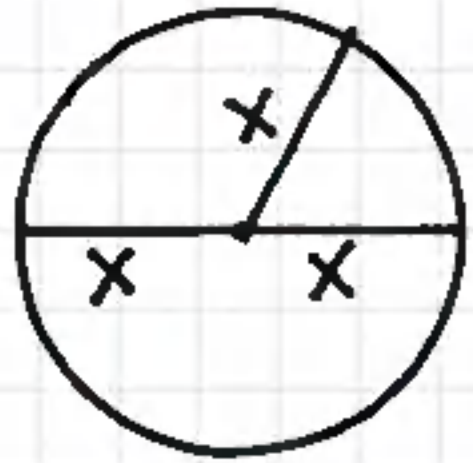
Solution: step 1] Draw picture of solid and area slice.





Cross-sectional slice at height y is a circular disk with radius $= x$ where $r-h \le y \le r$

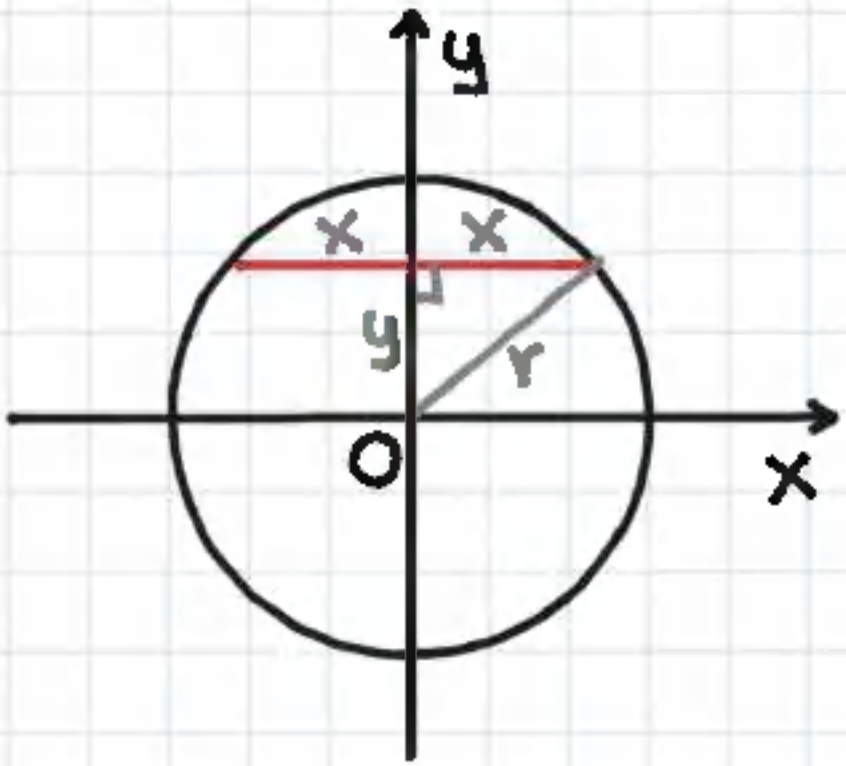


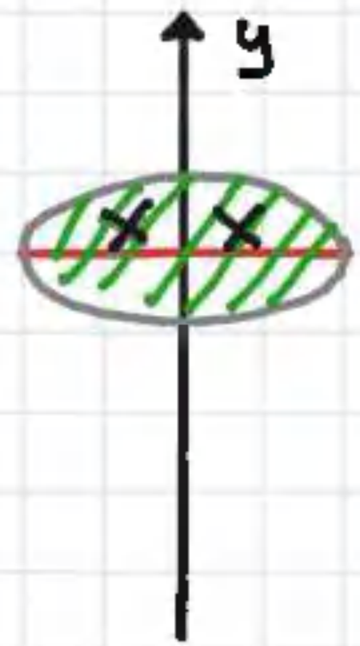


Area slice at height y is a circle of radius x

Key Concept: Area slice at height y is a circular disk with radius x perpendicular to the y axis, so we need to find $A(y)$ and "add" up horizontal circular area slices in the direction of the y axis to obtain Volume of spherical cap.

Step 2] Find a formula for $A(y)$





Slice Area $= \pi(\text{Radius})^2 = \pi x^2$

but we need slice area to be a function of y NOT x

so let's apply equation of circle with radius r

centered at the origin : $x^2 + y^2 = r^2$

$x^2 + y^2 = r^2 \Rightarrow x^2 = r^2 - y^2$

slice Area $= \pi x^2$

Circle equation $x^2 + y^2 = r^2 \Rightarrow x^2 = r^2 - y^2$

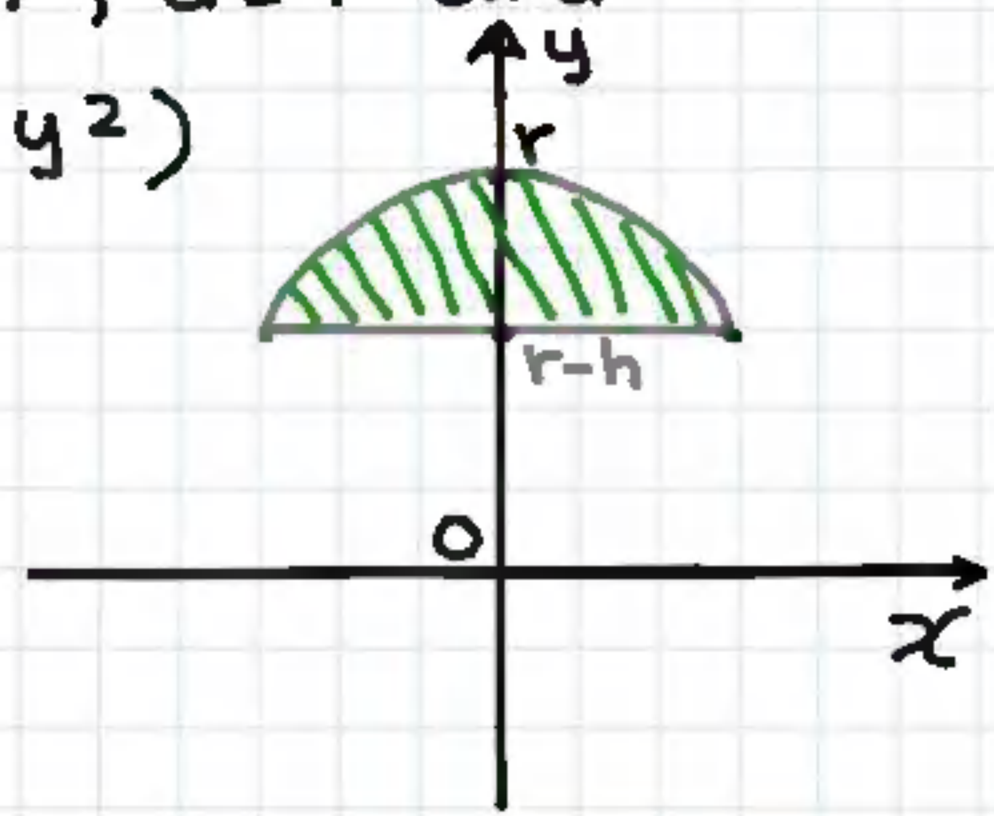
hence $A(y) = \pi x^2 = \pi(r^2 - y^2)$

Now that we have found area slice $A(y)$ this will form the integrand of the volume formula.

$V = \int_c^d A(y)\,dy$ where $c = r-h$, $d = r$ and

$A(y) = \pi(r^2 - y^2)$

where $r-h < y < r$



step 3] Integrate Area slice $A(y)$ to find volume

$$V = \int_c^d A(y)\,dy \quad \text{where } c = r-h,\ d = r,\ A(y) = \pi(r^2 - y^2)$$

$$V = \int_{r-h}^{r} \pi(r^2 - y^2)\,dy = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy$$

$$V = \pi\left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r} = \pi\left[r^3 - \frac{r^3}{3} - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right)\right]$$

$$V = \pi\left[\cancel{r^3} - \frac{r^3}{3} - \cancel{r^3} + r^2 h + \frac{(r-h)^3}{3}\right]$$

$$V = \pi\left[\frac{(r-h)^3}{3} - \frac{r^3}{3} + r^2 h\right]$$

$$V = \pi\left[\frac{(r-h)^3}{3} - \frac{r^3}{3} + \frac{r^2h}{1}\right] = \pi\left[\frac{(r-h)^3 - r^3 + 3r^2h}{3}\right]$$

$$(r-h)^3 = r^3 - 3r^2h + 3rh^2 - h^3$$

$$V = \pi\left[\frac{\cancel{r^3} - \cancel{3r^2h} + 3rh^2 - h^3 - \cancel{r^3} + \cancel{3r^2h}}{3}\right]$$

$$V = \frac{\pi\left[3rh^2 - h^3\right]}{3} = \frac{\pi h^2\left[3r - h\right]}{3} = \pi h^2\left(r - \frac{h}{3}\right)$$

$$V = \pi h^2\left(r - \frac{h}{3}\right)$$

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The base of a solid is an elliptical region in the xy plane with bounded curve 4x^2+y^2=4. Cross-sections perpendicular to the x axis are isosceles right triangles with hypotenuse in the base. Find the Volume ?

Volumes by Slicing 5

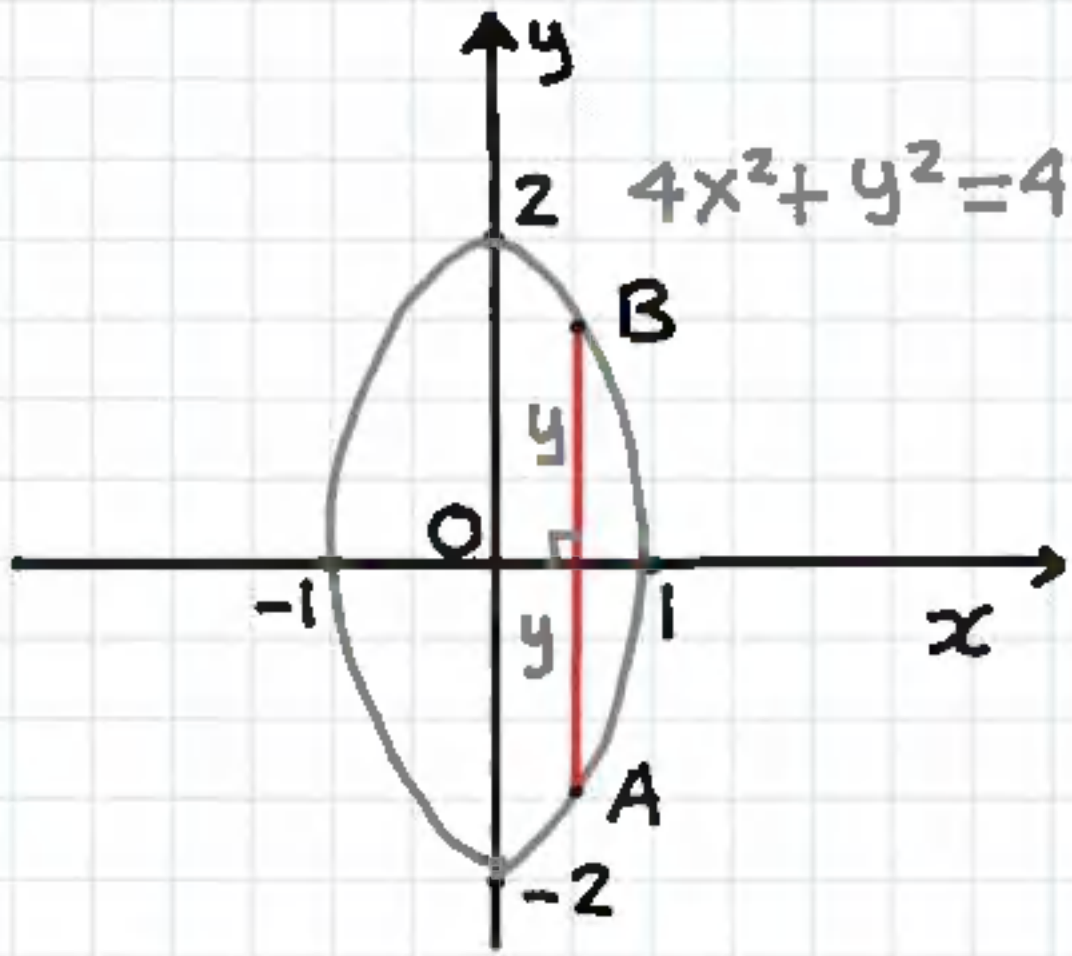
Ex] The base of a solid is an elliptical region in the xy plane with boundary curve $4x^2+y^2=4$. Cross-sections perpendicular to the x axis are isosceles right triangles with hypotenuse in the base. Find the volume of the solid.

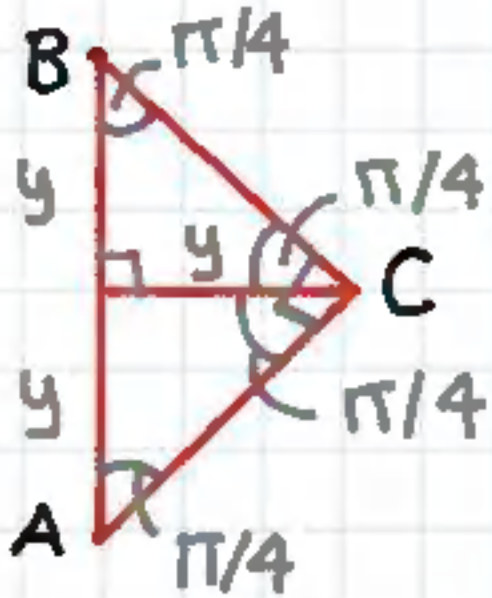
step 1] Draw picture of base and area slice.

$4x^2+y^2=4 \Rightarrow x=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$

$y=0 \Rightarrow 4x^2=4 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

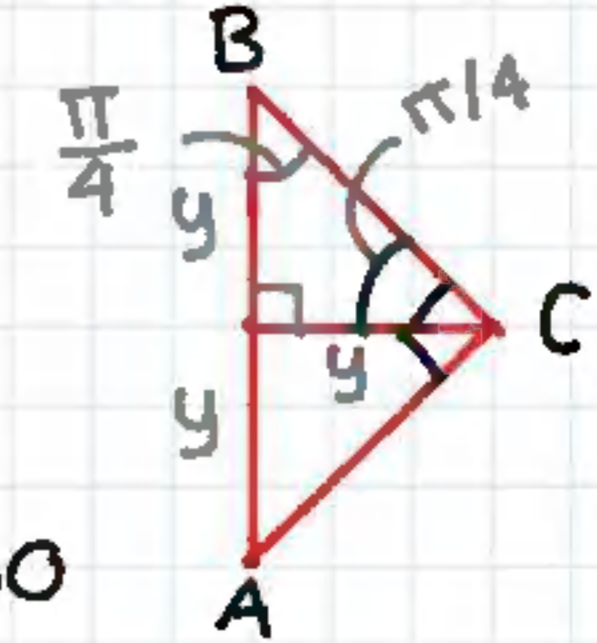
X and y intercepts of ellipse are $(1,0),(-1,0)$
$(0,2),(0,-2)$





Line segment $\overline{AB}$ is the hypotenuse of the area slice and since triangle ABC is an is isoceles triangle and angle C is 90° the height of the triangle bisects angle C and therefore the height of the triangle has length y.

Step 2| Find a formula for $A(x)$



Since triangle ABC is isoceles with hypotenuse in the base

$\overline{BC} = \overline{AC} \Rightarrow \angle B = \angle A$

$\angle A + \angle B + \angle C = 180 \Rightarrow 2\angle B + \angle C = 180$

but $\angle C = 90°$ since $\overline{AB}$ is hypotenuse

$2\angle B + 90 = 180 \Rightarrow 2\angle B = 90 \Rightarrow \angle B = 45° = \pi/4$ rads

and since height of triangle bisects angle C we obtain the usual 45-45-90 special triangle.

$\therefore$ Slice area $= \frac{1}{2}(\text{base}) * \text{height} = \frac{1}{2}|AB| \times \text{height}$

slice area $= \frac{1}{2}(2y) \times y = y^2$

slice area $= y^2$

But we need slice area to be a function of X NOT Y

So let's apply the elliptical boundary $4x^2 + y^2 = 4$

hence $A(x) = y^2 = 4 - 4x^2$

$A(x) = 4 - 4x^2$

Key Concept: Area slices are perpendicular to the x axis so we need to find $A(x)$ and "add" up vertical Isosceles area slices in the direction of the x axis to obtain volume of solid.

$A(x) = 4 - 4x^2$ where $-1 \leq x \leq 1$

step 3| Integrate Area slice $A(x)$ to find volume.

$$V = \int_a^b A(x)\,dx \quad \text{where } a=-1,\ b=1,\ A(x) = 4-4x^2$$

$$V = \int_{-1}^{1} (4-4x^2)\,dx = 2\int_0^1 (4-4x^2)\,dx$$

even Symmetry
Integrand $A(x)$
is even function

$$V = 2\left[4x - \frac{4x^3}{3}\right]_0^1 = 2\left[4 - \frac{4}{3} - (0-0)\right]$$

$$V = 2\left[\frac{4}{1} - \frac{4}{3}\right] = 2\left[\frac{12-4}{3}\right] = \frac{16}{3}$$

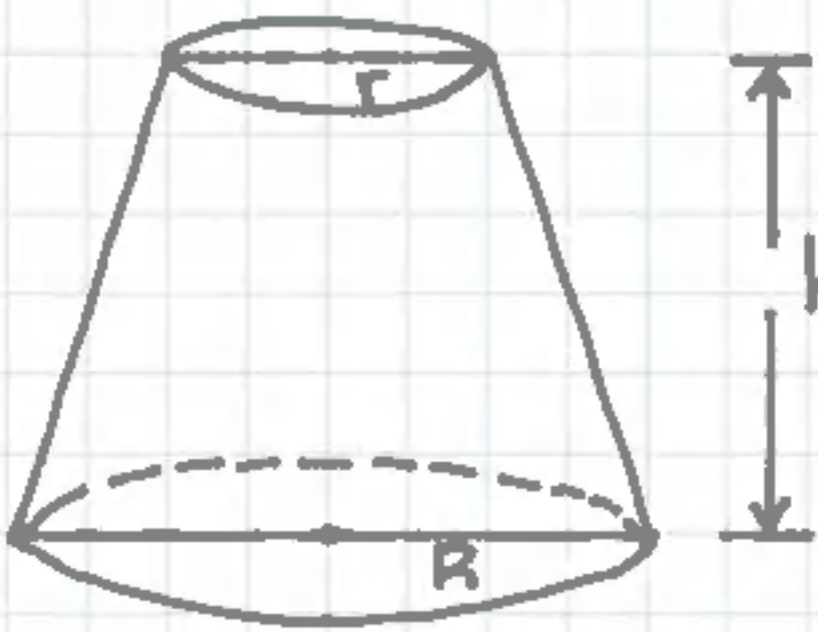
Volume of Solid = 16/3

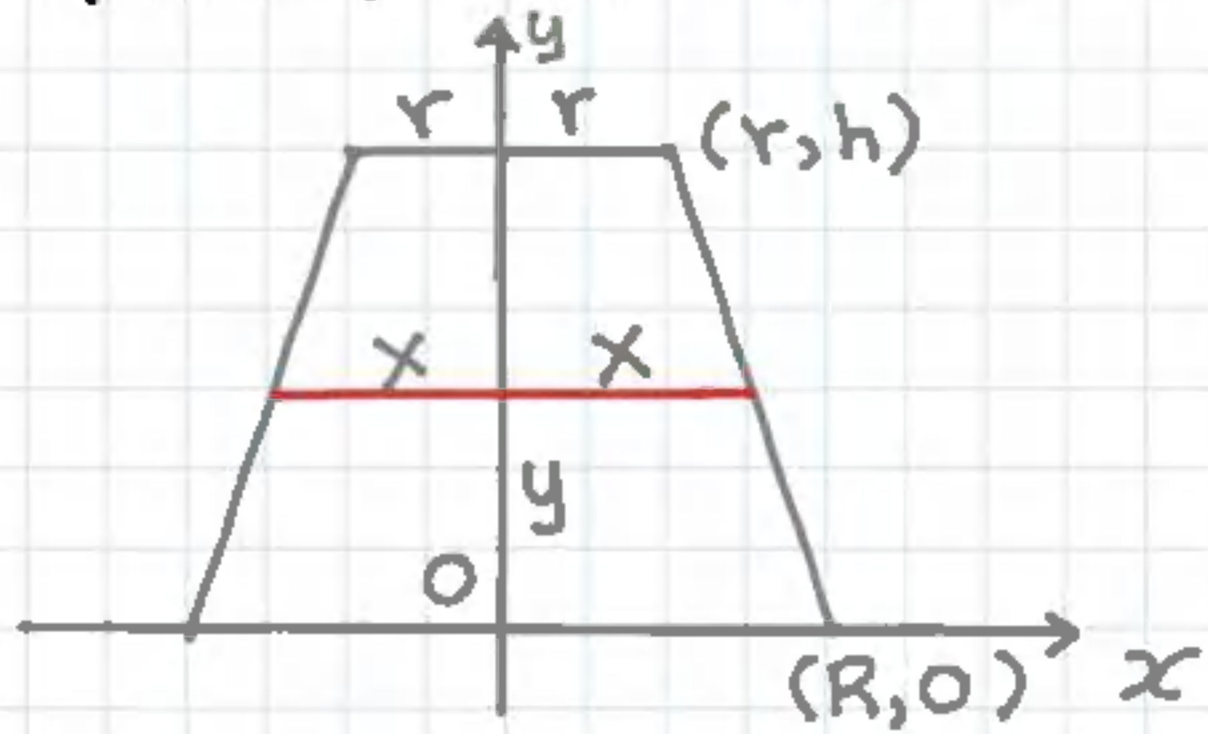
Find the volume of a right circular cone with height h, lower base radius R, and top radius r. (Top of the cone is cut off parallel to the base) solved example

Volumes by Slicing 6

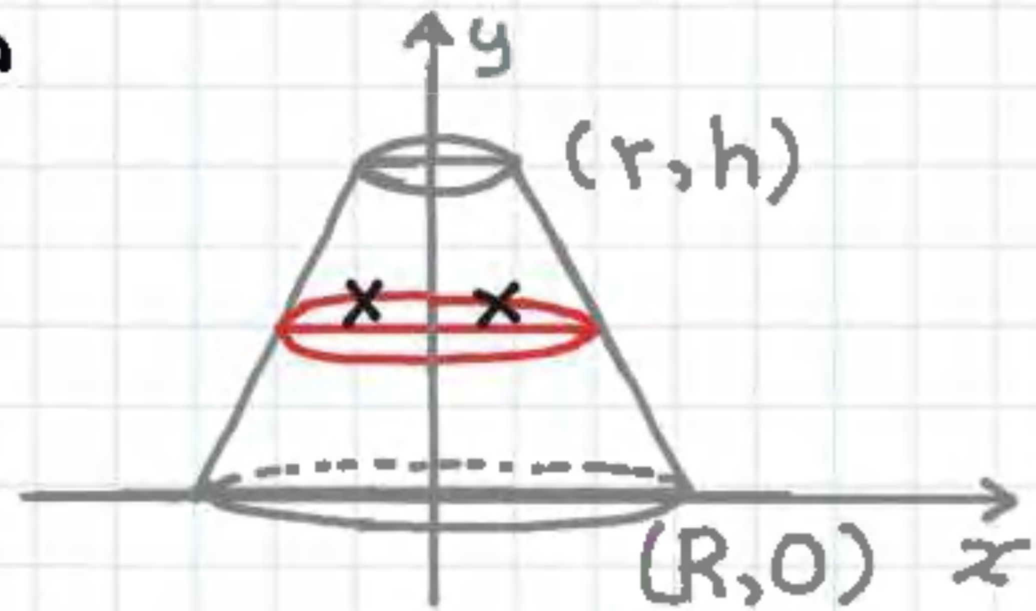
Ex] Find the volume of a right circular cone with height h, lower base radius R, and top radius r. (Top of the cone is cut off parallel to the base)

Solution : step1] Draw picture of solid and area slice.





cross-sectional slice at height y is a circular disk with radius $= x$ where $0 \le y \le h$



Let's solve for x in terms of y

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{h}{r-R}(x - R)$$

$$x = R + \left(\frac{r-R}{h}\right)y$$

Key Concept: Area slice at height y is a circular disk with radius x perpendicular to the y axis, so we need to find $A(y)$ and integrate $A(y)$ in the y direction to find volume of solid.

step 2| Find a formula for $A(y)$

slice Area $= \pi(\text{Radius})^2 = \pi x^2$

But we need slice area to be a function of y NOT x

So let's apply equation of line joining bottom disk to top disk: $x = R + \left(\frac{r-R}{h}\right)y$

$$A(y) = \pi x^2 = \pi\left(R + \left(\frac{r-R}{h}\right)y\right)^2$$

Now that we have found area slice $A(y)$ this will form the integrand of the volume formula.

$V = \int_c^d A(y)\,dy$ where $c=0$, $d=h$

step 3 | Integrate Area slice $A(y)$ to find volume

$$V = \int_c^d A(y)\,dy \quad \text{where } c=0,\ d=h,\ A(y) = \pi\left(R + \frac{(r-R)}{h}y\right)^2$$

$$V = \int_0^h \pi\left(R + \left(\frac{r-R}{h}\right)y\right)^2 dy$$

U- Substitution

$$u = R + \left(\frac{r-R}{h}\right)y$$

$$du = \left(\frac{r-R}{h}\right)dy$$

$$dy = \left(\frac{h}{r-R}\right)du$$

Let's change y limits to u limits

$$y=0 \Rightarrow u = R + \left(\frac{r-R}{h}\right)y \Rightarrow u = R$$

$$y=h \Rightarrow u = R + \left(\frac{r-R}{h}\right)y \Rightarrow u = r$$

$$V = \pi\left(\frac{h}{r-R}\right)\int_R^r u^2\,du$$

$$V = \pi \int_0^h \left(R + \left(\frac{r-R}{h}\right)y\right)^2 dy = \pi\left(\frac{h}{r-R}\right)\int_R^r u^2\,du$$

$$V = \pi\left(\frac{h}{r-R}\right)\left[\frac{u^3}{3}\right]_R^r = \frac{\pi h}{3(r-R)}\left[r^3 - R^3\right]$$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$ difference of cubes

$$V = \frac{\pi h}{3(r-R)}\left[r^3 - R^3\right] = \frac{\pi h \cancel{(r-R)}(r^2+rR+R^2)}{3\cancel{(r-R)}}$$

$$V = \frac{\pi h}{3}\left(R^2 + rR + r^2\right)$$

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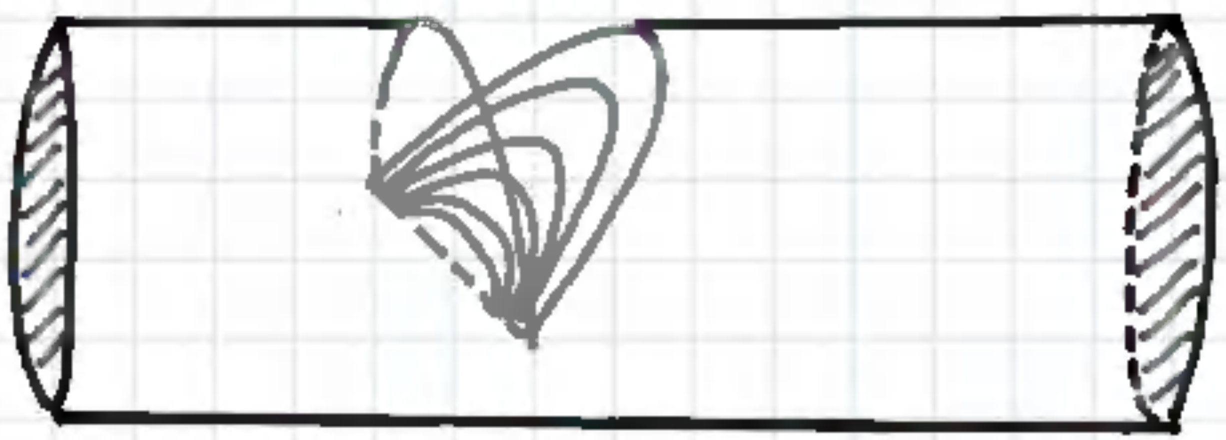
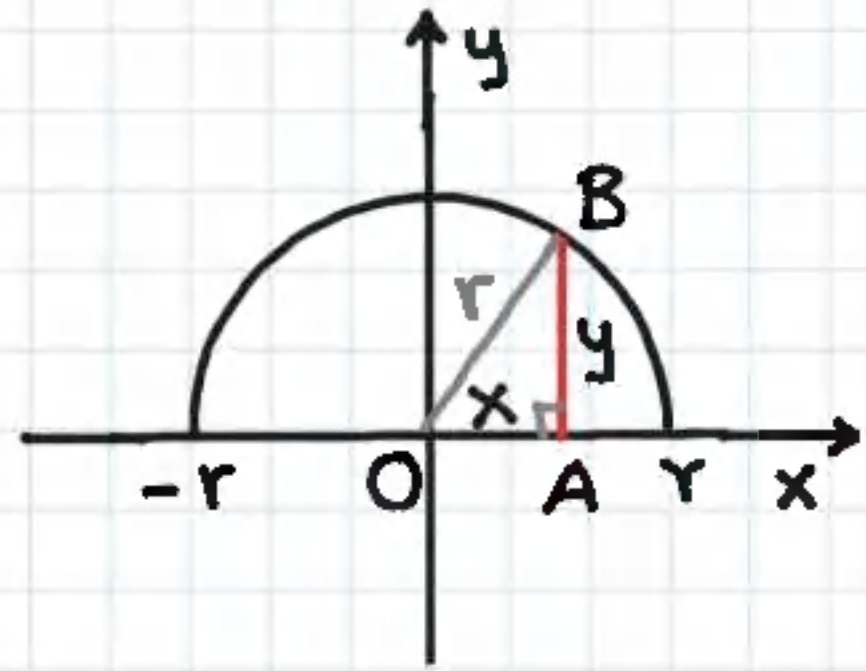
Volumes by Slicing 7

Ex] A wedge is cut out of a log with the shape of a circular cylinder of radius r. The wedge is removed from the log by a vertical cut Perpendicular to the axis of cylinder and another cut at an angle of 45° along the diameter of the log. Find the volume of wedge removed.

Step 1] Draw picture of solid and area slice





Let's place the x axis along the diameter where the vertical cut and the 45° cut meet. Area slice perpendicular to x axis at distance x from the origin is the triangle ABC

whose base is $|\overline{AB}| = y = \sqrt{r^2 - x^2}$

whose height $|\overline{BC}| = y\tan(\pi/4) = \sqrt{r^2 - x^2}$

Note : $\tan(\pi/4) = 1$

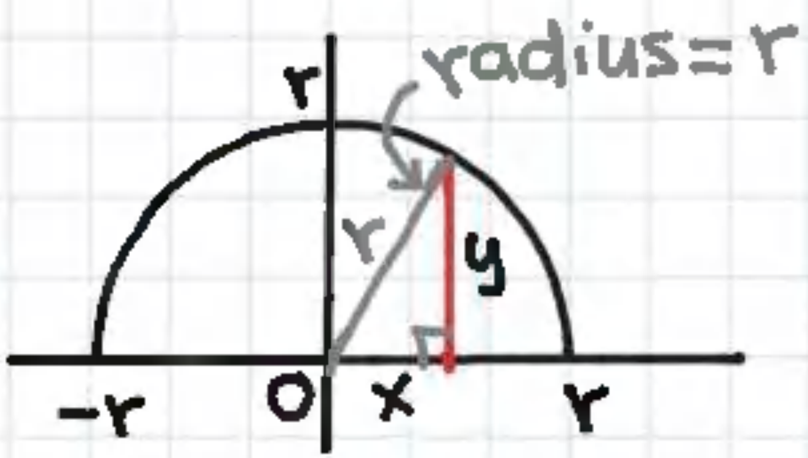
Step 2| Find a formula for $A(x)$

Area of triangle $= \frac{1}{2}$ (base) x height

$$A(x) = \frac{1}{2} |\overline{AB}| |\overline{BC}| = \frac{1}{2} y . y = \frac{1}{2} \left(\sqrt{r^2 - x^2}\right)^2$$

$$A(x) = \frac{1}{2} (r^2 - x^2)$$

Note: since we have placed the X axis along the diameter, then the base of the wedge is a semicircle with equation $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$



Step 3] Integrate Area slice $A(x)$ to find Volume

$$V = \int_a^b A(x)\,dx \quad \text{where } a=-r,\ b=r,\ A(x)=\frac{1}{2}(r^2-x^2)$$

$$V = \int_{-r}^{r} \frac{1}{2}(r^2-x^2)\,dx = 2\int_0^r \frac{1}{2}(r^2-x^2)\,dx$$

Apply Symmetry
Integrand $A(x)$
is Even function

$$V = \cancel{2}\cdot\frac{1}{\cancel{2}}\int_0^r (r^2-x^2)\,dx$$

$$V = \left[r^2x - x^3/3\right]_0^r = \left[r^3 - \frac{r^3}{3} - (0-0)\right]$$

$$V = \frac{2}{3}r^3$$