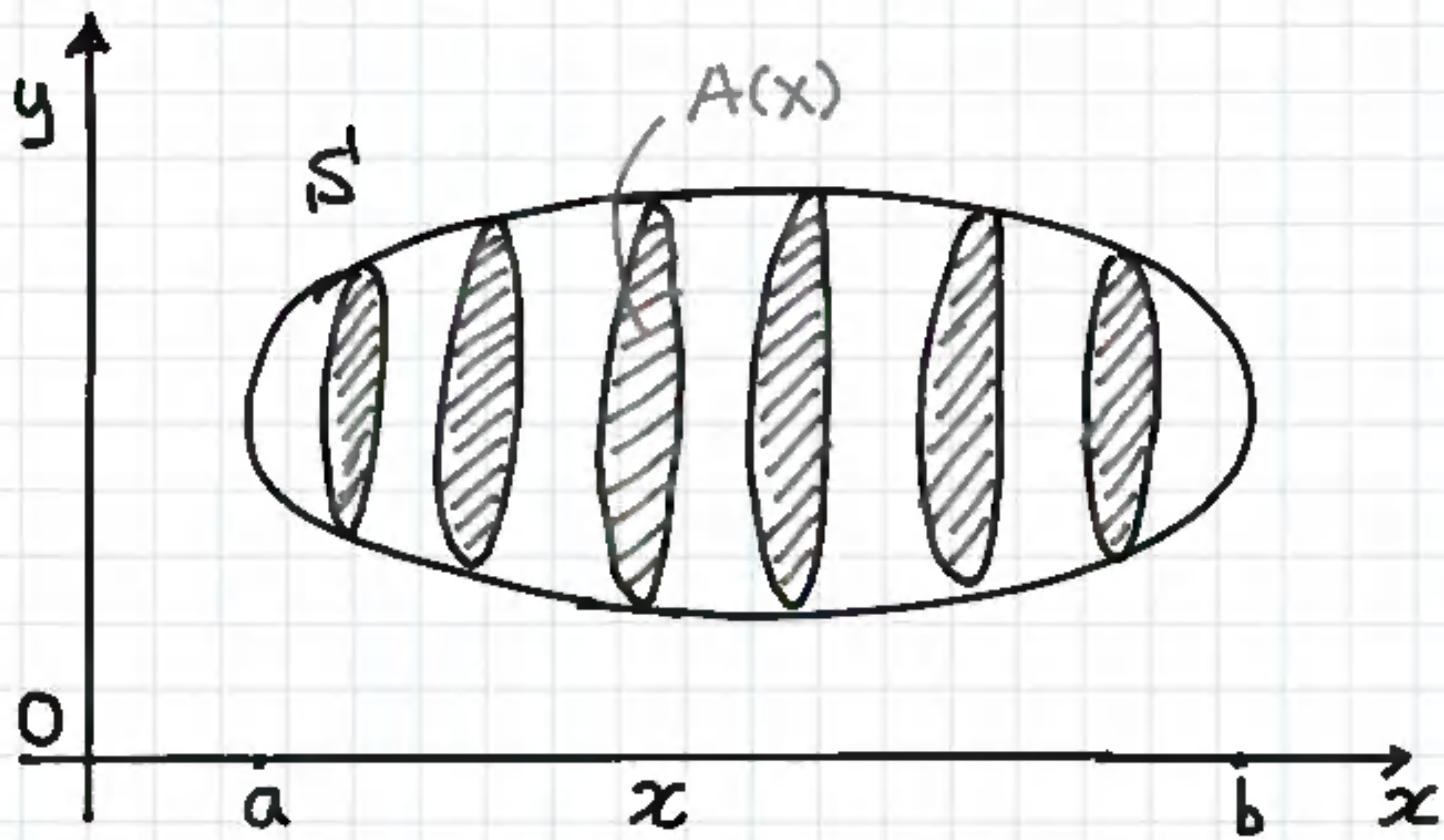
Volumes I (Introduction)

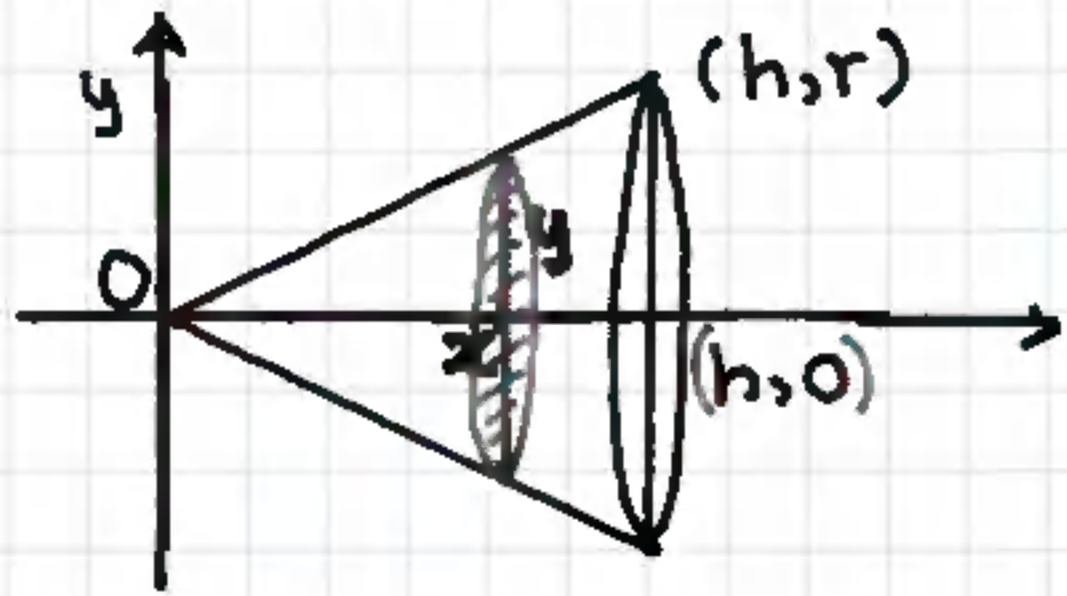


Definition: S is a 3-D solid extending from $x=a$ to $x=b$, and cross sections perpendicular to x axis defined by a contiuous function $A(x)$ then the volume of solid is $V=\int_a^b A(x)\,dx$

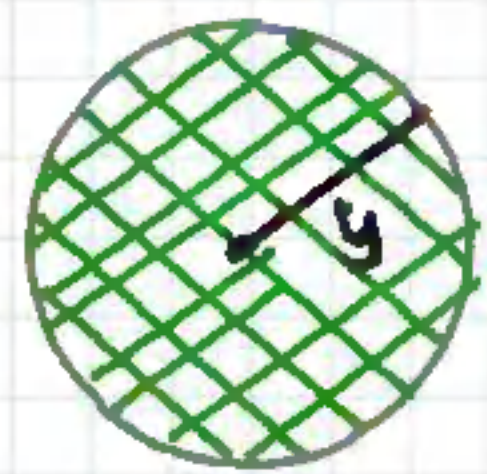
Prove that the volume of a cone of radius r and height h is V=(πr^2 *h)/3 solved example

Ex] Prove that the volume of a cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$

Solution: Place the cone with axis of symmetry along the x axis.



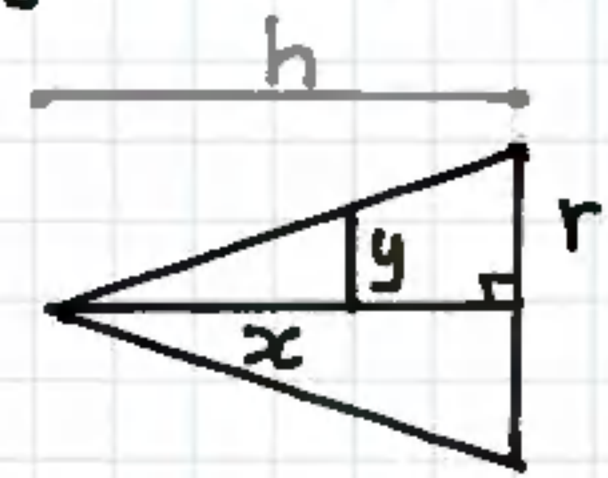
$A = \pi y^2$ (circle)



Areas of cross sectional slice perpendicular to x axis is $A = \pi y^2$ but we need cross sectional slice to be a function of x.

Goal: Volume $= \int_0^h A(x)\,dx$

lets apply similiar triangles to express y in terms of x



$$\frac{y}{x} = \frac{r}{h} \Rightarrow y = \frac{rx}{h}$$

$$A(x) = \pi y^2 = \pi\left(\frac{rx}{h}\right)^2 = \frac{\pi r^2}{h^2}x^2$$

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{\pi r^2}{h^2}x^2\,dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$V = \frac{\pi r^2}{h^2}\left.\frac{x^3}{3}\right|_0^h = \frac{\pi r^2}{3h^2}\cdot h^3 - 0 = \frac{\pi r^2 h}{3}$$

We have proven that the volume of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$ by setting up volume $= \int_a^b A(x)\,dx$

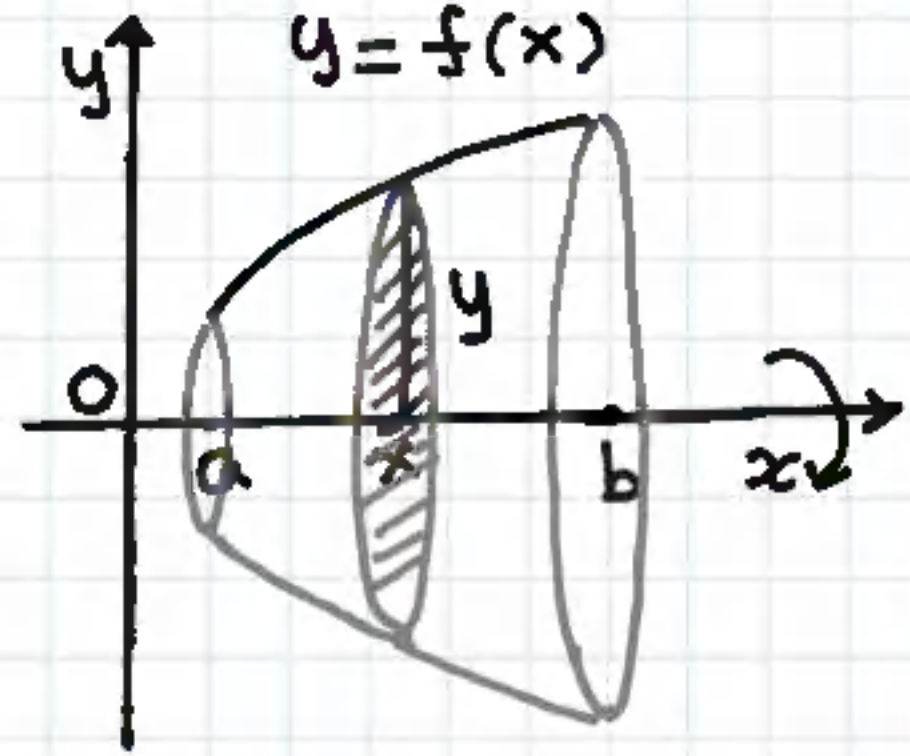
Summary :

$a=0$, $b=h$, $A(x) = \pi y^2 = \frac{\pi r^2}{h^2} x^2$

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{\pi r^2}{h^2} x^2\,dx = \frac{1}{3}\pi r^2 h$$

Key concept: Cross sectional slices A(x) must be perpendicular to the direction of the x axis since the area slices A(x) are summed up in direction of x axis.

Volumes 2 (Disk method)



Motivation: For computing volumes of solids of revolution all the cross sectional slices are circular disks with radius $f(x)$, therefore the circular cross sections that are perpendicular to the x axis with radius $y = f(x)$ have area:

$$A(x) = \pi(\text{radius})^2 = \pi y^2 = \pi[f(x)]^2$$

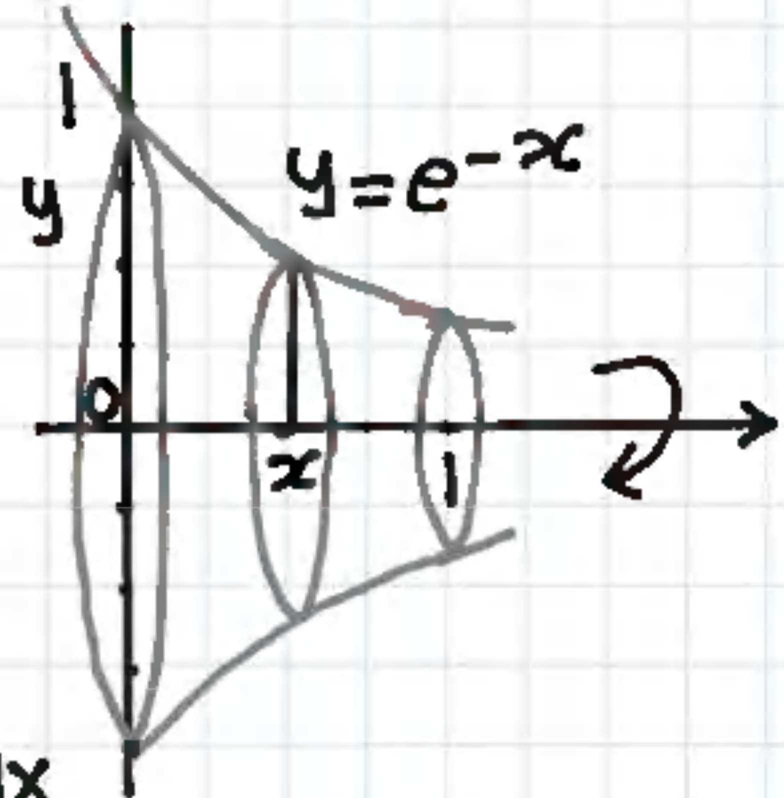
Therefore the volume of solid of revolution is:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx$$

Volume formula for revolving $f(x)$ for $a \le x \le b$ about the x axis.

Find the volume of solid of revolution that is generated by revolving f(x)=e^(-x) from x=0 to x=1 about the x axis solved example

Ex] Find the volume of solid of revolution that is generated by revolving $f(x)=e^{-x}$ from $x=0$ to $x=1$ about the x axis.

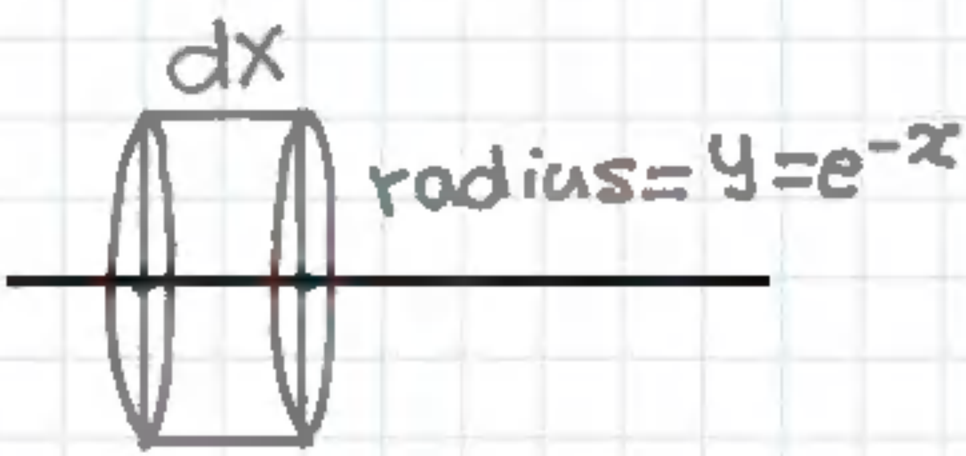


Solution: Cross sections $\perp$ to axis of rotation x axis are circular disks with radius $r=e^{-x}$

disk radius $r=y=e^{-x}$

disk Volume $dV=\pi r^2 dx=\pi(e^{-x})^2 dx$

Volume of Solid $V=\int_0^1 dV$



$V=\int_0^1 \pi e^{-2x} dx$

$$V = \pi \int_0^1 e^{-2x}\,dx$$

Apply U-Substitution

$u = -2x \quad du = -2\,dx$

$$V = \frac{\pi e^{-2x}}{-2}\Big|_0^1 = \frac{\pi}{-2}\left[e^{-2} - e^{-0}\right] = -\frac{\pi}{2}\left[e^{-2} - 1\right]$$

$$V = \frac{\pi}{2}\left[1 - e^{-2}\right]$$ Volume of solid of revolution

Summary: $V = \int_a^b \pi\,[f(x)]^2\,dx$

$a = 0$, $b = 1$, $f(x) = e^{-x}$

$$\therefore\ V = \pi \int_0^1 (e^{-x})^2\,dx = \pi \int_0^1 e^{-2x}\,dx = \frac{\pi e^{-2x}}{-2}\Big|_0^1$$

$$V = -\frac{\pi}{2}\left[e^{-2} - 1\right]$$

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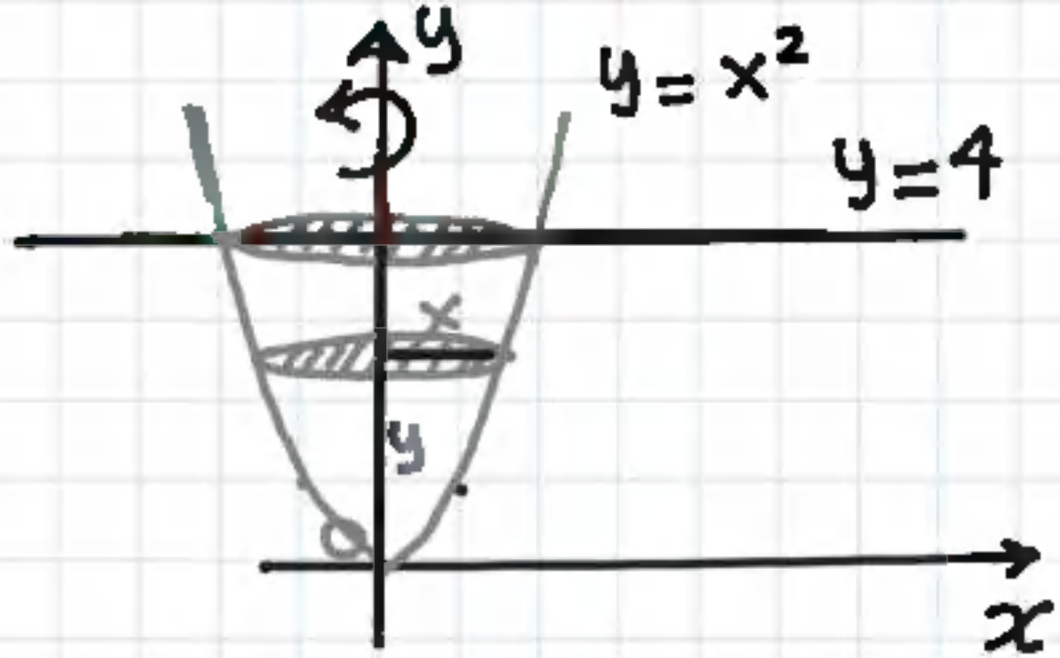
Find the volume of the solid obtained by rotating the region bounded by y=x^2,y=4 and x=0 about the y axis

solved example

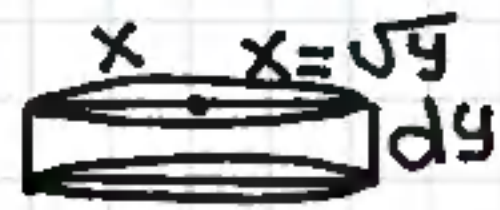
Volumes 3 (Disk method about the y axis)

Ex| Find the volume of the solid obtained by rotating the region bounded by $y=x^2$, $y=4$, and $x=0$ about the y axis.

Solution:



Since the bounded region is being revolved about the y axis we slice the circular cross sections perpendicular to the y axis and integrate with respect to y.



$y=x^2 \Rightarrow x=\pm\sqrt{y} \Rightarrow x=\sqrt{y}$ radius $= x = \sqrt{y}$

disk radius $r = x = \sqrt{y}$

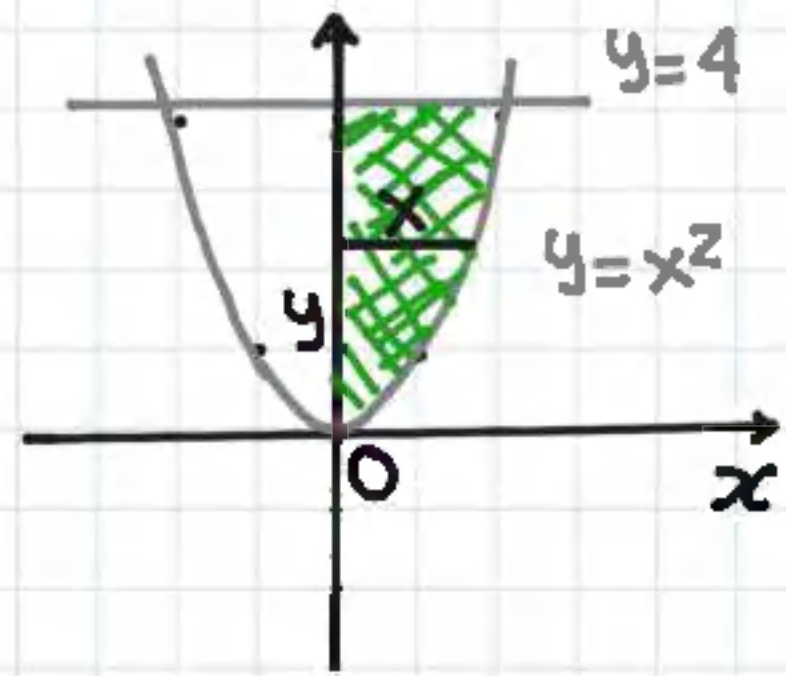
disk volume $dV = \pi x^2 dy = \pi(\sqrt{y})^2 dy$

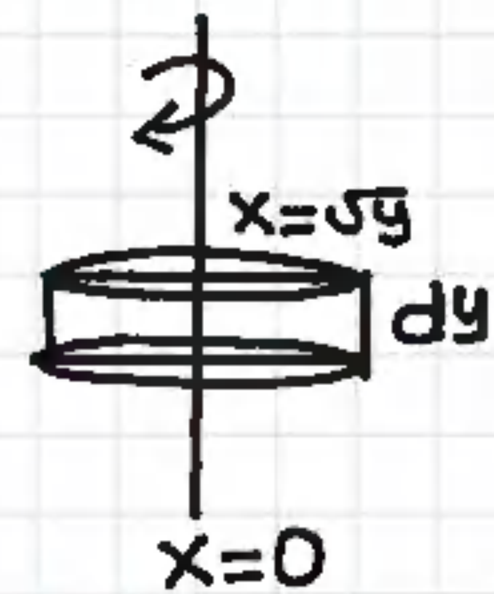
$dV = \pi y \, dy$

Volume $V = \int_0^4 \pi y \, dy$

$V = \pi \int_0^4 y \, dy = \frac{\pi y^2}{2} \Big|_0^4 = \frac{\pi}{2}(16-0)$

$V = 8\pi$

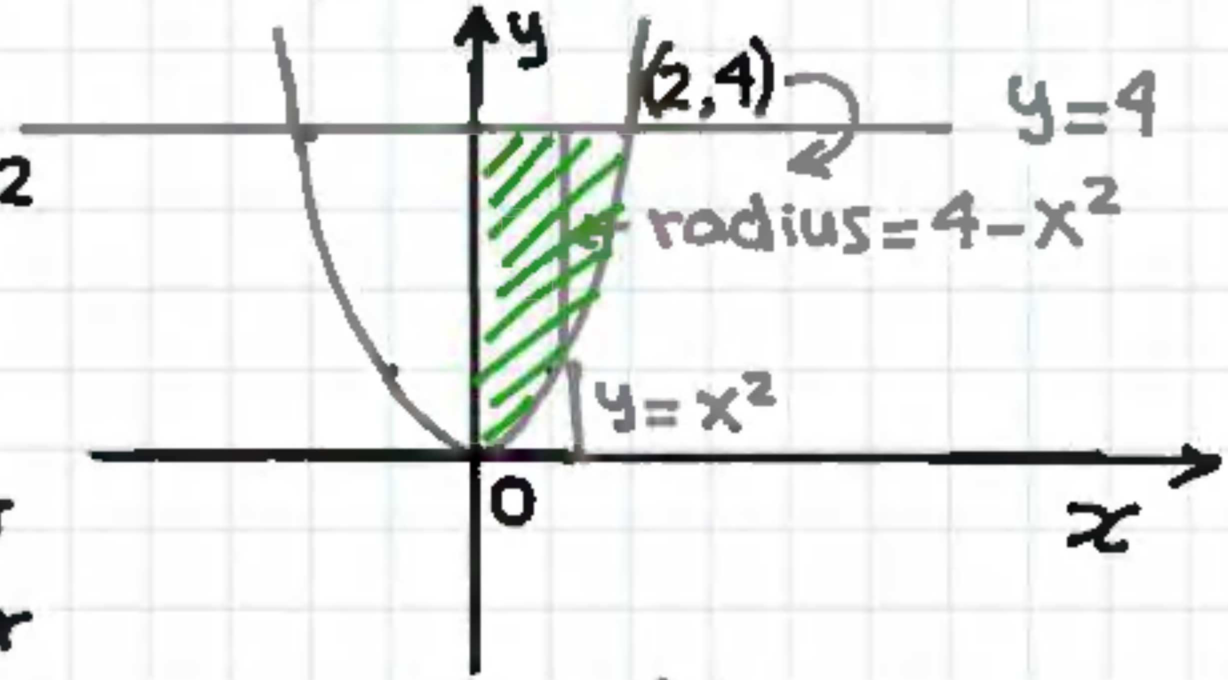




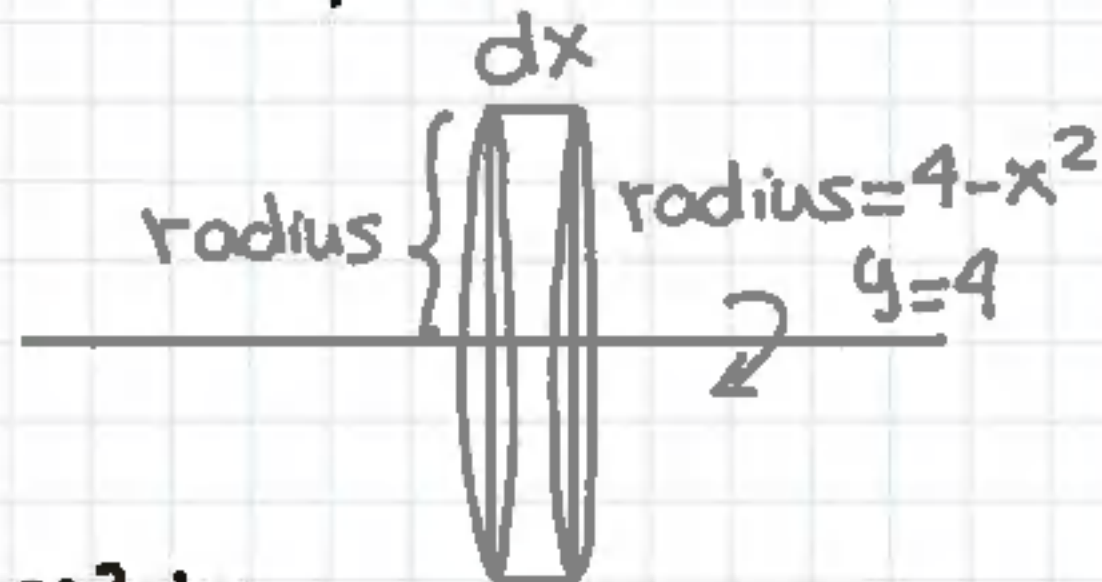
Find the volume of the solid obtained by revolving the region bounded by y=x^2,y=4 and x=0 about the line y=4 solved example

Ex] Find the volume of the solid obtained by revolving the region bounded by $y=x^2$, $y=4$ and $x=0$ about the line $y=4$

Solution: $x^2=4 \Rightarrow x=\pm 2 \Rightarrow x=2$



Since the bounded region is is being revolved about the line $y=4$ we slice the circular cross sections perpendicular to the x axis and integrate with respect to x.



disk radius : $r=4-x^2$

disk volume: $dV=\pi(4-x^2)^2dx$

Volume : $V=\int_0^2 dV=\int_0^2 \pi(4-x^2)^2dx$

$$V = \int_0^2 \pi(4-x^2)^2\,dx = \pi\int_0^2 (4-x^2)(4-x^2)\,dx$$

$$V = \pi\int_0^2 \left(16-4x^2-4x^2+x^4\right)dx = \pi\int_0^2 (16-8x^2+x^4)\,dx$$

$$V = \pi\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2 =$$

$$V = \pi\left[16(2) - \frac{8}{3}(2)^3 + \frac{2^5}{5}\right] - \pi[0-0+0]$$

$$V = \pi\left[32 - \frac{64}{3} + \frac{32}{5}\right] = \pi\left[\frac{480-320+96}{15}\right] = \frac{\pi(256)}{15}$$

$$V = \frac{256\pi}{15} \cong 53.62$$

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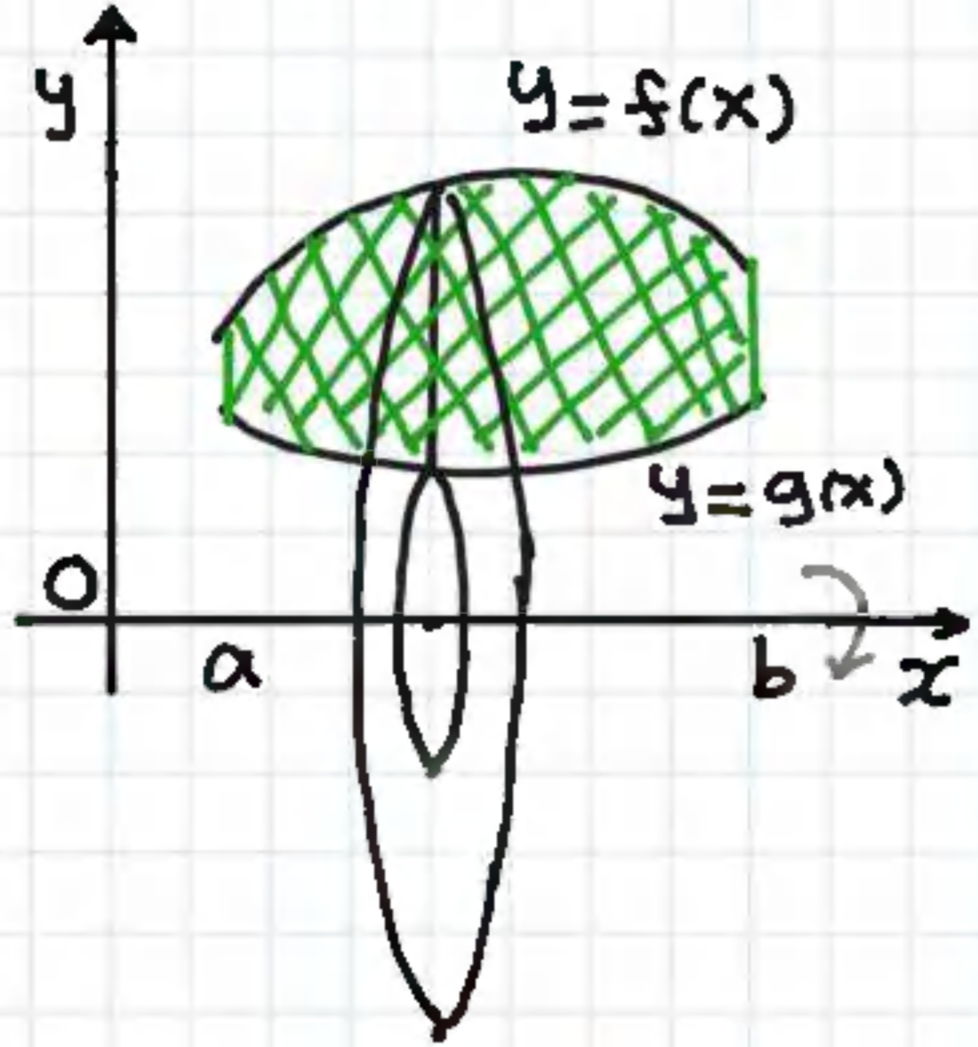
Volumes 4 (washer method)

Motivation: Let's modify the Disk method to find the volume of a solid of revolution bounded by the graphs of $f(x)$ and $g(x)$ where $f(x) \geq g(x)$ for x in $[a,b]$ rotated about the x axis.

Key concept: cross section thru the solid perpendicular to the x axis is a circular washer with outer radius $f(x)$ and inner radius $g(x)$.



Washer Volume

$$dV=(\pi R_{out}^2 - \pi R_{inner}^2)\,dx$$

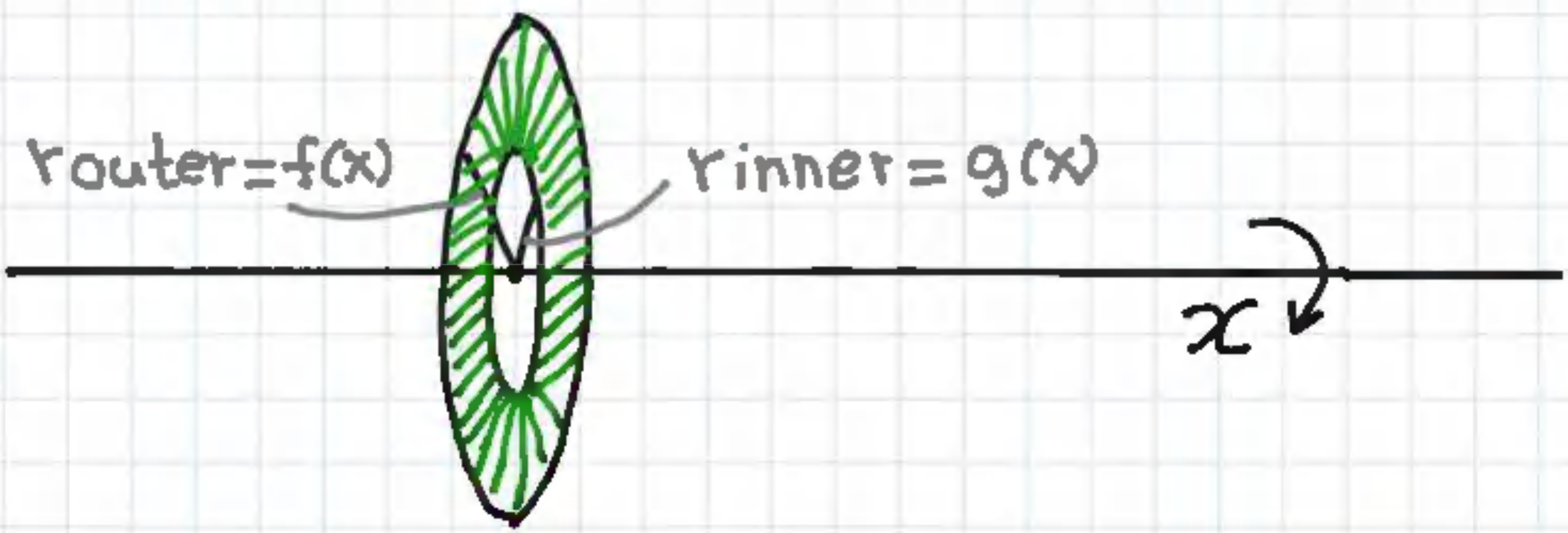
$$dV=(\pi[f(x)]^2 - \pi[g(x)]^2)\,dx$$

$dV = [\pi(f(x))^2 - \pi(g(x))^2]\,dx$ washer volume

$V = \int_a^b [\pi(f(x))^2 - \pi(g(x))^2]\,dx$ total volume

Key concept: $r_{outer} = f(x)$
$r_{inner} = g(x)$

Cross sectional slices $\perp$ to x axis are circular washers with inner radius $g(x)$ that generates a hole when revolved about the x axis.



The region bounded by the curves y=2x and y=x^2 is rotated about the x axis. Find the volume of the solid obtained.

Solved example

Ex] The region bounded by the curves $y=2x$ and $y=x^2$ is rotated about the x axis. Find the volume of the solid obtained.

Solution: Step 1] Find the points of intersection

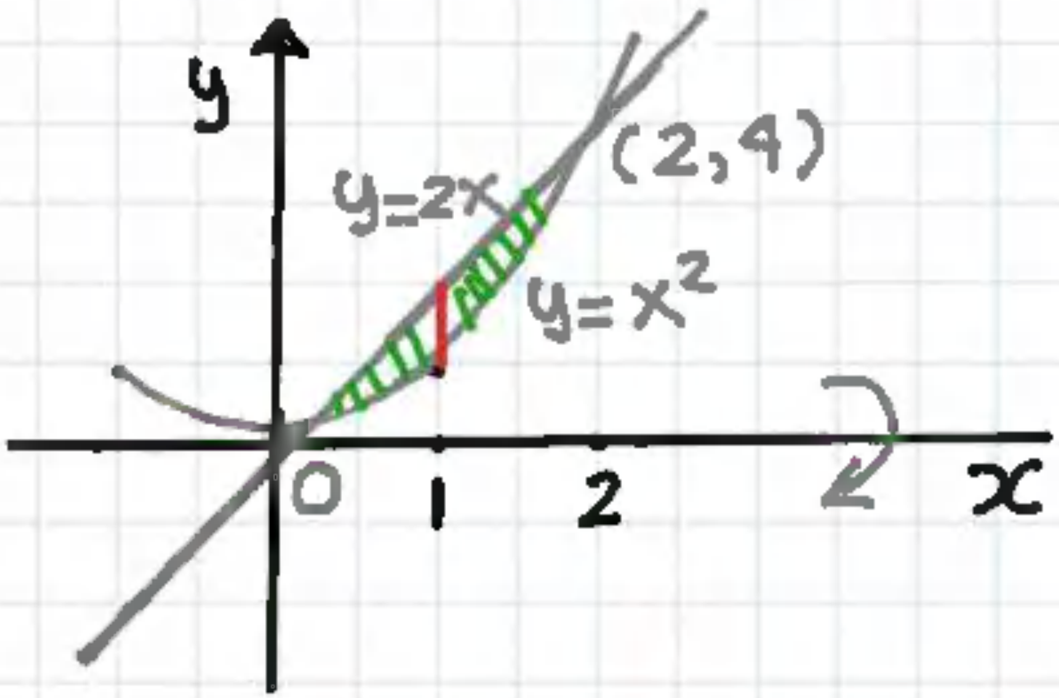
$y=2x \quad y=x^2 \quad$ Let $Y=Y \Rightarrow x^2=2x \Rightarrow x^2-2x=0$

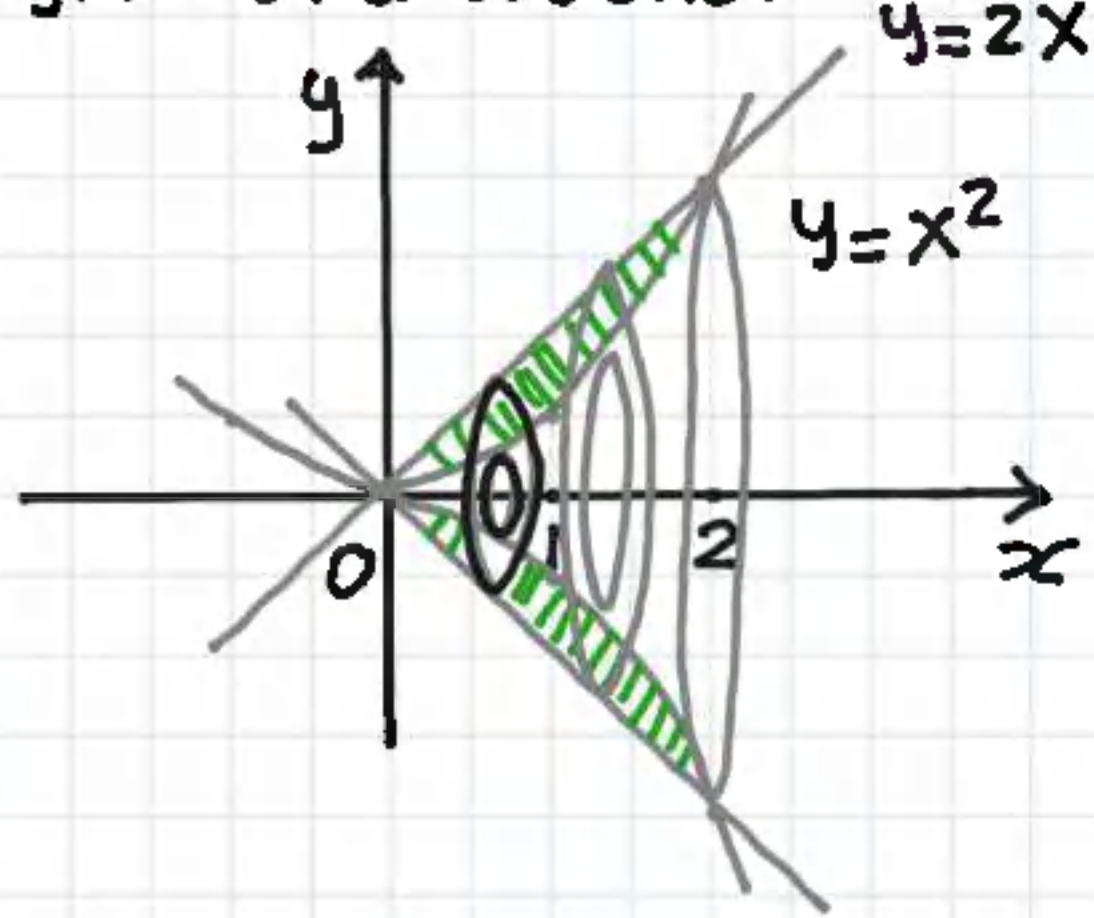
$x^2-2x=0 \Rightarrow x(x-2)=0 \Rightarrow x=0\ ;\ x-2=0 \Rightarrow x=2$

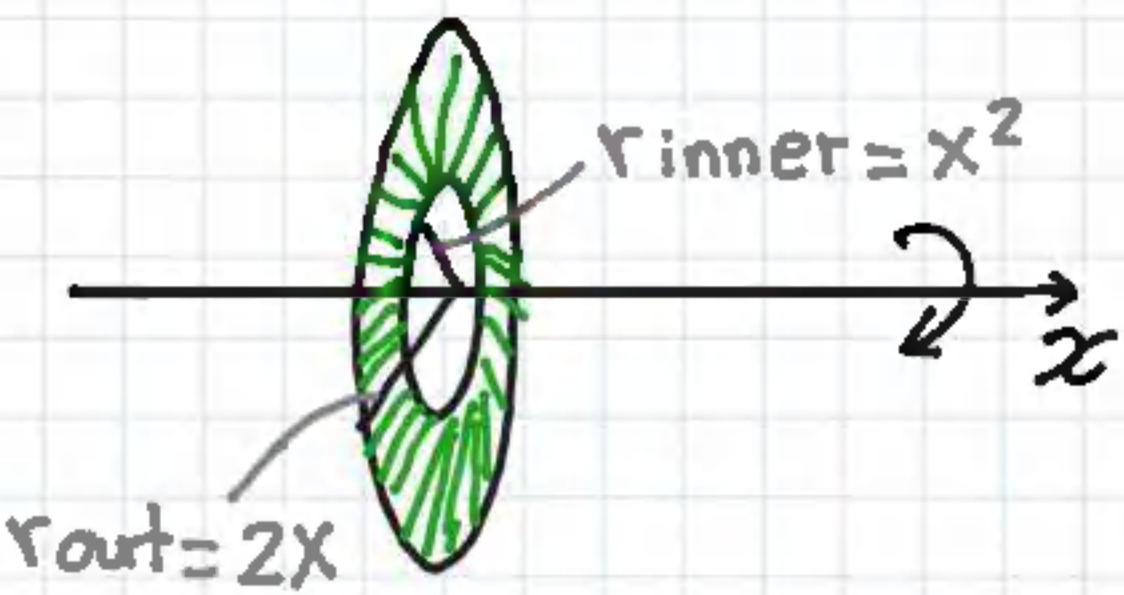
$x=0 \Rightarrow$ Plug into $y=2x \Rightarrow y=0$ hence $(0,0)$

$x=2 \Rightarrow$ plug into $y=x^2 \Rightarrow y=4$ hence $(2,4)$

step2] Sketch bounded Region and washer







cross sectional area

$A(x) = \pi\{r_{out}^2 - r_{inner}^2\}$

$A(x) = \pi\{(2x)^2 - (x^2)^2\}$

washer Volume

$dV = \pi\{(2x)^2 - (x^2)^2\}\,dx$

step 3] Set up definite integral and find the volume

$$V = \pi \int_0^2 \left[r_{outer}^2 - r_{inner}^2 \right] dx$$

$$V = \pi \int_0^2 \left[(2x)^2 - (x^2)^2 \right] dx$$

$$V = \pi \int_0^2 \left[4x^2 - x^4 \right] dx = \pi \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2$$

$$V = \pi \left[\frac{32}{3} - \frac{32}{5} \right] - \pi [0 - 0] = \pi \left[\frac{5(32) - 3(32)}{15} \right]$$

$$V = \pi \left[\frac{64}{15} \right] \cong 13.40$$

Volumes 5 (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y=x^2$ and $y=x$ about the line $y=2$.

Solution: step 1] Find the points of intersection

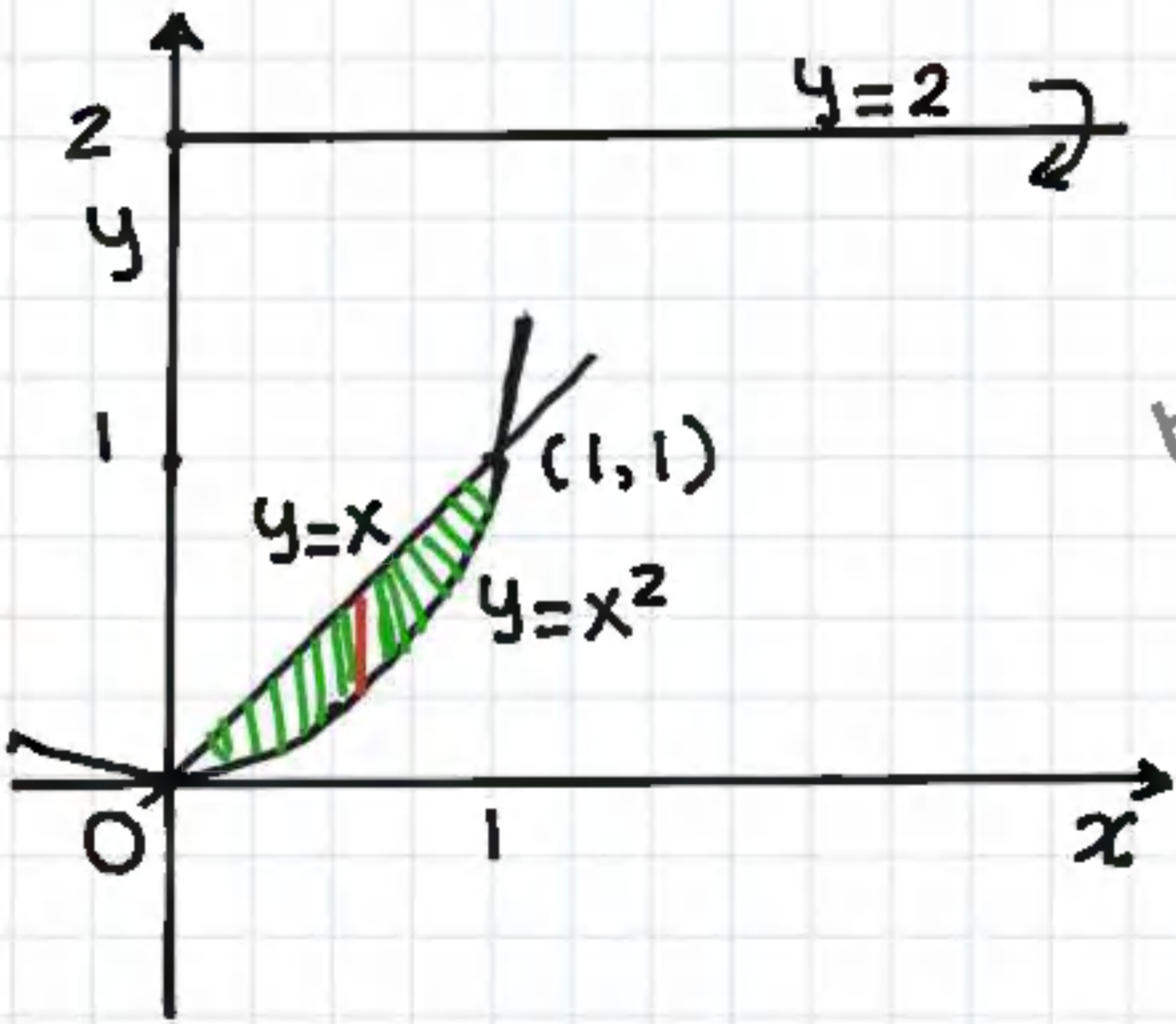
$y=x \quad , y=x^2 \quad$ Let $Y=Y \Rightarrow x^2=x \Rightarrow x^2-x=0$

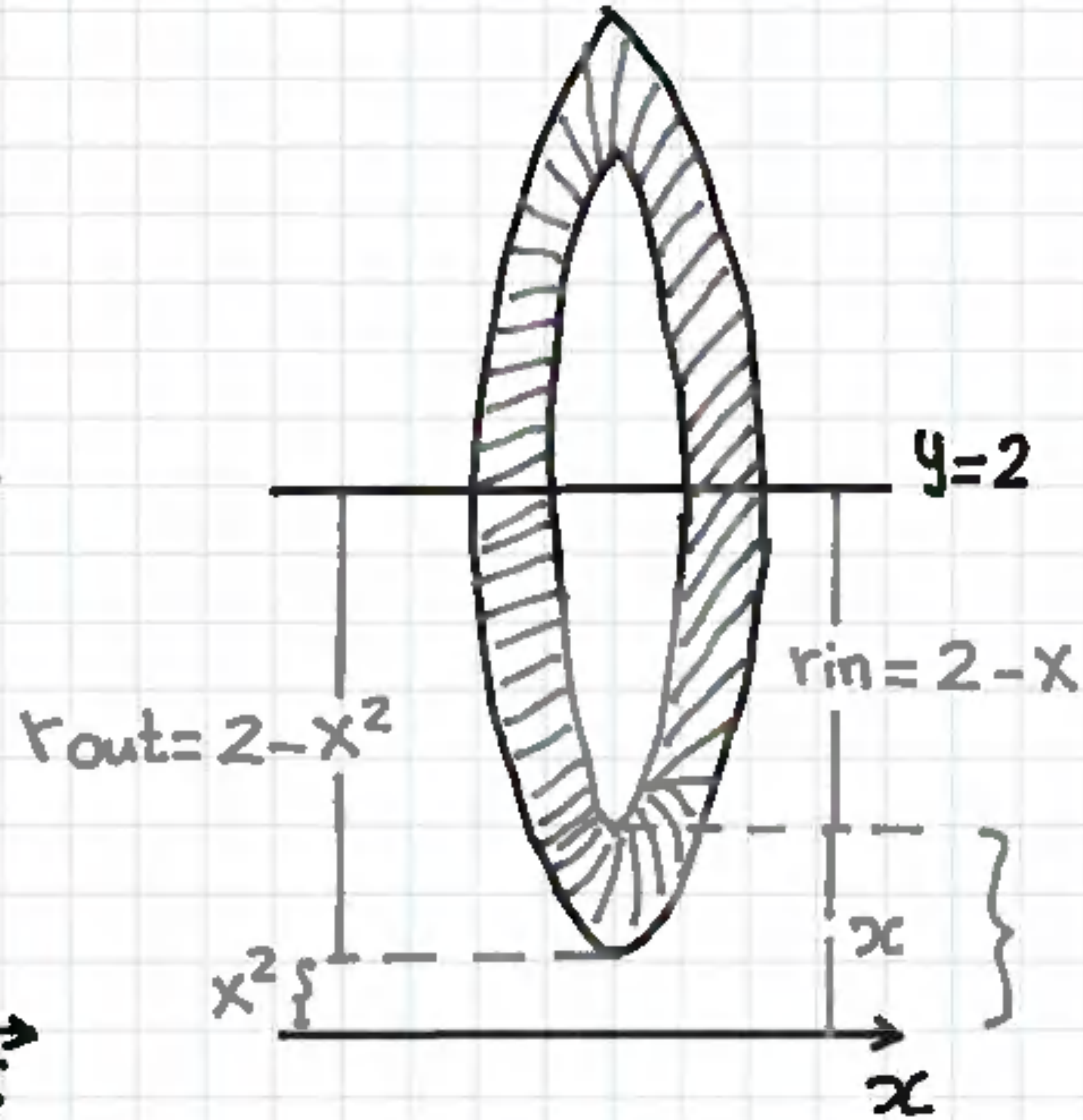
$x^2-x=0 \Rightarrow x(x-1)=0 \Rightarrow x=0 \,,\, x-1=0 \Rightarrow x=1$

$x=0 \Rightarrow$ plug into $y=x \Rightarrow y=0$ hence $(0,0)$

$x=1 \Rightarrow$ plug into $y=x^2 \Rightarrow y=1$ hence $(1,1)$

step 2 | sketch bounded Region and typical washer





router $= 2 - x^2$; rinner $= 2 - x$

cross sectional area

$$A(x) = \pi\{ \text{rout}^2 - \text{rinner}^2 \}$$

$$A(x) = \pi\{ (2-x^2)^2 - (2-x)^2 \}$$

washer volume

$$dV = \pi\{ (2-x^2)^2 - (2-x)^2 \}\, dx$$

Total Volume

$$V = \int_0^1 \pi\{ (2-x^2)^2 - (2-x)^2 \}\, dx$$

Step 3| Set up definite integral and find the volume

$$V = \pi \int_0^1 \left[(2-x^2)^2 - (2-x)^2 \right] dx$$

$$V = \pi \int_0^1 \left([4 - 4x^2 + x^4] - [4 - 4x + x^2] \right) dx$$

$$V = \pi \int_0^1 \left(\cancel{4} - 4x^2 + x^4 - \cancel{4} + 4x - \underline{x^2} \right) dx$$

$$V = \pi \int_0^1 \left(x^4 - 5x^2 + 4x \right) dx$$

$$V = \pi \left[\frac{x^5}{5} - \frac{5x^3}{3} + \frac{4x^2}{2} \right]_0^1$$

Apply F.T.C

Step 3] Cont.

$$V = \pi\left[\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2\right]_0^1 = \pi\left[\frac{1}{5} - \frac{5}{3} + 2\right] - \pi[0 - 0 + 0]$$

$$V = \pi\left[\frac{3 - 25 + 30}{15}\right] = \frac{8\pi}{15} \cong 1.68$$

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Find the volume of the solid obtained by rotating the region bounded by the curves y=sinx y=2 sinx and the lines x=0,x=π,y=0 about the x axis solved example

Volumes 6 (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sin x$, $y = 2\sin x$ and the lines $x=0$, $x=\pi$, $y=0$ about the x axis.

Solution: step 1] Find the points of intersection

$y = \sin x$, $y = 2\sin x \Rightarrow Y = Y \Rightarrow \sin x = 2\sin x \Rightarrow \sin x = 0$

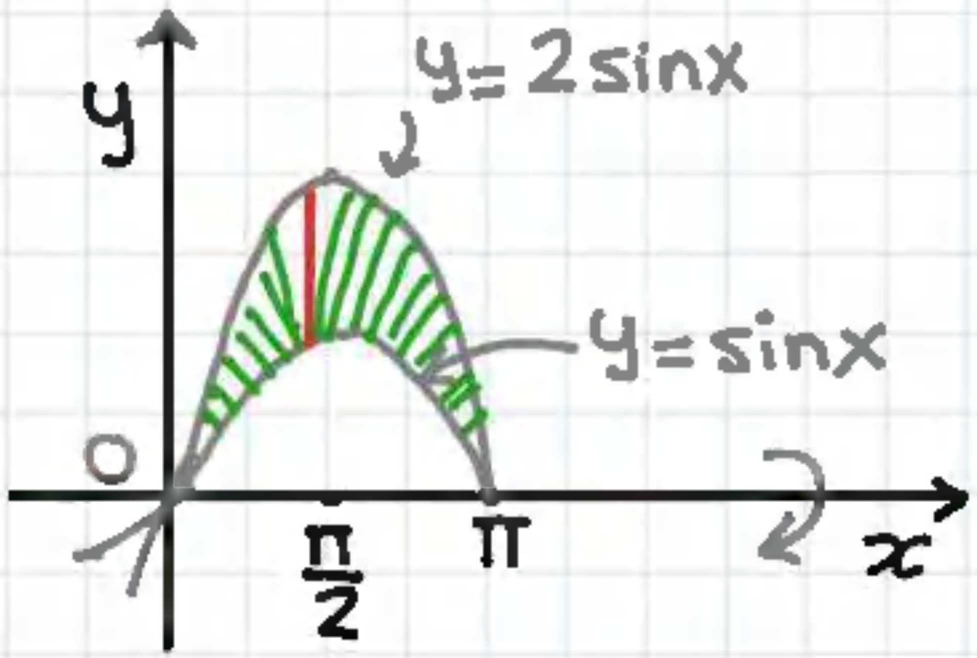
$\sin x = 0 \Rightarrow x = 0,\ x = \pi$

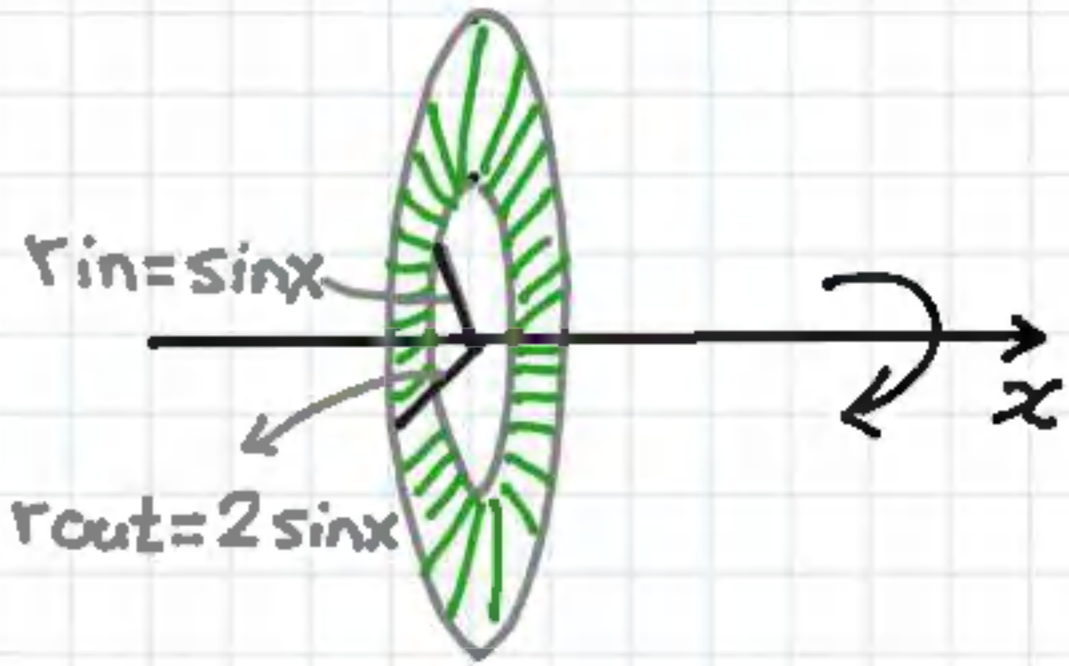
$x = 0 \Rightarrow$ Plug into $y = \sin x \Rightarrow y = 0$ hence $(0,0)$

$x = \pi \Rightarrow$ Plug into $y = 2\sin x \Rightarrow y = 0$ hence $(\pi, 0)$

Note: $\sin x = 0$ has solutions $x = n\pi$ but we are only interested in x between $[0, \pi]$

step 2] Sketch bounded region and washer





cross sectional area

$A(x) = \pi \{ r_{out}^2 - r_{in}^2 \}$ $\qquad$ $A(x) = \pi \{ (2\sin x)^2 - (\sin x)^2 \}$

washer Volume

$dV = \pi \{ (2\sin x)^2 - (\sin x)^2 \} dx$

step 3] Set up definite integral and find the volume

$$V = \int_0^{\pi} \pi\left[(2\sin x)^2 - (\sin x)^2 \right] dx$$

$$V = \pi \int_0^{\pi} \left[4\sin^2 x - \sin^2 x \right] dx$$

$$V = \pi \int_0^{\pi} 3\sin^2 x \, dx = 3\pi \int_0^{\pi} \sin^2 x \, dx$$

Apply Trig. Identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$$V = 3\pi \int_0^{\pi} \frac{1}{2}(1 - \cos(2x)) dx$$

$$V = \frac{3\pi}{2} \int_0^{\pi} (1 - \cos(2x)) \, dx$$

Step 3] cont.

$$V = \frac{3\pi}{2}\int_0^{\pi} (1 - \cos(2x))\,dx$$

Apply U-Sub

$$\int \cos(2x)\,dx$$

$u = 2x \quad du = 2\,dx$

$dx = \frac{du}{2}$

$$V = \frac{3\pi}{2}\left[x - \frac{\sin(2x)}{2} \right]_0^{\pi}$$

$$V = \frac{3\pi}{2}\left[\pi - \frac{\overset{=0}{\sin(2\pi)}}{2} - \left(0 - \frac{\overset{=0}{\sin 0}}{2}\right) \right]$$

$$V = \frac{3\pi}{2}\left[\pi - 0 - 0 + 0 \right] = \frac{3\pi^2}{2}$$

$$V = \frac{3\pi^2}{2} \cong 14.8$$

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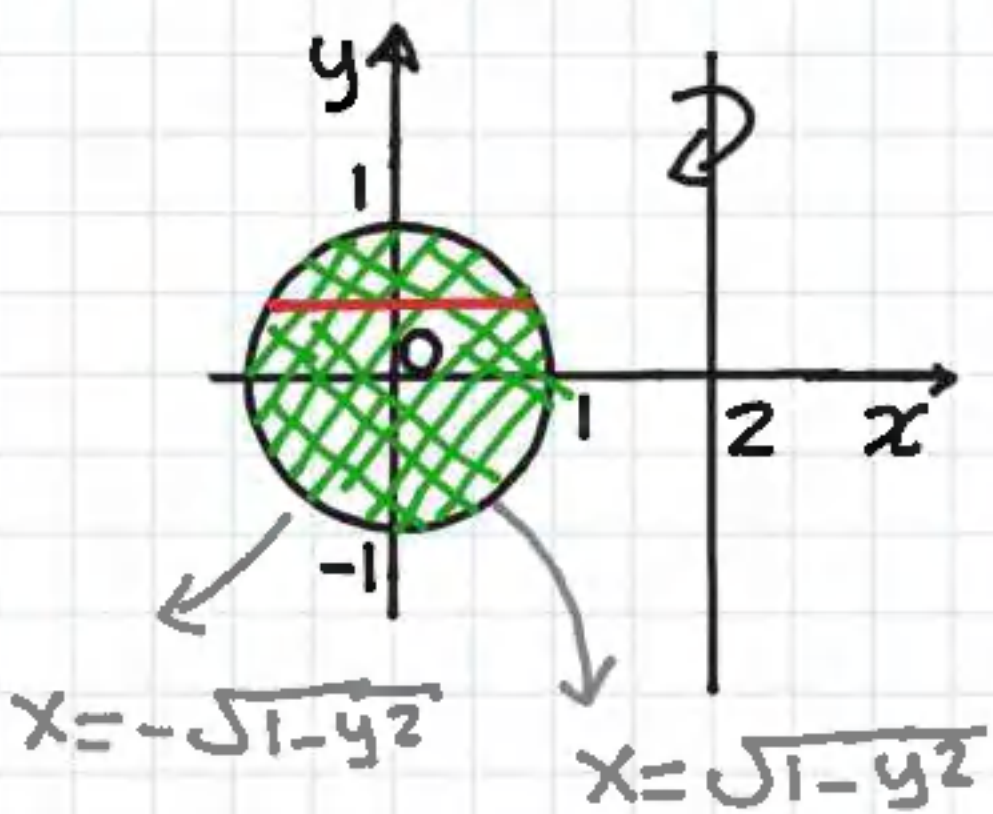
Find the volume of the solid obtained by rotating the entire unit circle x^2+y^2<=1 about the line x=2

solved example

Volumes7 (washer method)

Ex] Find the volume of the solid obtained by rotating the entire unit circle $x^2+y^2 \leq 1$ about the line $x=2$.

Step 1] Sketch bounded region and washer

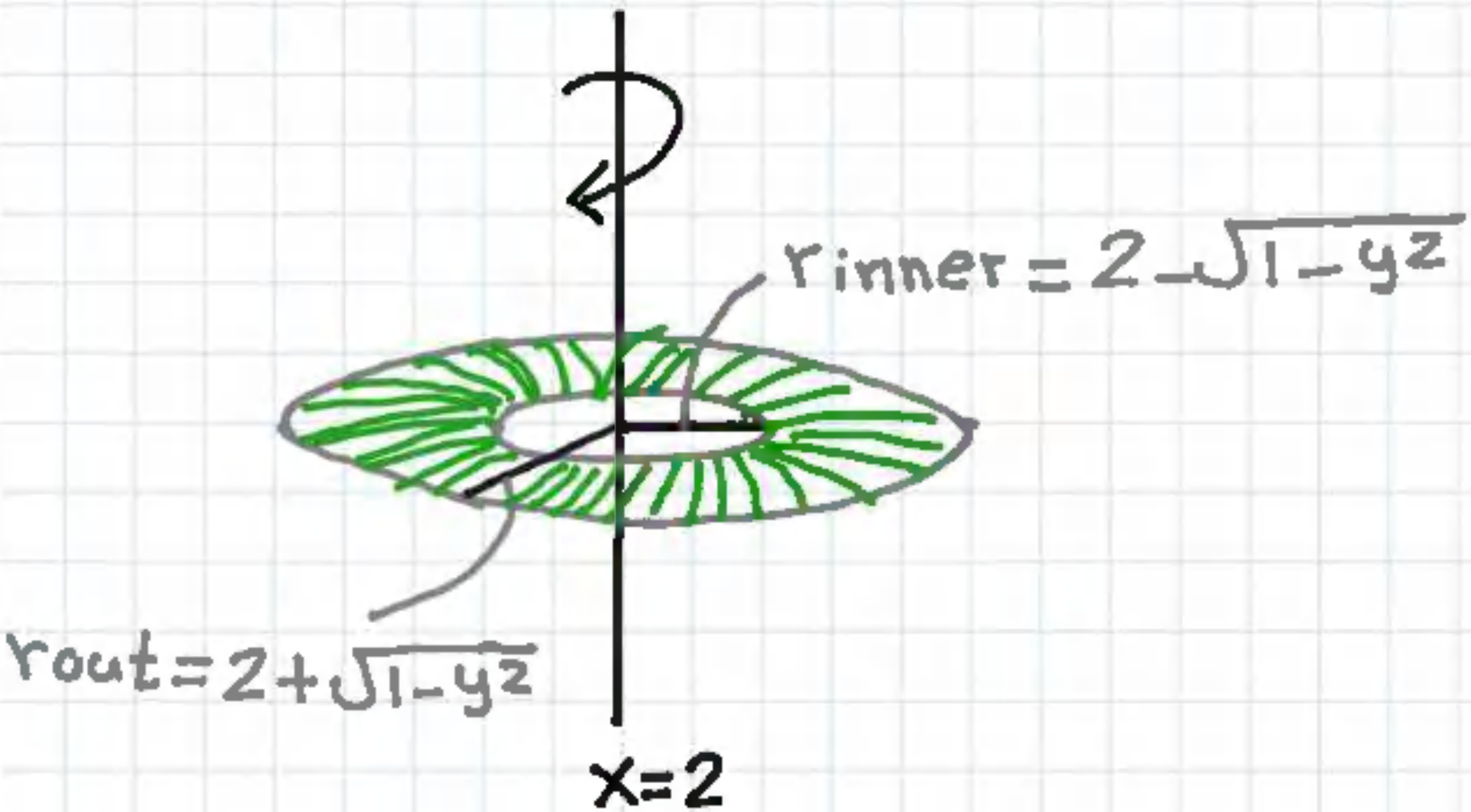


$x^2+y^2=1$

$x=\pm\sqrt{1-y^2}$

$r_{outer}=2--\sqrt{1-y^2}$

$r_{inner}=2-\sqrt{1-y^2}$



Washer Volume

$$dV = \pi \left\{ \left(2+\sqrt{1-y^2}\right)^2 - \left(2-\sqrt{1-y^2}\right)^2 \right\} dy$$

step 3 | Set up definite integral and find the volume

$$V = \pi \int_{-1}^{1} \left[\left(2+\sqrt{1-y^2}\right)^2 - \left(2-\sqrt{1-y^2}\right)^2 \right] dy$$

$$V = \pi \int_{-1}^{1} \left[\left(4+4\sqrt{1-y^2}+1-y^2\right) - \left(4-4\sqrt{1-y^2}+1-y^2\right) \right] dy$$

$$V = \pi \int_{-1}^{1} \left[\cancel{4} + \underbrace{4\sqrt{1-y^2}} + \cancel{1-y^2} - \cancel{4} + \underbrace{4\sqrt{1-y^2}} - \cancel{(1-y^2)} \right] dy$$

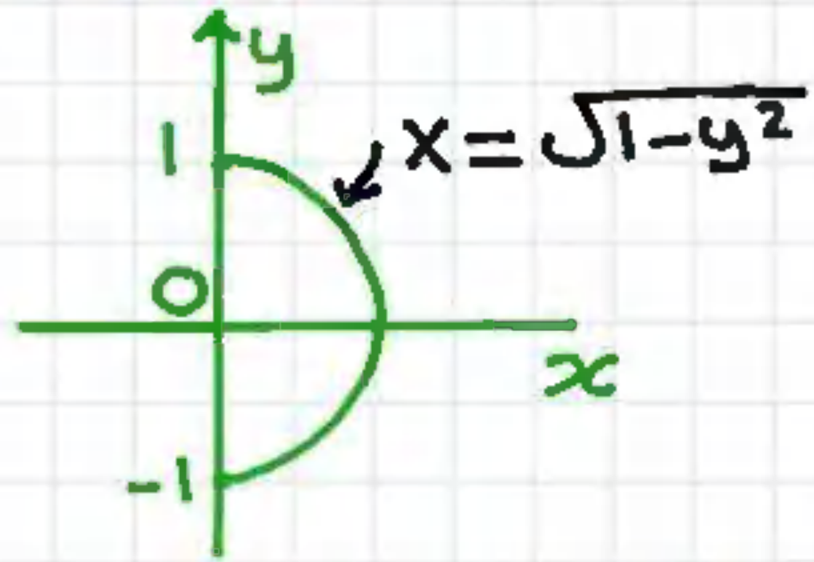
$$V = \pi \int_{-1}^{1} 8\sqrt{1-y^2}\, dy = 8\pi \int_{-1}^{1} \sqrt{1-y^2}\, dy$$

$$V = 8\pi \int_{-1}^{1} \sqrt{1-y^2}\, dy$$

Step 3

$$V = 8\pi \int_{-1}^{1} \sqrt{1-y^2}\, dy$$

Let's apply circle geometry to solve integral



$$V = 8\pi\left(\frac{\pi}{2}\right) = 4\pi^2$$

$$A = \frac{\pi (r)^2}{2} = \frac{\pi (1)^2}{2} = \frac{\pi}{2}$$

Note: The integrand $\sqrt{1-y^2}$ is a semicircle with radius 1 and is the right half of a unit circle and hence

$$\int_{-1}^{1} \sqrt{1-y^2}\, dy = \frac{\pi}{2}$$

Area of semicircle of radius 1

Find the volume of solid obtained by rotating the region bounded by y=1-x^3 and the lines x=-1,y=0 about the line y=2 solved example

Volumes 8 (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curve $y=1-x^3$ and the lines $x=-1, y=0$ about @ $y=2$.

Solution: step 1] Find the points of intersection

$y=1-x^3$ and $x=-1$ $\Rightarrow$ plug $x=-1$ into $y=1-x^3$

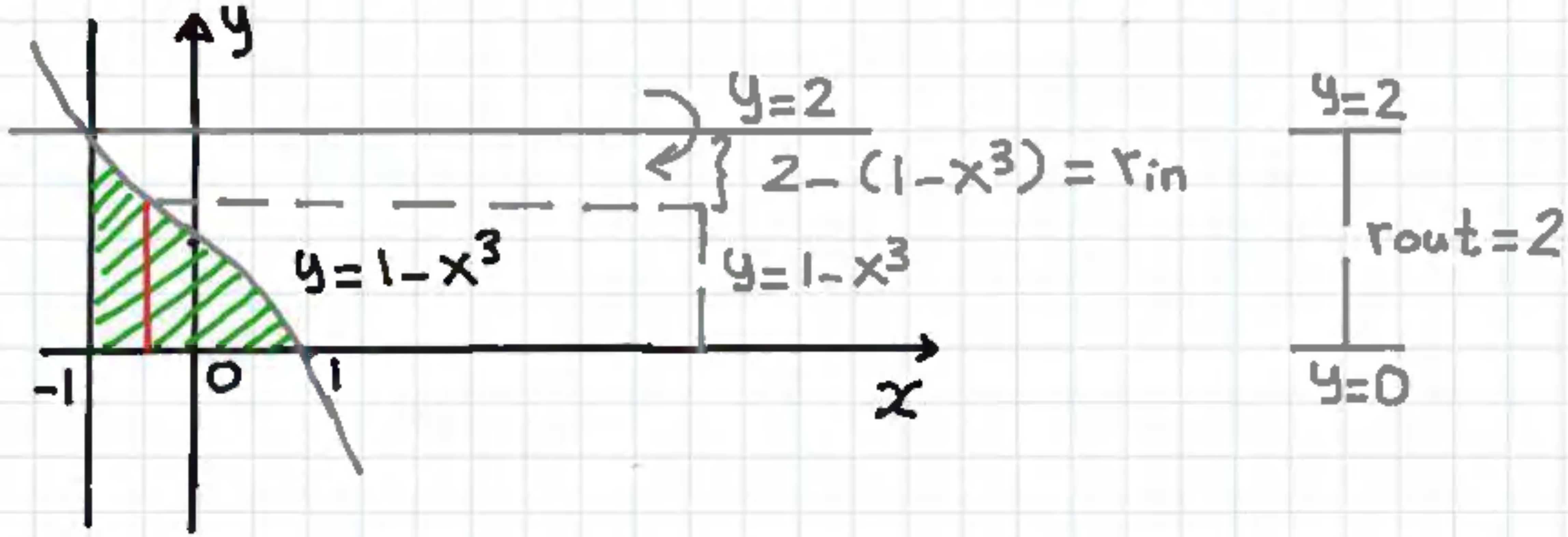
$x=-1 \Rightarrow y=1-(-1)^3=2$ hence $(-1,2)$

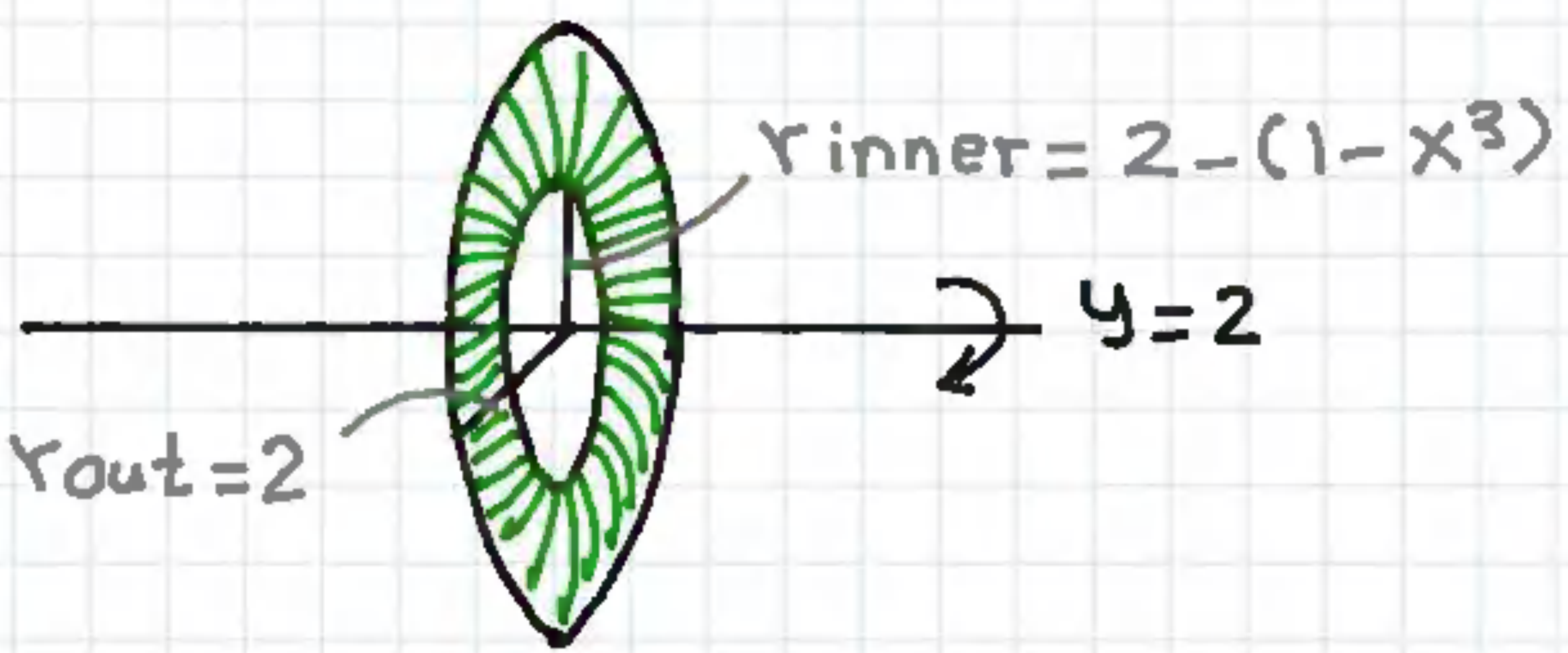
$y=1-x^3$ and $y=0 \Rightarrow y=y \Rightarrow 1-x^3=0 \Rightarrow x^3=1$

$x=1 \quad y=0$

$\therefore$ Intersection points are $(-1,2)$ and $(1,0)$

step 2] Sketch bounded region and typical washer





$r_{outer} = 2$; $r_{inner} = 2-(1-x^3)$

cross sectional area

$$A(x) = \pi \{ r_{out}^2 - r_{in}^2 \}$$

$$A(x) = \pi \{ 2^2 - (2-(1-x^3))^2 \}$$

washer volume

$$dV = \pi \{ 2^2 - (2-(1-x^3))^2 \} dx$$

Total Volume

$$V = \int_{-1}^{1} \pi \{ 2^2 - (2-(1-x^3))^2 \} dx$$

Step 3 | Set up definite integral and find volume

$$V = \pi \int_{-1}^{1} \left\{ (2)^2 - (2-(1-x^3))^2 \right\} dx$$

$$V = \pi \int_{-1}^{1} \left\{ 4 - (1+x^3)^2 \right\} dx$$

$$V = \pi \int_{-1}^{1} (4 - (1 + 2x^3 + x^6))\, dx$$

$$V = \pi \int_{-1}^{1} (3 - 2x^3 - x^6)\, dx$$

$$V = \pi \int_{-1}^{1} \left[3 - 2x^3 - x^6 \right] dx$$

$$V = \pi \left[3x - \frac{2x^4}{4} - \frac{x^7}{7} \right]_{-1}^{1}$$

$$V = \pi \left[3 - \frac{1}{2} - \frac{1}{7} \right] - \pi \left[-3 - \frac{1}{2} + \frac{1}{7} \right]$$

$$V = 3\pi - \cancel{\frac{\pi}{2}} - \frac{\pi}{7} + 3\pi + \cancel{\frac{\pi}{2}} - \frac{\pi}{7}$$

$$V = \frac{6\pi}{1} - \frac{2\pi}{7} = \frac{42\pi - 2\pi}{7} = \frac{40\pi}{7} \cong 17.95$$

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Find the volume of the solid obtained by rotating the region bounded by the curves y=x, y=4x and the line y=1 about the x axis solved example

Volumes 9 (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y=x$, $y=4x$ and the line $y=1$ about the x axis.

Solution: step 1] Find the points of intersection

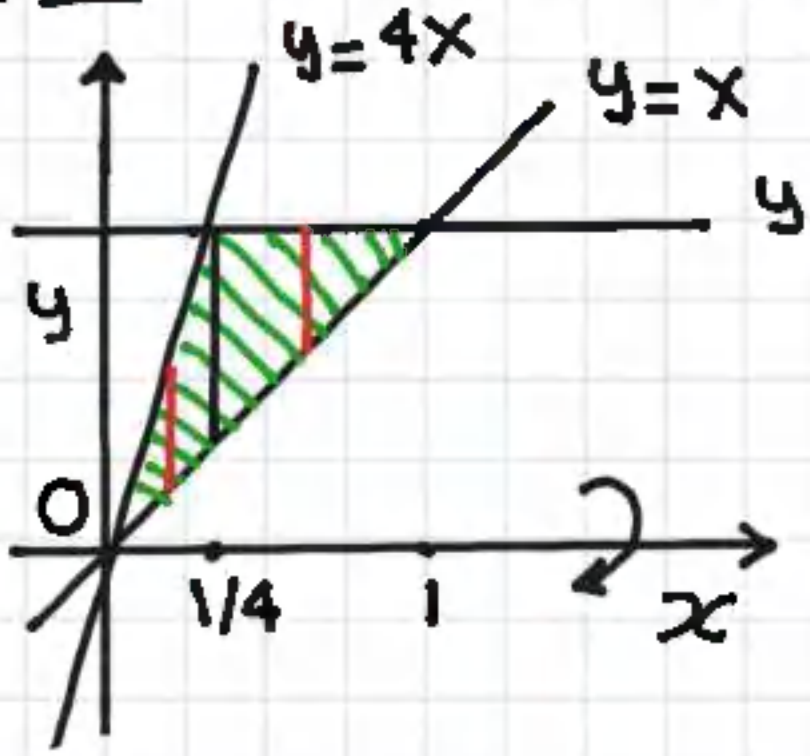
$y=1 \Rightarrow y=x \Rightarrow y=y \Rightarrow 1=x \Rightarrow y=x \Rightarrow y=1 \Rightarrow (1,1)$

$y=1 \Rightarrow y=4x \Rightarrow y=y \Rightarrow 1=4x \Rightarrow x=1/4 \Rightarrow y=1 \Rightarrow (\frac{1}{4},1)$

$y=x$, $y=4x \Rightarrow 4x=x \Rightarrow 3x=0 \Rightarrow x=0 \Rightarrow y=x=0$

$\therefore$ Intersection points are $(1,1)$, $(1/4,1)$, $(0,0)$

step 2 | Sketch bounded region and typical washer

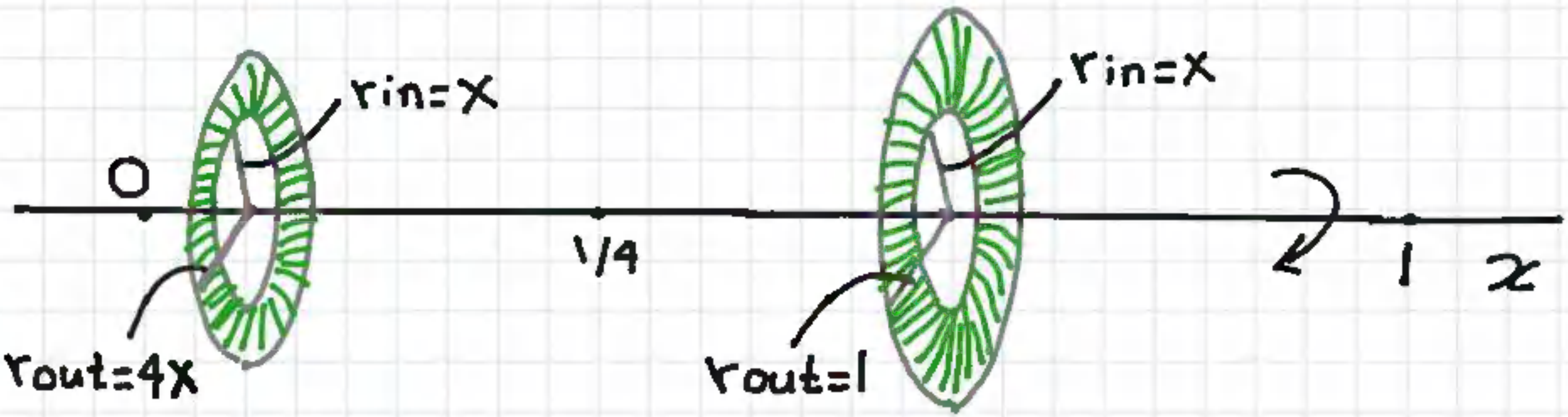


when $0 \le x \le 1/4$

$r_{inner} = x$; $r_{outer} = 4x$

however when $\frac{1}{4} \le x \le 1$

$r_{inner} = x$; $r_{outer} = 1$



when $0 \le x \le 1/4 \Rightarrow r_{inner} = x$; $r_{outer} = 4x$

when $\frac{1}{4} \le x \le 1 \Rightarrow r_{inner} = x$; $r_{outer} = 1$

$\therefore 0 \le x \le \frac{1}{4} \Rightarrow A_1(x) = \pi \{ (4x)^2 - x^2 \}$

$1/4 \le x \le 1 \Rightarrow A_2(x) = \pi \{ 1^2 - x^2 \}$

$$V = \int_0^{1/4} A_1(x)\,dx + \int_{1/4}^{1} A_2(x)\,dx$$

$$V = \int_0^{1/4} \pi \{ (4x)^2 - x^2 \} dx + \int_{1/4}^{1} \pi \{ 1^2 - x^2 \} dx$$

Step 3| Set up definite integral and find volume

$$V = \int_0^{1/4} \pi\left[(4x)^2 - x^2\right]dx + \int_{1/4}^{1} \pi\left[1^2 - x^2\right]dx$$

$$V = \pi \int_0^{1/4} \left[16x^2 - x^2\right]dx + \pi \int_{1/4}^{1} \left[1^2 - x^2\right]dx$$

$$V = \pi \int_0^{1/4} 15x^2\,dx + \pi \int_{1/4}^{1} (1 - x^2)\,dx$$

$$V = \left[\frac{15\pi x^3}{3}\right]_0^{1/4} + \pi\left[x - \frac{x^3}{3}\right]_{1/4}^{1}$$ Apply F.T.C

$$V = 5\pi(1/4)^3 - 5\pi(0)^3 + \pi\left[1 - \frac{1}{3} - \left(\frac{1}{4} - \frac{1}{192}\right)\right]$$

$$V = 5\pi(1/4)^3 - 5\pi(0)^3 + \pi\left[\frac{2}{3} - (1/4 - 1/192)\right]$$

$$V = \frac{5\pi}{64} + \frac{2\pi}{3} - \frac{\pi}{4} - \frac{\pi}{192} = \frac{15\pi + 128\pi - 48\pi - \pi}{192}$$

$$V = \frac{94\pi}{192} \cong 1.54$$

Basic skills how to sketch a hyperbola x^2 -y^2 =1 solved example

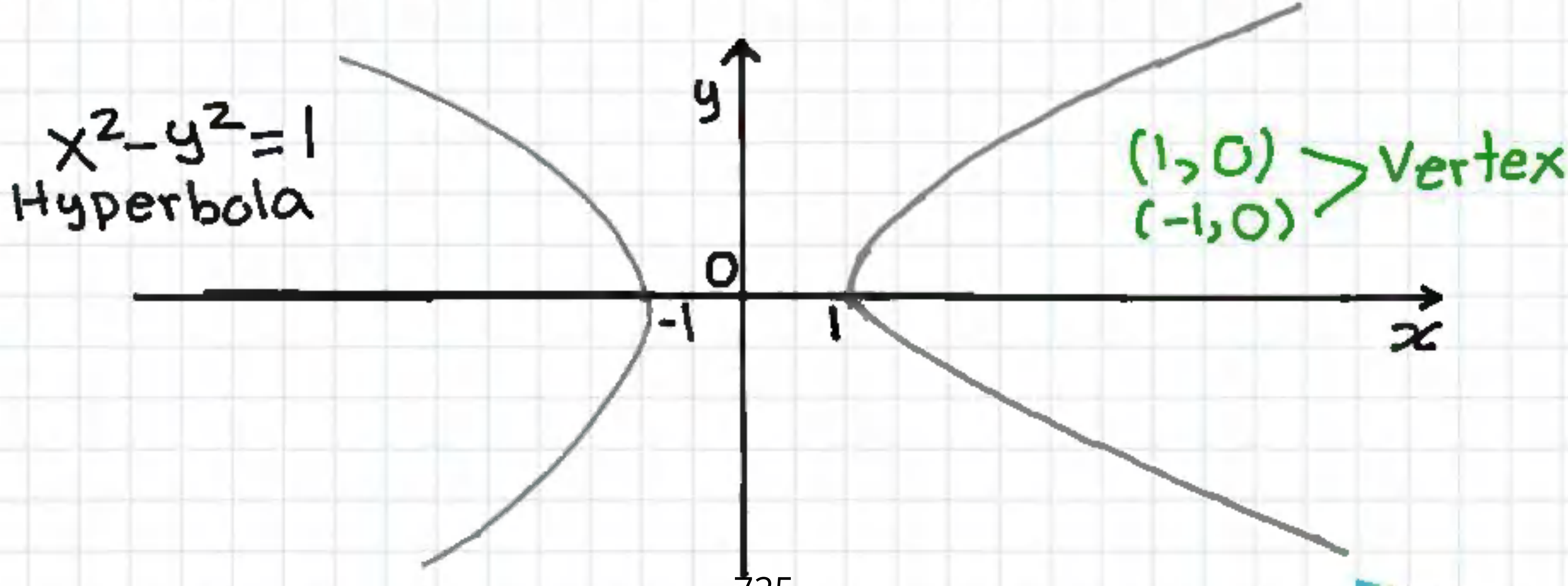
Volumes 10 (washer method)

Basic Skills Review before we start!

How to sketch a hyperbola $x^2-y^2=1$

Choose $y=0$ and solve for x

$y=0 \Rightarrow x^2=1 \Rightarrow \sqrt{x^2}=\sqrt{1} \Rightarrow |x|=1 \Rightarrow x=\pm 1 \quad y=0$



Set up a definite integral for the volume of solid obtained by rotating the region bounded by the hyperbola x^2 - y^2 =1 , x=3 rotated about x=-2 solved example

Volumes 10 (washer method)

Set up, but DONT EVALUATE, a definite integral for the volume of the solid obtained by rotating the region bounded by the hyperbola $x^2-y^2=1$ $x=3$ rotated about $x=-2$

Solution: Step 1] Find the points of intersection

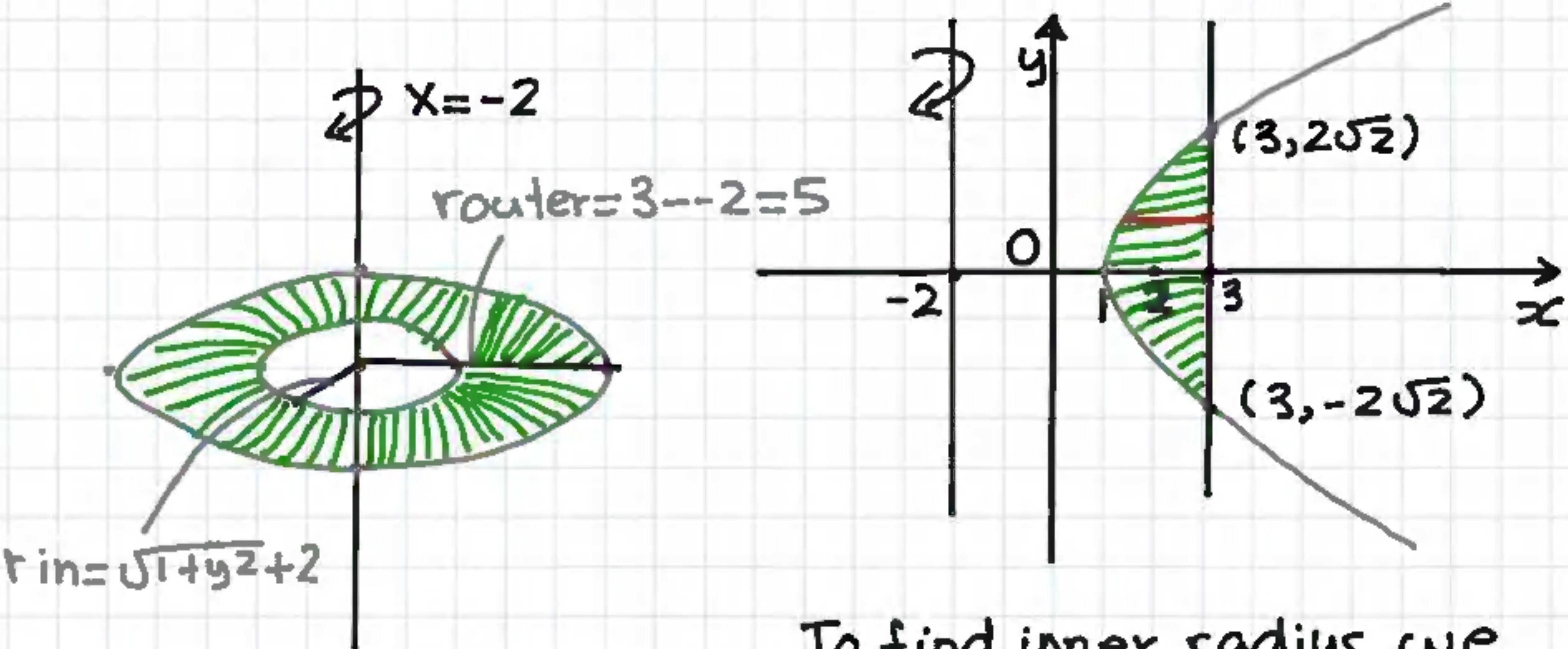
$x^2-y^2=1$ and $x=3 \Rightarrow 3^2-y^2=1 \Rightarrow -y^2=-8$

$y^2=8 \Rightarrow \sqrt{y^2}=\sqrt{8} \Rightarrow |y|=\sqrt{8} \Rightarrow |y|=\sqrt{4\times 2}$

$|y|=\sqrt{4}\sqrt{2}=2\sqrt{2} \Rightarrow y=\pm 2\sqrt{2}$

$\therefore$ Intersection points are: $(3, 2\sqrt{2})$ and $(3, -2\sqrt{2})$

step 2 | Sketch bounded region and typical washer



Note: Cross sectional slice must be ⊥ perpendicular to axis of rotation $x=-2$

To find inner radius we need to solve for x in terms of y; $x^2-y^2=1$

$$x^2 = 1+y^2 \Rightarrow x = \pm\sqrt{1+y^2}$$

$$x = \sqrt{1+y^2}$$

$$r_{outer} = 3 - -2 = 3 + 2 = 5$$

$$r_{inner} = \sqrt{1+y^2} - -2 = \sqrt{1+y^2} + 2$$

$$A(y) = \pi \left\{ r_{out}^2 - r_{in}^2 \right\}$$

$$A(y) = \pi \left\{ (5)^2 - \left(\sqrt{1+y^2} + 2 \right)^2 \right\}$$

washer Volume

$$dV = \pi \left\{ (5)^2 - \left(\sqrt{1+y^2} + 2 \right)^2 \right\} dy$$

$$V = \int_{-2\sqrt{2}}^{2\sqrt{2}} A(y)\, dy \qquad V = \int_{-2\sqrt{2}}^{2\sqrt{2}} dV$$

Step 3 | Set up definite integral for the volume

$$dV = \pi \left\{ (5)^2 - \left(\sqrt{1+y^2}+2\right)^2 \right\} dy$$

$$V = \int_{-2\sqrt{2}}^{2\sqrt{2}} \pi \left\{ (5)^2 - \left(\sqrt{1+y^2}+2\right)^2 \right\} dy$$

Note: Although we did not solve this integral it can still be done, the only "hard" part of the integrand is $\int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{1+y^2}\, dy$; apply trig. substitution

$y = \tan\theta \qquad dy = \sec^2\theta \, d\theta \qquad 1+\tan^2\theta = \sec^2\theta$

Find the volume of solid obtained by rotating the region bounded by y=2/√(3+x^2) and y=|x| about the x axis solved example

Volumes II (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y = 2/\sqrt{3+x^2}$ and $y = |x|$ about the x axis.

Solution: step 1] Find the intersection points

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

for $x \geq 0 \Rightarrow Y = Y \Rightarrow \dfrac{4}{3+x^2} = x^2 \Rightarrow 4 = x^4 + 3x^2$

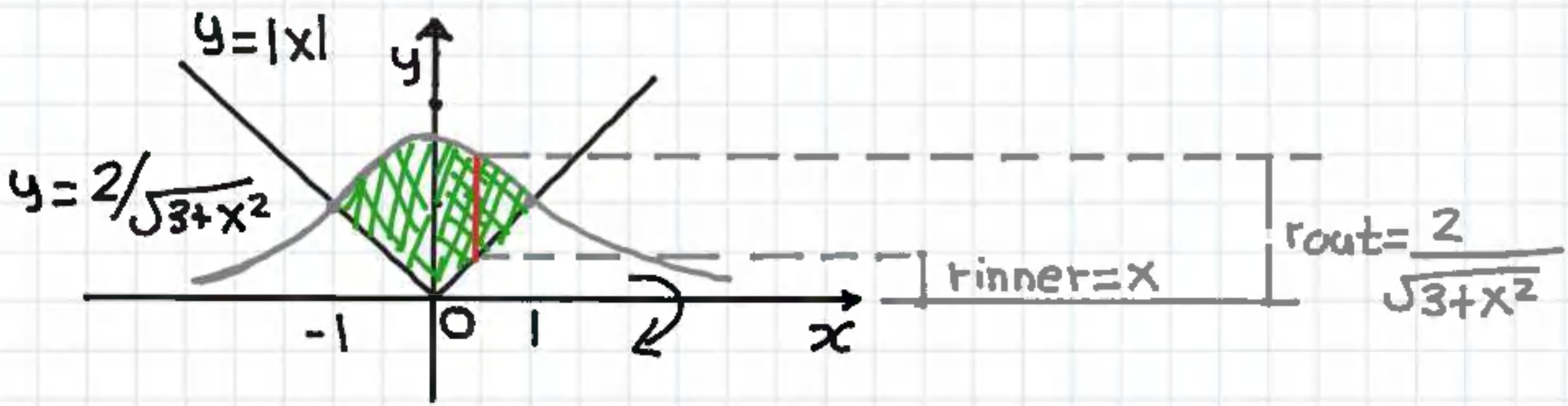
$x^4 + 3x^2 - 4 = 0 \Rightarrow$ Guess $x = 1 \Rightarrow 1 + 3 - 4 = 0$ ✓

Points of intersection $x=1 \Rightarrow y=x \Rightarrow y=1 \Rightarrow (1,1)$

Similiarly we can find the other intersection point

$x=-1$, $y=1 \Rightarrow (-1,1)$

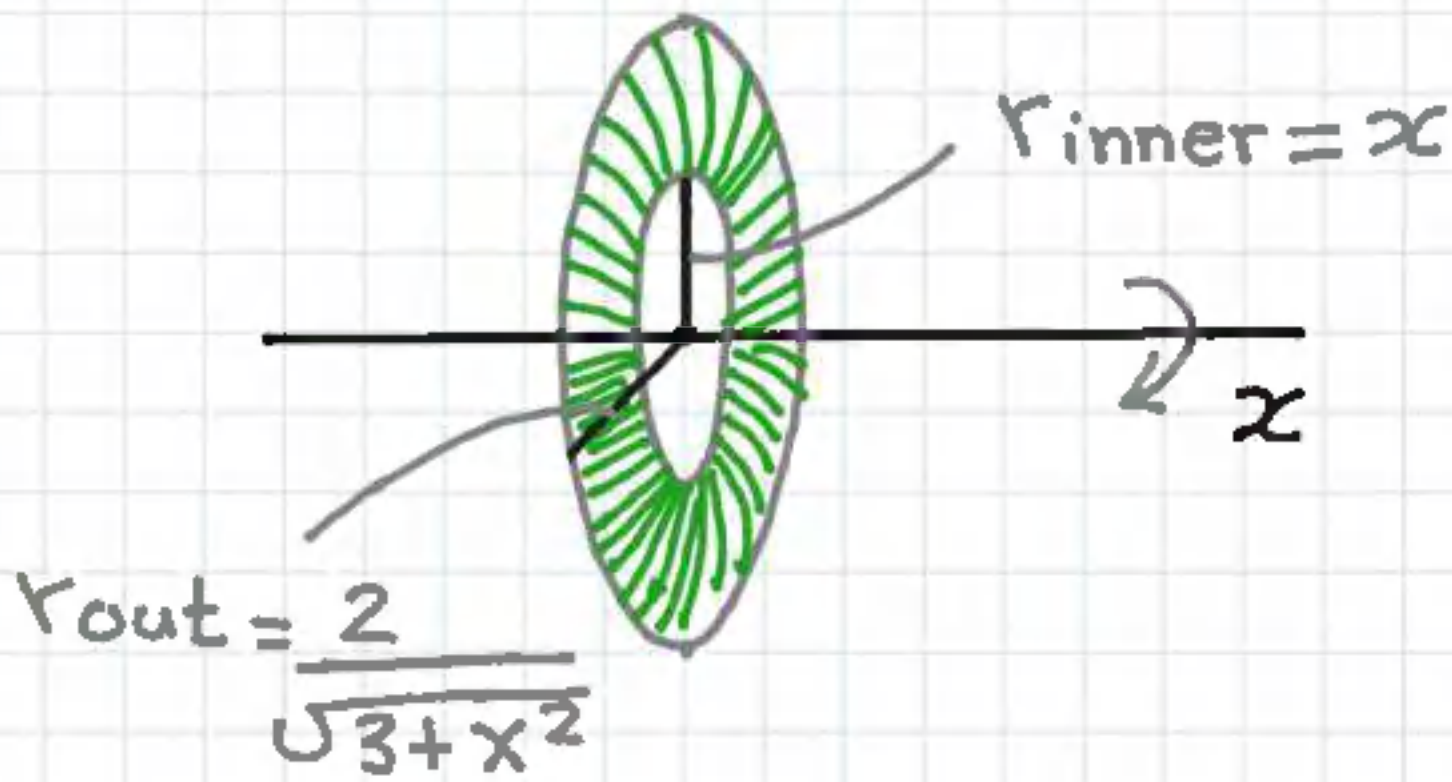
Step 2| Sketch bounded region and typical washer



r_{inner} = distance from $y=x$ to $y=0$

r_{outer} = distance from $y=2/\sqrt{3+x^2}$ to $y=0$

For $x \geq 0$



For $x \geq 0$ $\quad r_{inner} = x \quad ; \quad r_{out} = \frac{2}{\sqrt{3+x^2}}$

For $x < 0$ $\quad r_{inner} = -x \quad ; \quad r_{out} = \frac{2}{\sqrt{3+x^2}}$

For $x \geq 0 \Rightarrow r_{in} = x$; $r_{out} = 2/\sqrt{3+x^2}$

$$A_1(x) = \pi \{ r_{out}^2 - r_{in}^2 \}$$

$$A_1(x) = \pi \left\{ \left(\frac{2}{\sqrt{3+x^2}} \right)^2 - (x)^2 \right\} \qquad 0 \leq x \leq 1$$

For $x < 0 \Rightarrow r_{in} = -x$; $r_{out} = 2/\sqrt{3+x^2}$

$$A_2(x) = \pi \left\{ \left(\frac{2}{\sqrt{3+x^2}} \right)^2 - (-x)^2 \right\} \qquad -1 \leq x < 0$$

$$V = \int_{-1}^{0} A_2(x)\,dx + \int_{0}^{1} A_1(x)\,dx$$

Since bounded region is symmetric about y axis

$$V = 2 \int_{0}^{1} A_1(x)\,dx$$

step 3] Set up definite integral and find volume

$$V = 2\int_0^1 A_1(x)\,dx \Rightarrow V = 2\int_0^1 \left[\left(\frac{2}{\sqrt{3+x^2}}\right)^2 - x^2\right] dx$$

$$V = 2\int_0^1 \left(\frac{4}{3+x^2} - x^2\right) dx$$ Symmetry about y axis

$$V = 8\int_0^1 \frac{1}{3+x^2}\,dx - 2\int_0^1 x^2\,dx$$ split up integrals

Let's apply integral formula:

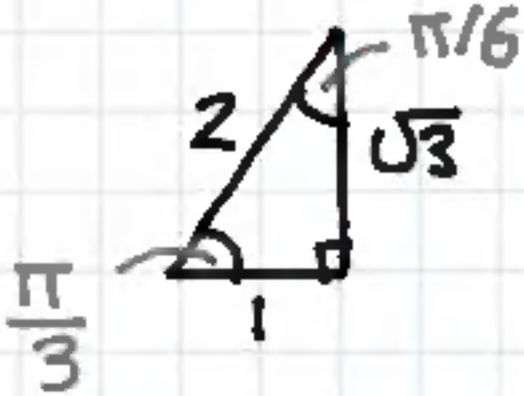
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}(x/a) + C$$

In this case $a^2 = 3 \Rightarrow a = \sqrt{3}$

$$V = 8\int_0^1 \frac{1}{3+x^2}\,dx - 2\int_0^1 x^2\,dx$$

$$V = 8\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\Bigg|_0^1 - \frac{2x^3}{3}\Bigg|_0^1$$

$$V = \frac{8}{\sqrt{3}}\left[\tan^{-1}(1/\sqrt{3}) - \underset{\substack{\| \\ 0}}{\tan^{-1}0}\right] - \frac{2}{3}\left[1-0\right]$$



<u>30-60-90 triangle</u>

$$V = \frac{8}{\sqrt{3}}\left[\frac{\pi}{6} - 0\right] - \frac{2}{3} = \frac{8\pi}{6\sqrt{3}} - \frac{2}{3} \cong 1.75$$

Find the volume of solid obtained by rotating the region bounded by the curves y=Ln(x) , y=Ln(x^2) and y=Ln(4) rotated about the y axis solved example

Volumes 12 (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y = \ln x$, $y = \ln x^2$ and $y = \ln 4$ rotated about the y axis.

step 1] Find the points of intersection

$y = \ln x$, $y = \ln 4 \Rightarrow Y = Y \Rightarrow \ln x = \ln 4 \Rightarrow e^{\ln x} = e^{\ln 4}$

$x = 4 \quad y = \ln 4$

$y = \ln x^2$, $y = \ln 4 \Rightarrow Y = Y \Rightarrow \ln x^2 = \ln 4 \Rightarrow e^{\ln x^2} = e^{\ln 4}$

$x^2 = 4 \Rightarrow x = 2, x = -2$ reject $x = -2$ not in domain of $\ln x$

$x = 2 \quad Y = \ln 4$

$Y = \ln x$, $Y = \ln x^2 \Rightarrow Y = Y \Rightarrow \ln x = \ln x^2 \Rightarrow e^{\ln x} = e^{\ln x^2}$

$x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0,\ x - 1 = 0 \Rightarrow x = 1$

reject $x=0$ not in domain of $y = \ln x$

$x=1$ plug into $y = \ln x \Rightarrow y = \ln 1 = 0 \Rightarrow x = 1,\ y = 0$

$\therefore$ Points of intersection of $y = \ln x$, $y = \ln x^2$, $y = \ln 4$

are $(2, \ln 4)$, $(4, \ln 4)$, $(1, 0)$

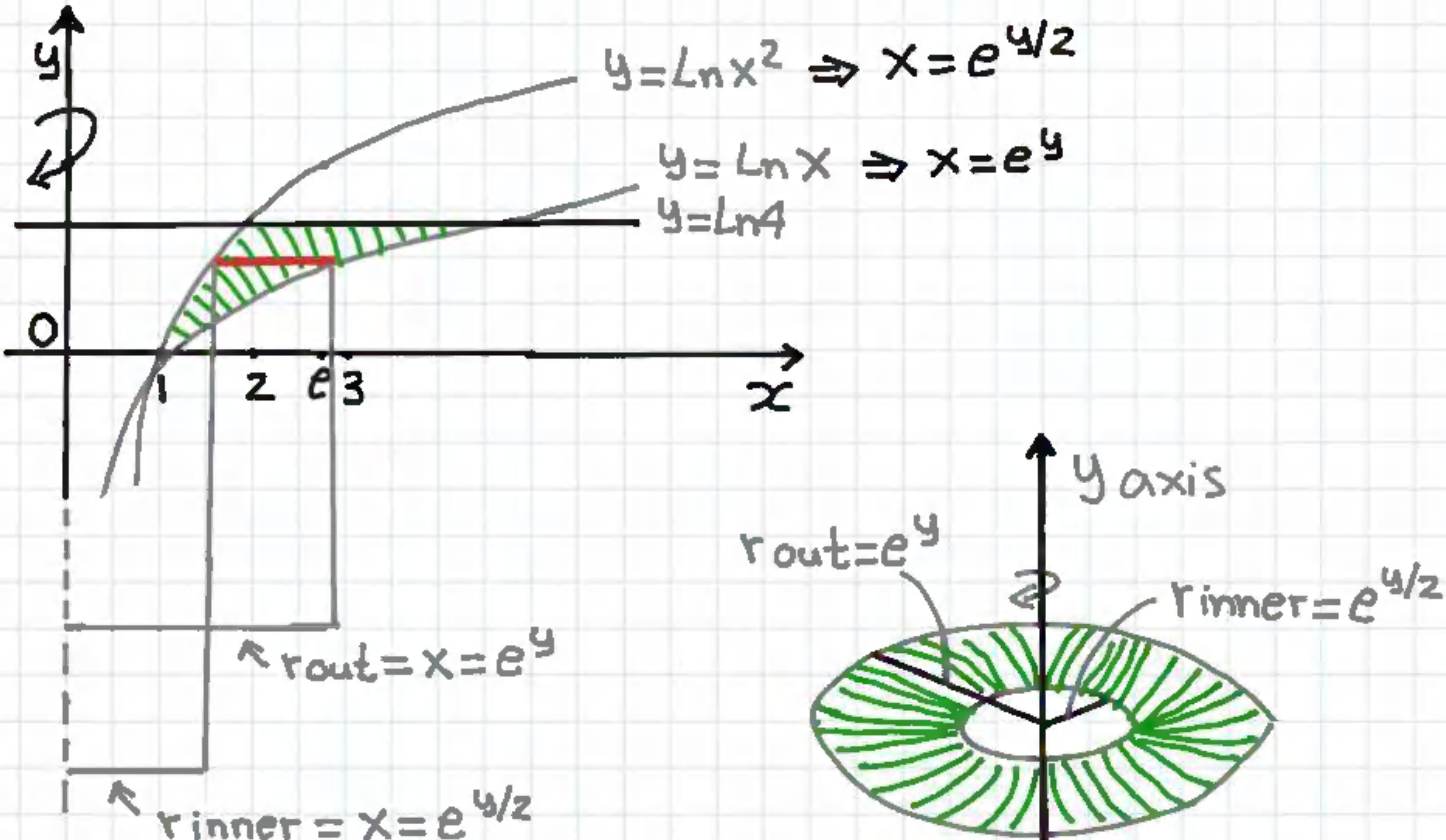
step 2| sketch bounded region and typical washer

Let us first solve for x in terms of y

$y = \ln x \Rightarrow e^y = e^{\ln x} \Rightarrow x = e^y$

$y = \ln x^2 \Rightarrow e^y = e^{\ln x^2} \Rightarrow x^2 = e^y \Rightarrow x = \pm e^{y/2} \Rightarrow x = e^{y/2}$

$y = \ln x^2 \Rightarrow x = e^{y/2}$



Note: horizontal slice ⊥ (Perp.) to axis of rotation y axis.

$r_{outer} = e^y$; $r_{inner} = e^{y/2}$

$A(y) = \pi \{ r_{out}^2 - r_{in}^2 \}$

$A(y) = \pi \{ (e^y)^2 - (e^{y/2})^2 \}$

washer Volume

$dV = \pi \{ (e^y)^2 - (e^{y/2})^2 \} \, dy$

Total Volume

$$V = \int_0^{\ln 4} \pi \{ (e^y)^2 - (e^{y/2})^2 \} \, dy$$

$$V = \pi \int_0^{\ln 4} [e^{2y} - e^y] \, dy$$

Step 3 | Set up definite integral and find volume

$$V = \pi \int_0^{\ln 4} \left[(e^{y})^2 - (e^{y/2})^2 \right] dy$$

$$V = \pi \int_0^{\ln 4} \left[e^{2y} - e^{y} \right] dy$$

Apply U-Subst.

$\int e^{2y}\, dy$

$u = 2y \quad du = 2dy$

$dy = \frac{du}{2}$

$$V = \pi \left[\frac{e^{2y}}{2} - e^{y} \right]_0^{\ln 4}$$

$$V = \pi \left[\frac{e^{2\ln 4}}{2} - e^{\ln 4} \right] - \pi \left[\frac{e^{0}}{2} - e^{0} \right]$$

$$V = \pi \left[\frac{e^{\ln 4^2}}{2} - e^{\ln 4} \right] - \pi \left[\frac{1}{2} - 1 \right]$$

$$V = \pi\left[\frac{16}{2} - 4\right] - \pi\left[\frac{1}{2} - 1\right]$$

$$V = \frac{4\pi}{1} + \frac{\pi}{2} = \frac{8\pi + \pi}{2} = \frac{9\pi}{2} \cong 14.14$$

Basic Skills review

$$e^{\ln x} = x \quad ; \quad \ln e^{x} = x \quad ; \quad \ln x^{r} = r \ln x$$

$$\ln 1 = 0 \quad ; \quad \ln e = 1 \quad ; \quad e^{\ln x^2} = x^2$$

$$x^2 = e^{y} \Rightarrow x = \pm (e^{y})^{1/2} = \pm e^{y/2}$$

Find the volume of the solid obtained by rotating the region bounded by the curves y=arctanx ,y=π/4,x=0 rotated about the line x=2 solved example

Volumes 13 (washer method)

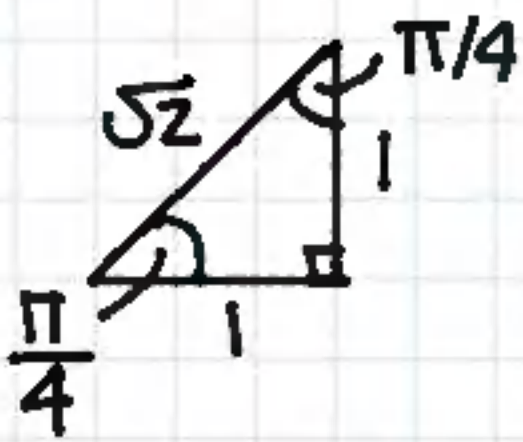
Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y = \tan^{-1} x$, $y = \pi/4$, $x = 0$ rotated 2 about $x = -2$

step 1] Find the points of intersection

$y = \tan^{-1} x$, $y = \pi/4 \Rightarrow Y = Y \Rightarrow \tan^{-1} x = \pi/4$

$\tan(\tan^{-1} x) = \tan(\pi/4) \Rightarrow x = \tan\frac{\pi}{4} = 1 \Rightarrow x = 1 , y = \pi/4$

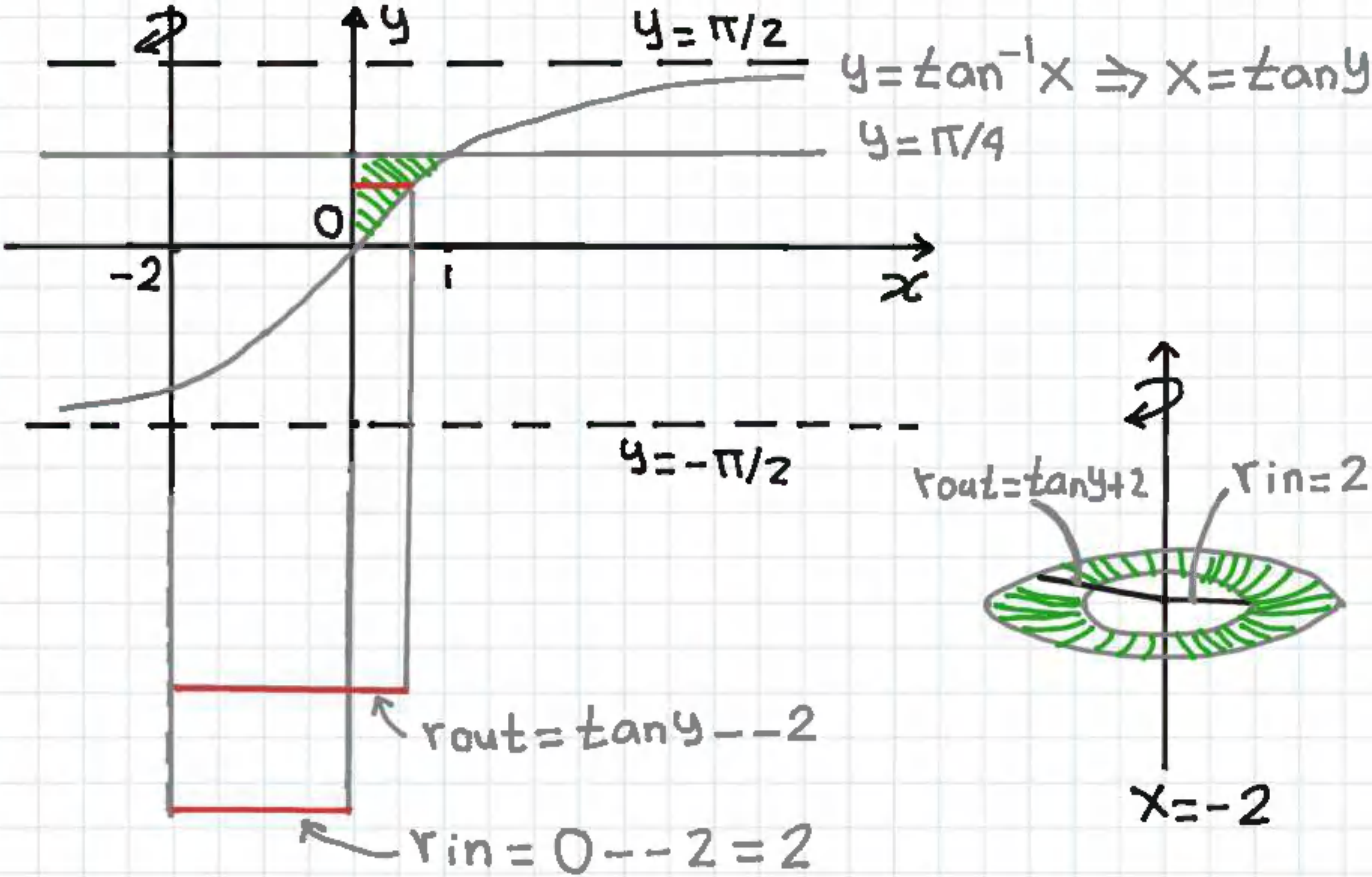
$y = \tan^{-1} x$, $x = 0 \Rightarrow y = \tan^{-1} 0 = 0 \Rightarrow x = 0 \quad y = 0$



45-45-90 triangle

$\tan(\pi/4) = 1$

Step 2 | Sketch bounded region and typical washer



$r_{out} = \tan y + 2$; $r_{inner} = 2$

$A(y) = \pi \{ r_{out}^2 - r_{in}^2 \}$ Cross sectional area

$A(y) = \pi \{ (\tan y + 2)^2 - (2)^2 \}$

washer Volume

$dV = \pi \{ (\tan y + 2)^2 - (2)^2 \} \, dy$

Total Volume of Solid

$$V = \pi \int_0^{\pi/4} \left[(\tan y + 2)^2 - (2)^2 \right] dy$$

Step 3 | Set up definite integral and find volume

$$V = \pi \int_0^{\pi/4} \left[(\tan y + 2)^2 - (2)^2 \right] dy$$

$$V = \pi \int_0^{\pi/4} \left[\tan^2 y + 4\tan y + \cancel{4} - \cancel{4} \right] dy$$

$$V = \pi \int_0^{\pi/4} \left[\tan^2 y + 4\tan y \right] dy$$

$$V = \pi \int_0^{\pi/4} \tan^2 y \, dy + 4\pi \int_0^{\pi/4} \tan y \, dy$$

$$V = \pi \int_0^{\pi/4} \tan^2 y\, dy + 4\pi \int_0^{\pi/4} \tan y\, dy$$

$$V = \pi \int_0^{\pi/4} (\sec^2 y - 1)\, dy + 4\pi \int_0^{\pi/4} \frac{\sin y}{\cos y}\, dy$$

u-Substitution
$u = \cos y \quad du = -\sin y\, dy$

$$V = \pi \Big[\tan y - y \Big]_0^{\pi/4} + 4\pi \Big[-\ln|\cos y| \Big]_0^{\pi/4}$$

$$V = \pi \Big[\underbrace{\tan(\pi/4)}_{=1} - \frac{\pi}{4} - (\underbrace{\tan 0}_{=0} - 0) \Big]$$

$$- 4\pi \Big[\ln|\underbrace{\cos(\pi/4)}_{=1/\sqrt{2}}| - \ln|\underbrace{\cos 0}_{=1}| \Big]$$

$$V = \pi\left[\overset{=1}{\tan(\pi/4)} - \pi/4 - (\overset{=0}{\tan 0} - 0)\right]$$
$$-4\pi\left[\ln\left|\underset{=1/\sqrt{2}}{\cos(\pi/4)}\right| - \ln\left|\underset{=1}{\cos 0}\right|\right]$$

$$V = \pi - \frac{\pi^2}{4} - 4\pi \ln(1/\sqrt{2}) + 4\pi \overset{=0}{\ln 1}$$

$$V = \pi - \frac{\pi^2}{4} - 4\pi \ln(1/\sqrt{2}) \cong 5.03$$

Basic Skills Review

$\tan 0 = 0$; $\tan(\pi/4) = 1$; $\cos(\pi/4) = 1/\sqrt{2}$

$\ln 1 = 0$; $\ln e = 1$; $\cos 0 = 1$

$1 + \tan^2 y = \sec^2 y$

Find the volume of the solid obtained by rotating the region bounded by the curves y=1/x and 2x+2y=5 about the line y=-1

Volumes 14 (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y=\frac{1}{x}$ and $2x+2y=5$ about the line $y=-1$.

Solution: step 1] Find the points of intersection

Subst. $y=\frac{1}{x}$ into $2x+2y=5 \Rightarrow 2x+\frac{2}{x}=5$

$2x^2+2=5x \Rightarrow 2x^2-5x+2=0$ apply quad. form.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow x=\frac{5\pm\sqrt{25-4(2)(2)}}{4}$$

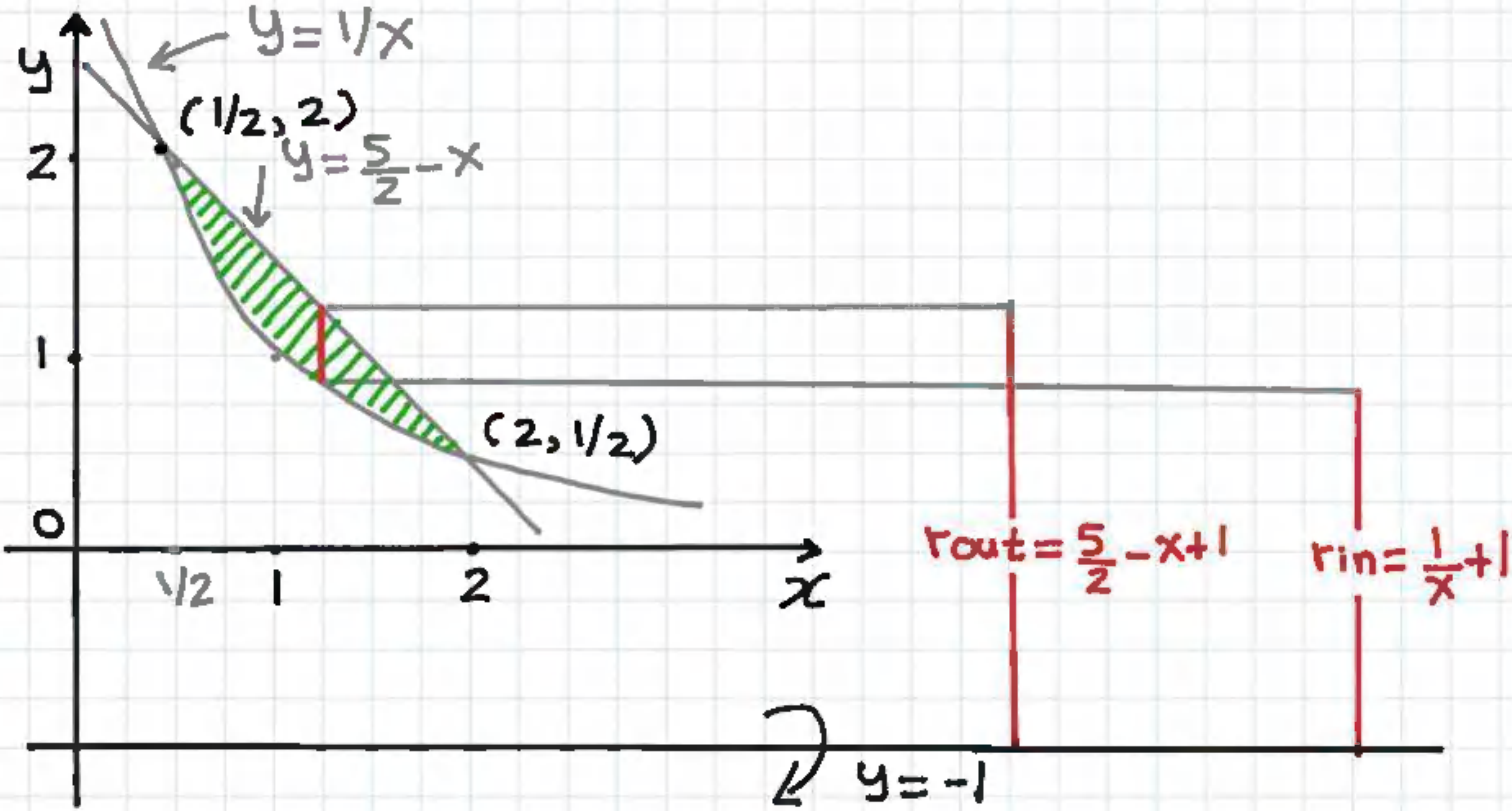
$$x=\frac{5\pm\sqrt{9}}{4}=\frac{5\pm 3}{4}=2,\frac{1}{2} \Rightarrow x=2 \quad y=\frac{1}{x}=1/2$$

$$x=1/2 \quad y=1/x=2$$

$\therefore$ $(2, 1/2)$ and $(1/2, 2)$

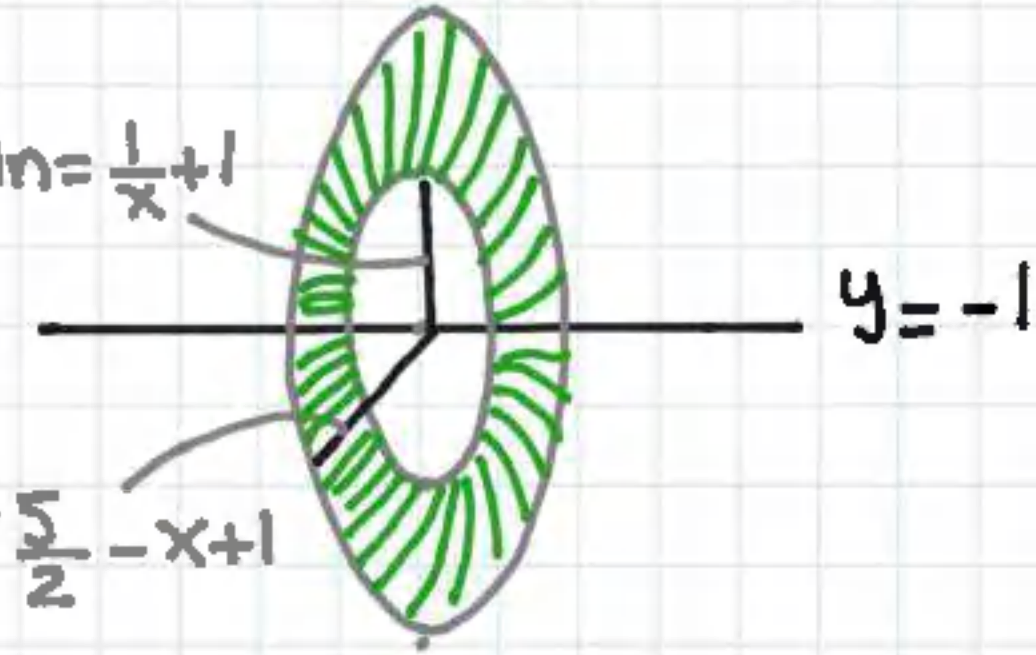
Step 2 | Sketch bounded Region and typical washer

$$2x + 2y = 5 \Rightarrow 2y = 5 - 2x \Rightarrow y = 5/2 - x$$



$r_{outer} = \frac{5}{2} - x + 1$

$r_{inner} = \frac{1}{x} + 1$



Washer Volume

$$dV = \pi\left\{ \left(\frac{5}{2} - x + 1\right)^2 - \left(\frac{1}{x} + 1\right)^2 \right\} dx$$

$$V = \pi \int_{1/2}^{2} \left[\left(\frac{5}{2} - x + 1\right)^2 - \left(\frac{1}{x} + 1\right)^2 \right] dx$$

$$V = \pi \int_{1/2}^{2} \left[(7/2 - x)^2 - (1/x + 1)^2 \right] dx$$

step 3 | Set up definite integral and find volume

$$V = \pi \int_{1/2}^{2} \left[(7/2 - x)^2 - (1/x + 1)^2 \right] dx$$

$$V = \pi \int_{1/2}^{2} \left[\frac{49}{4} - 2(7/2) \cdot x + x^2 - \left(\frac{1}{x^2} + \frac{2}{x} + 1 \right) \right] dx$$

$$V = \pi \int_{1/2}^{2} \left[\frac{49}{4} - 7x + x^2 - \frac{1}{x^2} - \frac{2}{x} - 1 \right] dx$$

$$V = \pi \int_{1/2}^{2} \left[\frac{45}{4} - 7x + x^2 - x^{-2} - \frac{2}{x} \right] dx$$

$$V = \pi \left[\frac{45}{4} x - \frac{7x^2}{2} + \frac{x^3}{3} - \frac{x^{-1}}{-1} - 2 \ln|x| \right]_{1/2}^{2}$$

$$V = \pi\left[\frac{45}{4}x - \frac{7x^2}{2} + \frac{x^3}{3} + \frac{1}{x} - 2\ln|x|\right]_{1/2}^{2}$$

$$V = \pi\left[\frac{90}{4} - 14 + \frac{8}{3} + \frac{1}{2} - 2\ln 2\right]$$

$$-\pi\left[\frac{45}{8} - \frac{7}{8} + \frac{1}{24} + 2 - 2\ln(1/2)\right]$$

Calculator ready answer

After few more steps...

$$V = \frac{\pi(39 - 32\ln 2)}{8} \cong 6.6$$

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Find the volume of the solid obtained by rotating the region bounded by the curves y-√x+1 ,y=x^3+1 revolved about the line x=2 solved example

Volumes 15 (washer method)

Ex] Find the volume of the solid obtained by rotating the region bounded by the curves $y=\sqrt{x}+1$, $y=x^3+1$ about the line $x=2$

step 1] Find the points of intersection

$y=\sqrt{x}+1$, $y=x^3+1 \Rightarrow Y=Y \Rightarrow \sqrt{x}+1=x^3+1$

$\sqrt{x}+1=x^3+1$ Let's Guess!

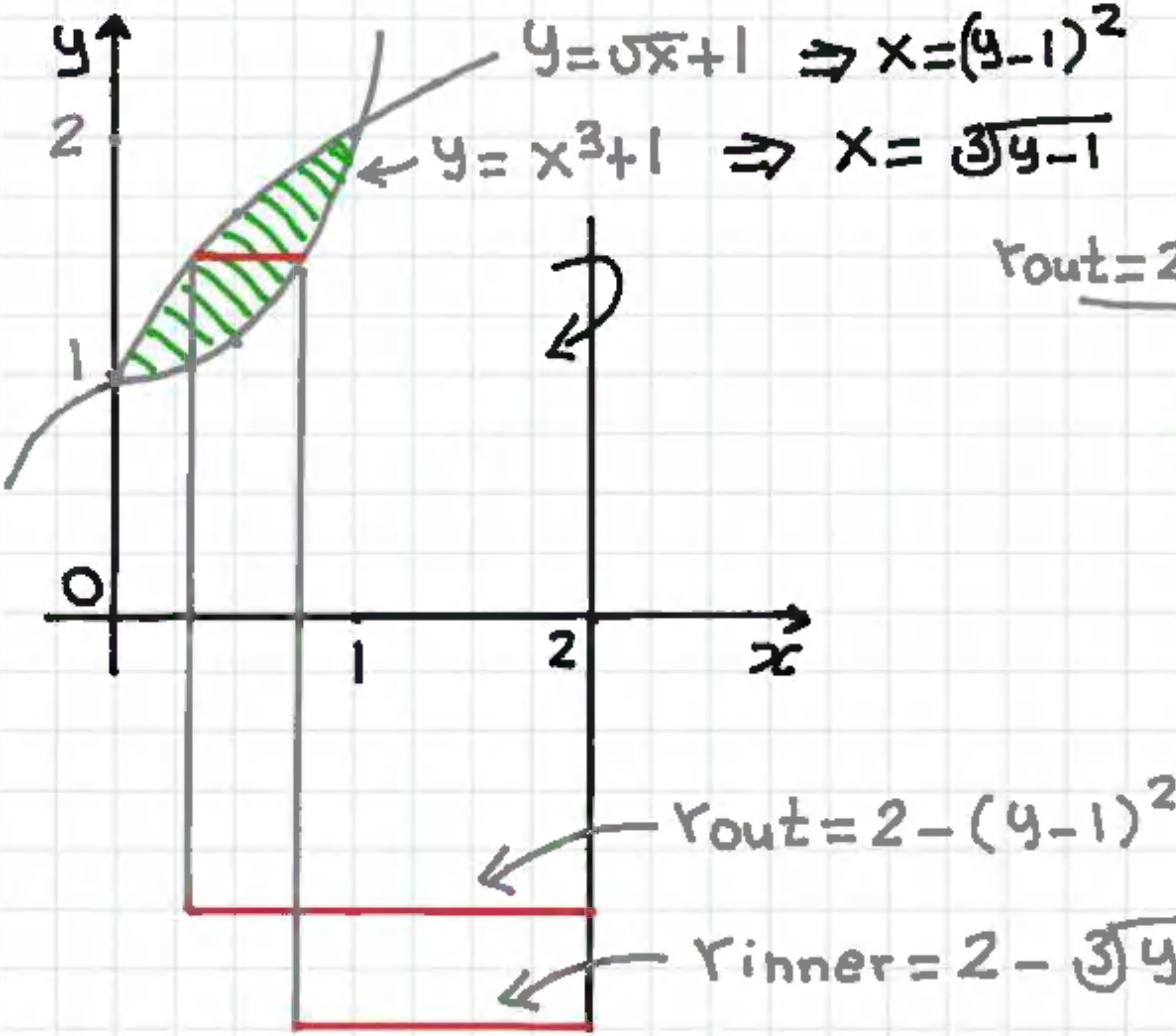
$x=0 \Rightarrow 0+1=0+1$ ✓

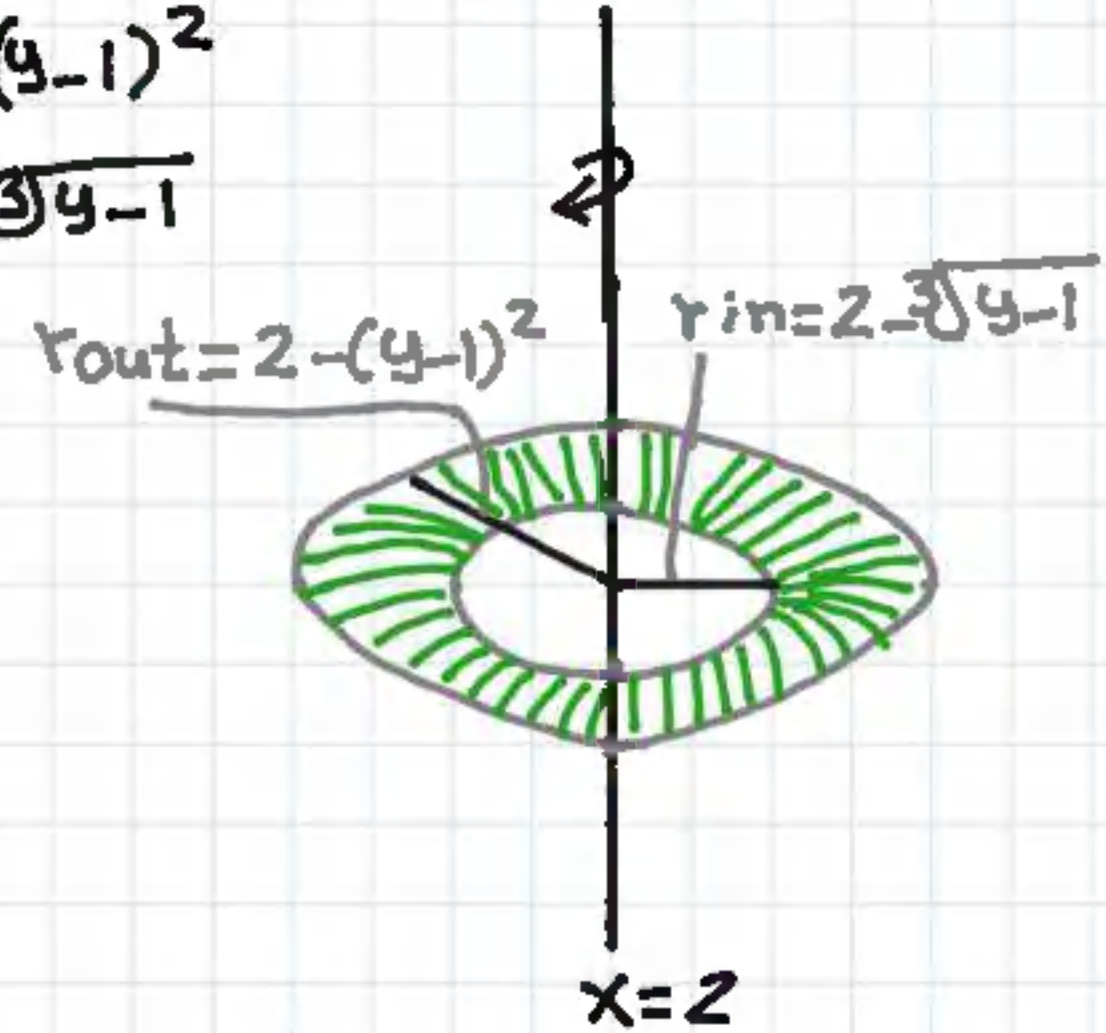
$x=1 \Rightarrow \sqrt{1}+1=1^3+1 \Rightarrow 2=2$ ✓

$x=0 \Rightarrow y=\sqrt{x}+1 \Rightarrow y=1 \Rightarrow x=0 , y=1$

$x=1 \Rightarrow y=x^3+1 \Rightarrow y=2 \Rightarrow x=1 , y=2$

Step 2 | Sketch bounded region and typical washer





$r_{out} = 2-(y-1)^2$; $r_{in} = 2-\sqrt[3]{y-1}$

$$A(y) = \pi\left\{ r_{out}^2 - r_{in}^2 \right\}$$

$$A(y) = \pi\left\{ (2-(y-1)^2)^2 - (2-\sqrt[3]{y-1})^2 \right\}$$

washer volume at a point y where $1 \le y \le 2$

$$dV = \pi\left\{ (2-(y-1)^2)^2 - (2-\sqrt[3]{y-1})^2 \right\} dy$$

Total Volume of solid

$$V = \pi \int_1^2 \left[(2-(y-1)^2)^2 - (2-\sqrt[3]{y-1})^2 \right] dy$$

step 3| Set up definite integral and find volume

$$V=\pi\int_1^2 \left[(2-(y-1)^2)^2-(2-\sqrt[3]{y-1})^2 \right] dy$$

$$V=\pi\int_1^2 \left[4-4(y-1)^2+(y-1)^4-(4-4\sqrt[3]{y-1}+(y-1)^{2/3}) \right] dy$$

$$V=\pi\int_1^2 \left[-4(y-1)^2+(y-1)^4+4(y-1)^{1/3}-(y-1)^{2/3} \right] dy$$

$$V=\pi\left[\frac{-4(y-1)^3}{3}+\frac{(y-1)^5}{5}+4(y-1)^{4/3}\cdot\frac{3}{4}-(y-1)^{5/3}\cdot\frac{3}{5} \right]_1^2$$

$$V=\pi\left[-\frac{4}{3}+\frac{1}{5}+\cancel{4}\cdot\frac{3}{\cancel{4}}-\frac{3}{5}-(0) \right]=\pi\left[-\frac{4}{3}-\frac{2}{5}+3 \right]$$

$$V = \pi\left[-\frac{4}{3} - \frac{2}{5} + \frac{3}{1}\right] = \pi\left[\frac{-20-6+45}{15}\right] = \frac{19\pi}{15} \cong 3.98$$

Integration Review U-Substitution

$$\int (y-1)^4\,dy = \frac{(y-1)^5}{5} + C \qquad \text{Let } u = y-1 \quad du = dy$$

$$\int (y-1)^{1/3}\,dy = \frac{(y-1)^{4/3}}{4/3} + C \qquad \text{Let } u = y-1 \quad du = dy$$

$$\int (ay+b)^n\,dy = \frac{(ay+b)^{n+1}}{(n+1)(a)} + C$$

Let $u = ay+b$

$du = a\,dy$

$dy = \dfrac{du}{a}$

where $n \neq -1$