Work Integrals 1

⇒ Constant Force acts in direction of motion of an object thru a distance d, we define the work done on the object to be Work = Force × distance

Ex) A 1000 kg car is being lifted by a crane to a height of 10 metres. Find the work done by the crane?

Solution: Units are in SI metric system

$m = 1000$ kg $d = 10$ metres $g = 9.8\ m/s^2$

$W = F.d = mgd = (1000)(9.8)(10) = 98000$ J

Note: Joules is unit of work

1 Joule = 1 Newton. metre

Ex] How much work is done in lifting a 200 pound metal bar 100 feet off the ground ?

Solution: In the U.S system pound (lb) is the weight of the object and hence it is already a unit of force.

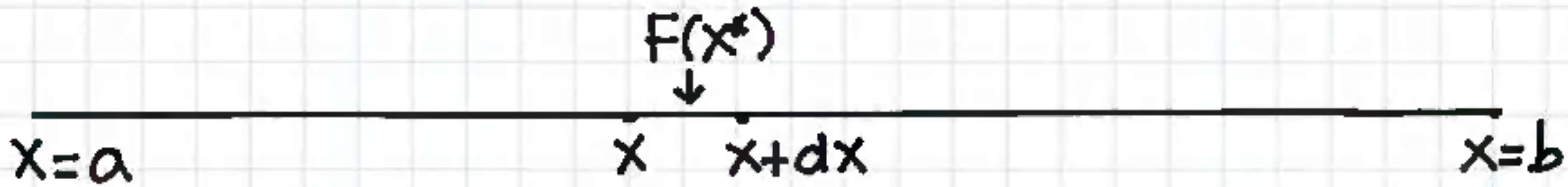
Therefore: $W = F.d = (200)(100) = 20000$ Ft-Lb

Summary: S.I system $1 J = N \cdot m$

U.S System 1 unit of work = Ft-lb

Theory: Lets consider a variable force that acts on an object along the x axis in the positive direction and let's assume that the force $F(x)$ varies continuously as it moves the object from $x=a$ to $x=b$

Consider an infinitesimally short distance dx thru which $F(x)$ moves the object



where $x^* \in (x, x+dx)$ then $dW = F(x^*)dx$

as $dx \to 0 \Rightarrow x^* \to x$ and $W = \int_a^b F(x)\,dx$

Work done in moving the object from $x=a$ to $x=b$

Ex| When a particle is x metres from the origin a variable force given by $F(x) = x + e^{-x} + \sin(\pi x/2)$ Newton moves the object from $x = \frac{1}{2}$ m to $x = 2$m

Find the work done on the object?

Solution: $W = \int_a^b F(x)\,dx$ where $a = 1/2$, $b = 2$

and $F(x) = x + e^{-x} + \sin(\pi x/2)$

$$W = \int_{1/2}^{2} \left(x + e^{-x} + \sin(\pi x/2) \right) dx$$

Recall: $\int e^{cx}\,dx = \frac{e^{cx}}{c} + k$

U-Substitution
$u = cx \quad du = c\,dx$

$$W = \int_{1/2}^{2} \left(x + e^{-x} + \sin(\pi x/2) \right) dx$$

$$W = \left[\frac{x^2}{2} - e^{-x} - \cos(\pi x/2) \cdot \frac{2}{\pi} \right]_{1/2}^{2} \quad \text{Apply F.T.C}$$

$$W = \left[2 - e^{-2} - \frac{2}{\pi} \overset{= -1}{\cos(\pi)} - \left(1/8 - e^{-1/2} - 2/\pi \ \overset{= 1/\sqrt{2}}{\cos(\pi/4)} \right) \right]$$

$$W = \left[2 - e^{-2} + \frac{2}{\pi} - \frac{1}{8} + e^{-1/2} + \frac{2}{\pi} \cdot \frac{1}{\sqrt{2}} \right] \text{ Joules}$$

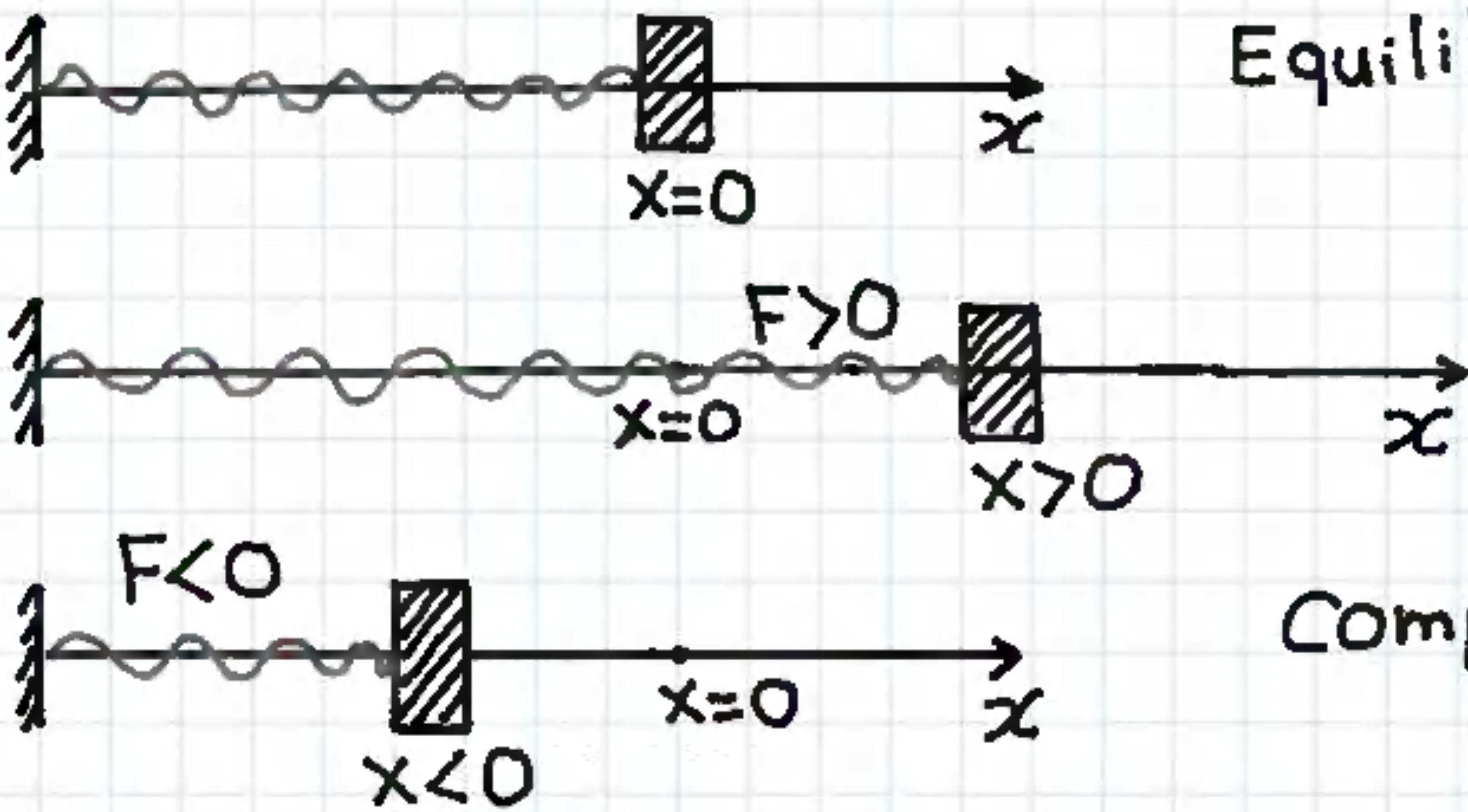
Notes: $\int \sin(\pi x/2)\, dx = -\frac{2}{\pi} \cos(\pi x/2) + C$

U-Subst. Let $u = \pi x/2 \Rightarrow du = \pi/2\, dx \Rightarrow dx = \frac{2}{\pi} du$

Work Integrals 2

Hooke's Law: The force $f(x)$ required to stretch a spring x units beyond its natural length is:

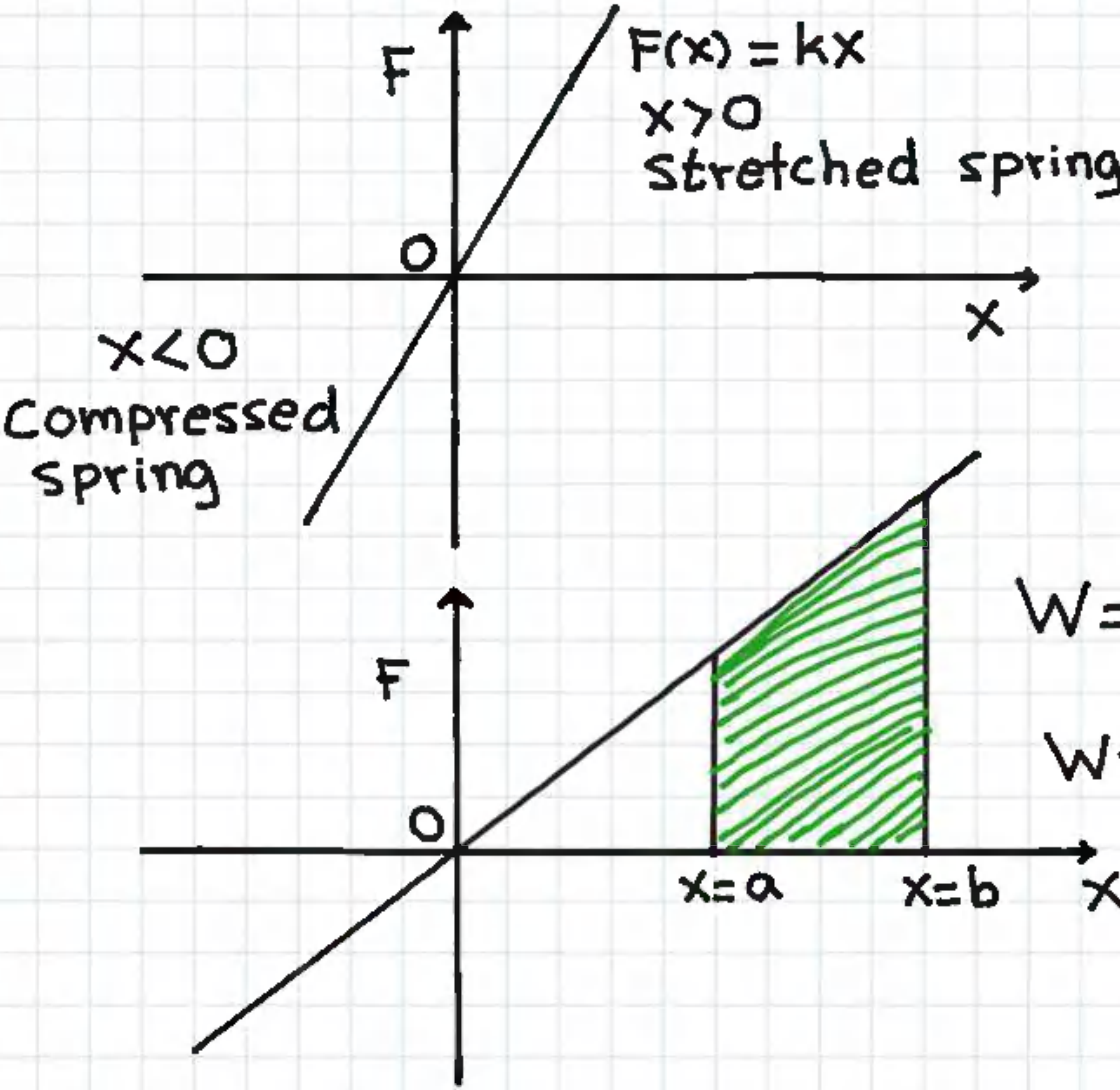
$f(x) = kx$ where x is the stretch beyond equilibrium and k is spring constant



Equilibrium $x=0$
Force $=0$

Stretched $x>0$
Force >0

Compressed $x<0$
Force <0



$$W = \int_a^b F(x)\,dx$$

$$W = \int_a^b kx\,dx$$

Ex1] A force of 20 N is required to stretch a spring 10 cm from its equilibrium position.

a) Find the spring constant k

b) How much work is required to stretch the spring 20 cm from its equilibrium position.

c) How much work is required to compress the spring 10 cm from its equilibrium position.

D) Compute the extra work needed to stretch the spring an additional 15 cm when it has already been stretched 5 cm beyond equilibrium.

Ex 1/a] Find the spring constant K

Hookes Law: $F(x) = Kx \Rightarrow F(0.1) = k(0.1m) = 20N$

$\Rightarrow K = \frac{20}{0.1} = 200$ N/m spring constant

Ex 1/b] Find Work to stretch the spring 20 cm from its equilibrium position. (equilibrium means $x=0$)

Solution: By Hooke's law $F(x) = 200x$

The work required to stretch the spring from $x=0$ to $x=0.20$ m is:

$$W = \int_a^b F(x)dx = \int_0^{0.2} 200x\,dx = \frac{200x^2}{2}\Big|_0^{0.2} = 100x^2\Big|_0^{0.2}$$

$$= 100(0.2)^2 - 100(0) = 4 \text{ Joules}$$

Ex 1/C] Find the work needed to compress the spring 10 cm from its equilibrium position.

Solution : Since we are compressing the spring we must integrate F(x) from $x=0$ to $x=-0.1$ cm

$$W = \int_0^{-0.1} 200x\,dx = \frac{200x^2}{2}\Bigg|_0^{-0.1} = 100(-0.1)^2 - 0$$
$$= 1 \text{ Joule}$$

Therefore the work required to compress the spring 10 cm or 0.1 m from its equilibrium position is 1 Joule.

Ex 1/d) Compute the extra work needed to stretch the spring an additional 15 cm when the spring has already been stretched 5 cm beyond equilibrium.

Solution: X is stretch of spring beyond equilibrium.

∴ We must integrate F(x) from $x = 0.05$ m to $x = 0.2$ m

$$W = \int_a^b F(x)\,dx = \int_{0.05}^{0.2} 200x\,dx = \frac{200x^2}{2}\Big|_{0.05}^{0.2}$$

$$W = 100x^2\Big|_{0.05}^{0.2} = 100(0.2)^2 - 100(0.05)^2$$

$$W = 4 - 0.25 = 3.75 \text{ Joules}$$

Work Integrals 3

Ex] The work required to stretch a spring from 10 cm to 12 cm is 6 Joules. It takes an additional 10 Joules of work to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring.

Solution: $W = \int_a^b kx\,dx$

$$6 = \int_{0.10-L}^{0.12-L} kx\,dx \quad \text{and} \quad 10 = \int_{0.12-L}^{0.14-L} kx\,dx$$

Key concept: We have to subtract natural length of the spring from the position of the spring, since x represents stretch of spring beyond equilibrium. i.e) $x = 0.10 - L$, $x = 0.12 - L$, $x = 0.14 - L$

$$6 = \int_{0.10-L}^{0.12-L} kx\,dx \Rightarrow 6 = \frac{kx^2}{2}\Big|_{0.10-L}^{0.12-L}$$

$$6 = \frac{k}{2}\left[(0.12-L)^2 - (0.10-L)^2\right]$$

$$10 = \int_{0.12-L}^{0.14-L} kx\,dx \Rightarrow 10 = \frac{kx^2}{2}\Big|_{0.12-L}^{0.14-L}$$

$$10 = k/2\left[(0.14-L)^2 - (0.12-L)^2\right]$$

$$6 = \frac{k}{2}\left[(0.12 - L)^2 - (0.1 - L)^2\right]$$

$$10 = \frac{k}{2}\left[(0.14 - L)^2 - (0.12 - L)^2\right]$$

Strategy: Divide the two equations to solve for L

$$\frac{3}{5} = \frac{(0.12 - L)^2 - (0.1 - L)^2}{(0.14 - L)^2 - (0.12 - L)^2}$$

$a^2 - b^2 = (a-b)(a+b)$

$$\frac{3}{5} = \frac{(0.12 - L - (0.1 - L))(0.12 - L + (0.1 - L))}{(0.14 - L - (0.12 - L))(0.14 - L + (0.12 - L))}$$

$$\frac{3}{5} = \frac{\cancel{(0.02)}(0.22 - 2L)}{\cancel{(0.02)}(0.26 - 2L)}$$

$$\frac{3}{5} = \frac{0.22 - 2L}{0.26 - 2L}$$ Cross Multiply

$$3(0.26 - 2L) = 5(0.22 - 2L)$$

$$3(0.26) - 6L = 5(0.22) - 10L$$

$$4L = 5(0.22) - 3(0.26)$$

$$4L = 0.32 \Rightarrow L = \frac{0.32}{4} = 0.08 \text{ metres}$$

Therefore natural Length of spring is 8 cm.

Work Integrals 4

Ex| Find the work required to pump all the oil out of a cylindrical tank with a height of 2 m and a radius of 0.5 m. The oil is pumped out thru an outflow pipe 1 m above the top of the 2 m tank. Oil has density of 800 kg/m³

$g = 9.8\ m/s^2$.

Solution: $y=0$ Oil level at bottom of tank

$y=2$ Oil level at top of tank

radius $= 0.5$ m ; $g = 9.8\ m/s^2$; $\rho = 800\ kg/m^3$

$y = 3$ m Outflow pipe level

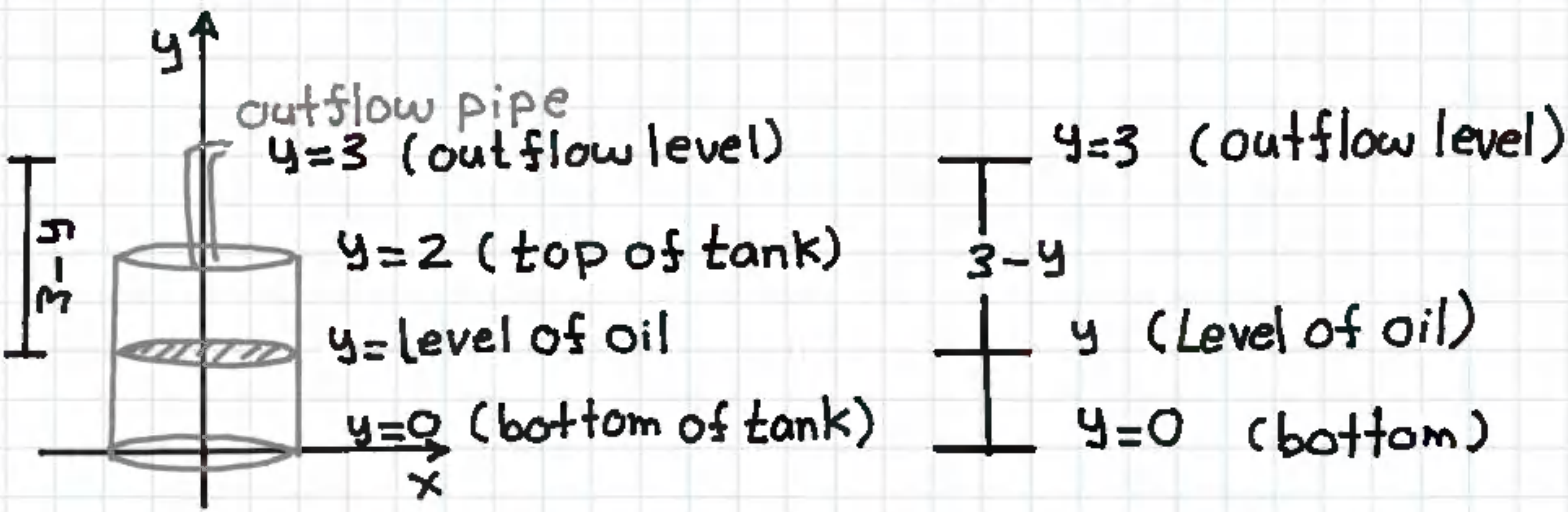
Step 1] Sketch a diagram and Define Variables

$y=0$ bottom of tank

$y=2$ top of tank

radius $= 0.5\,m$ $\quad \rho = 800\,\frac{kg}{m^3}$

$g = 9.8\,m/s^2$



Since the tank is a circular cylinder, all horizontal slices are circular disks of radius 0.5 m

$dV = \pi r^2 dy = \pi(0.5)^2 dy$ (Volume of thin slice)

$dm = \pi r^2 \rho\, dy = \pi(0.5)^2 \underline{(800)} dy$ (mass of thin slice)

$dF = g\, dm = \pi(0.5)^2 (800) \underline{(9.8)} dy$

(Force needed to overcome gravity acting on the mass of oil in this thin circular slice)

$dW = (3-y)\, dF = \pi(0.5)^2 (800)(9.8) \underline{\underline{(3-y)}} dy$

(Work required to pump out this thin slice of oil out of the outflow pipe $\underline{\underline{3-y}}$ metres.

Step2] Set up definite integral and find work required

$$dW = \pi(0.5)^2(800)(9.8)(3-y)\,dy$$

$$W = \int_{y=0}^{y=2} dW = \int_0^2 \pi(0.5)^2(800)(9.8)(3-y)\,dy$$

$$W = \pi(0.5)^2(800)(9.8)\int_0^2 (3-y)\,dy$$

$$W = 1960\pi\left[3y - \frac{y^2}{2}\right]_0^2$$

$$W = 1960\pi[6-2-(0-0)] = 1960\pi(4) = 7840\pi$$

$$W = 7840\pi \text{ J} \cong 24630.09 \text{ Joules}$$

Work required to pump oil out of the tank.

Work Integrals 5

Ex| An inverted Circular cone with height 8m and base radius 4 m. The tank is filled with water to a height of 6 m. Find the work required to empty the tank by pumping all the water over the edge of the tank. (Density of water is 1000 kg/m³)

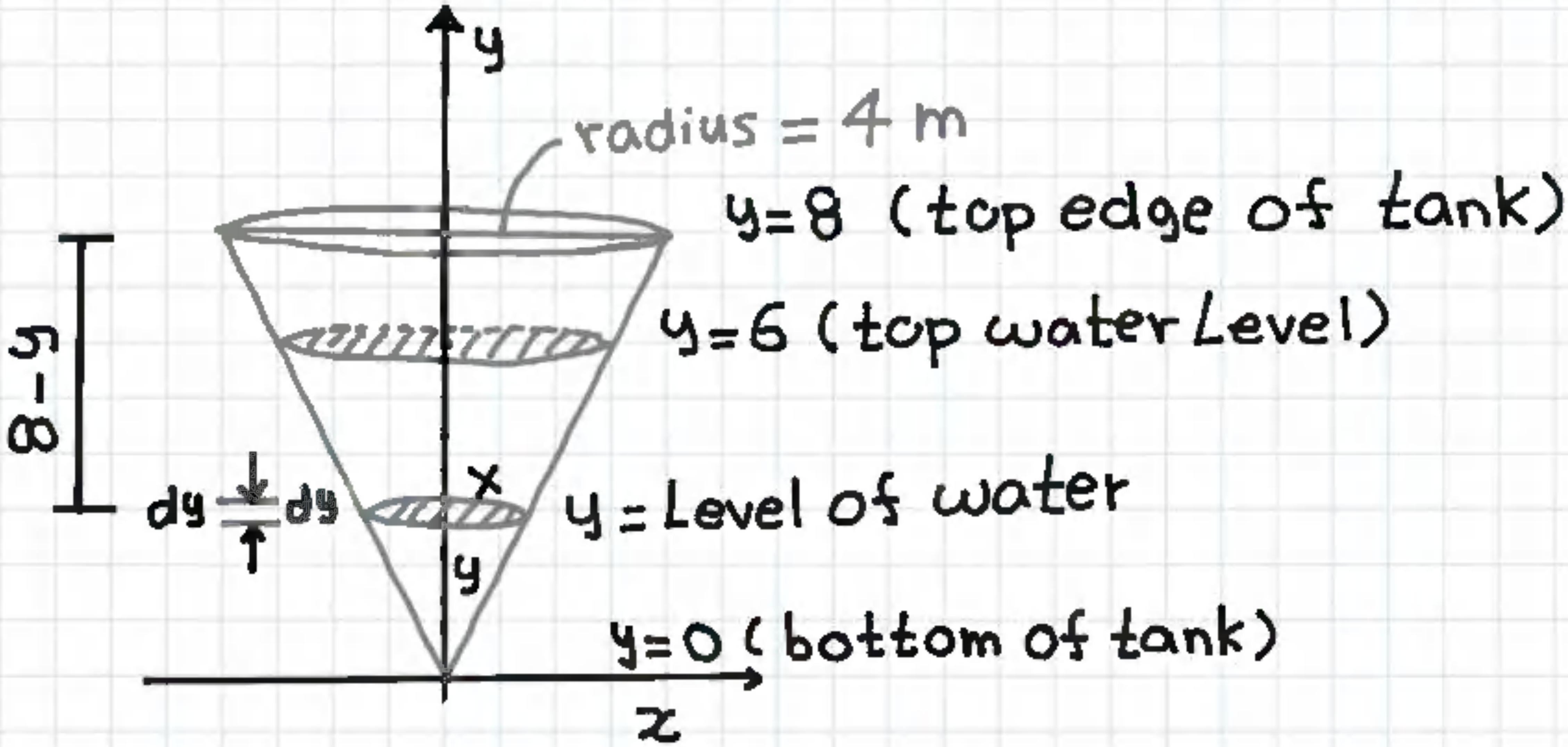
Solution: $y=0$ bottom of conical tank

$y=6\,m$ top water Level of conical tank

$y=8\,m$ top edge of conical tank

$\rho=1000\ kg/m^3$; $g=9.8\ m/s^2$

step 1] Sketch a diagram and define variables

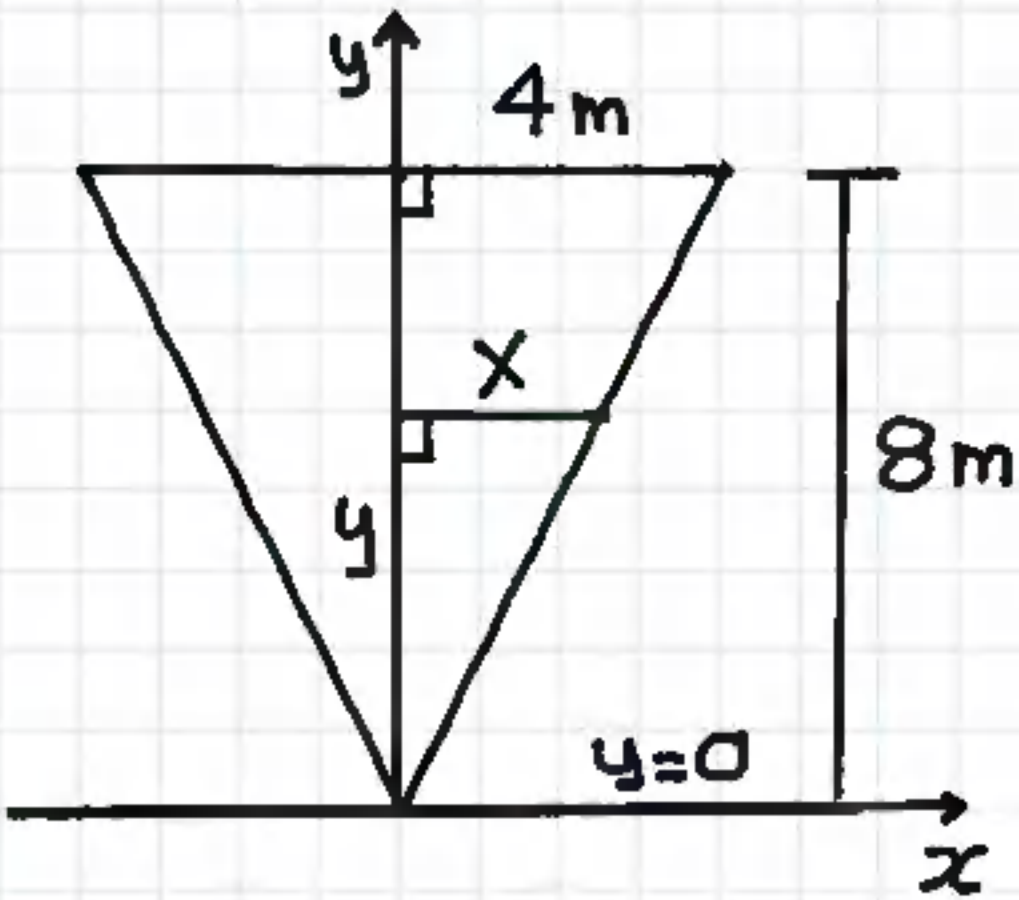


$\rho = 1000\ kg/m^3$; $g = 9.8\ m/s^2$; radius $= 4$ m

Since the tank is a circular cone, all horizontal slices are circular disks of radius x metres.

$dV = \pi x^2 dy$ (Volume of thin disc of water)

Let's apply similiar triangles to express x in terms of y.

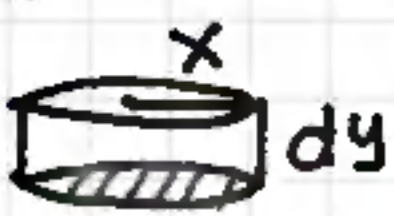


$$\frac{x}{y} = \frac{4}{8} \Rightarrow x = \frac{1}{2}y$$

$$dV = \pi x^2 dy = \pi\left(\frac{1}{2}y\right)^2 dy$$

$$dV = A(y)dy$$

$$dV = \frac{\pi}{4}y^2 dy$$



(Volume of thin disc of water)

$dV = \frac{\pi}{4} y^2 \, dy$ (Volume of thin disc of water)

$dm = \rho \, dV = \frac{\pi}{4} y^2 \underline{(1000)} \, dy$ (mass of thin disc)

$dF = g \, dm = \frac{\pi}{4} y^2 (1000) \underline{(9.8)} \, dy$

(Force needed to overcome gravity acting on the mass of water in this thin circular disk of water)

$dW = (8-y) \, dF = \frac{\pi}{4} y^2 (1000)(9.8) \underline{(8-y)} \, dy$

(Work required to pump out this thin disk of water over the edge of conical tank $\underline{\underline{8-y}}$ metres.)

step2| Set up definite integral and find work required

$$dW = \frac{\pi}{4} y^2 (1000)(9.8)(8-y)\,dy$$

$$W = \int_{y=0}^{y=6} dW = \int_0^6 \frac{\pi}{4}(1000)(9.8)\,y^2(8-y)\,dy$$

$$W = \frac{\pi}{4}(1000)(9.8)\int_0^6 y^2(8-y)\,dy$$

$$W = 2450\,\pi \int_0^6 (8y^2 - y^3)\,dy$$

$$W = 2450\,\pi \left[\frac{8y^3}{3} - \frac{y^4}{4}\right]_0^6$$

$$W = 2450\pi\left[\frac{8y^3}{3} - \frac{y^4}{4}\right]_0^6$$

$$W = 2450\pi\left[\frac{8(6)^3}{3} - \frac{6^4}{4} - (0-0)\right]$$

$$W = 2450\pi\left[\frac{6912 - 3888}{12}\right]$$

$$W = 2450\pi\left[\frac{3024}{12}\right] = 2450(252)\pi \text{ Joules}$$

$$W = 617400\pi \text{ J} \cong 1939619.3 \text{ Joules}$$

Important Note: $W = \int_{y=0}^{y=6} dW$

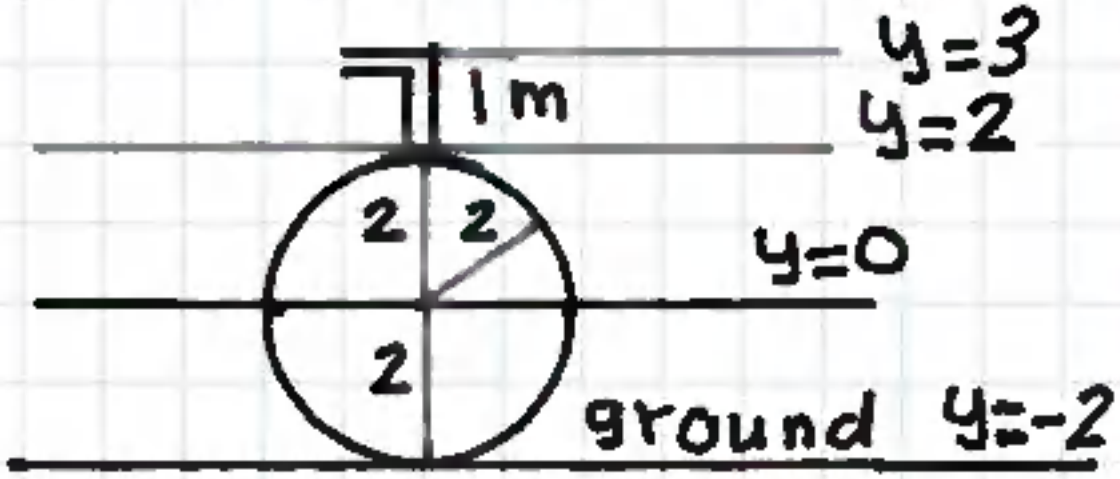
We integrate from $y=0$ to $y=6$ m because the conical tank is only filled up with water to a height of <u>6 metres</u> even though the height of the entire conical tank is <u>8 metres</u>

Key concept: Only integrate y values up to filled fluid levels and in this case the 8 m conical tank is only filled up to a height of 6 m with water.

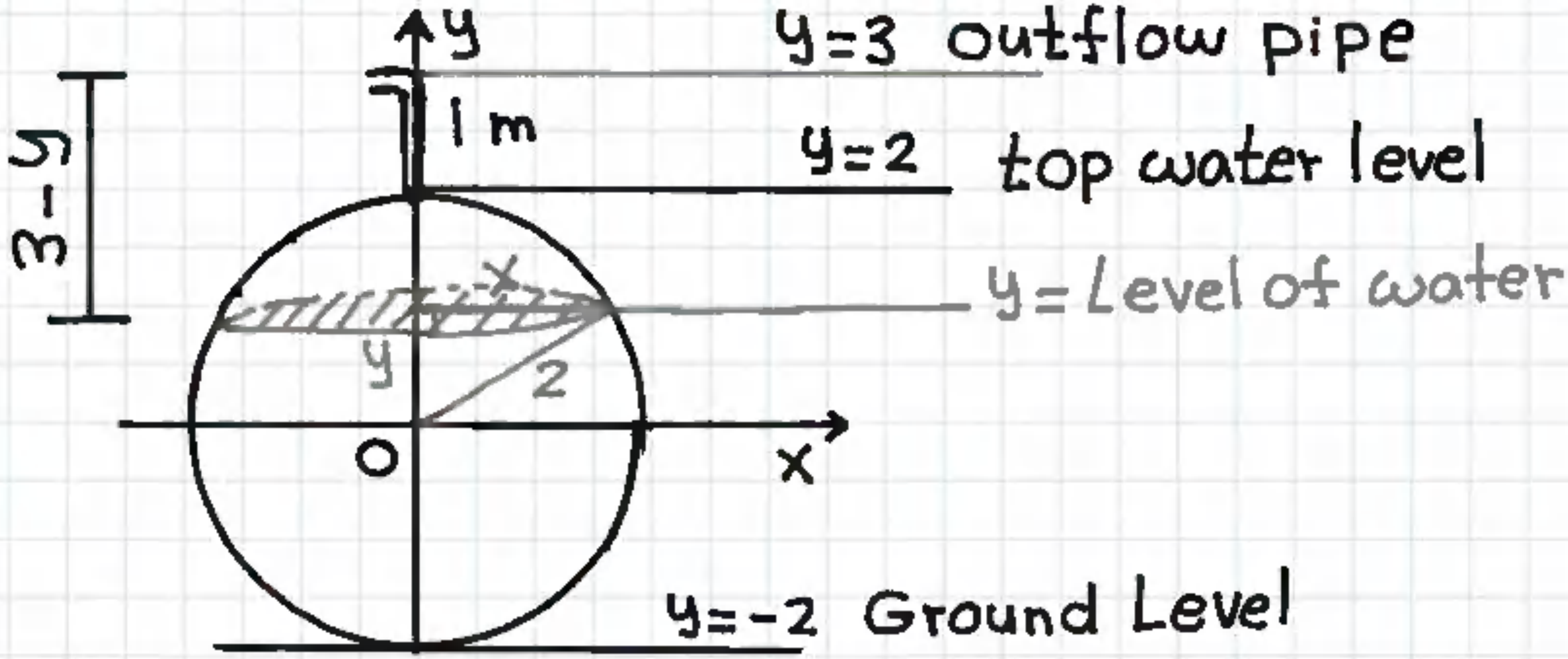
Work Integrals 6

Ex] A spherical tank with radius 2 m is full of water. Find the work required to pump all the water out of the outlet which is 1 m above the top of the spherical tank. ($g = 9.8 \, m/s^2$; $\rho_{water} = 1000 \, kg/m^3$)

Solution: To exploit the circular symmetry, we choose the origin to be the center of the spherical tank. so the height of the vertical circular cross section of the tank lies between $y=-2$ and $y=2$ with the outflow pipe at $y=3$.



Step 1| Sketch a diagram and define variables



Key Concept: Spherical tank lies between $y=-2$ and $y=2$ and every horizontal circular slice at height y has to be raised $\underline{3-y}$ metres to be pumped out of the outflow pipe.

Since the tank is a sphere, all horizontal slices perpendicular to the y axis are circular disks with radius x metres

$A(y) = \pi x^2$ (cross sectional area at height y)

Let's apply pythagoras to express x in terms of y.

$x^2 + y^2 = 4 \Rightarrow x^2 = 4 - y^2$

$A(y) = \pi(4 - y^2)$ (cross sectional area of slice)

$dV = A(y)\,dy$

$dV = \pi(4 - y^2)\,dy$ (Volume of thin disk of water)

$dm = \rho\,dV = \pi(4 - y^2)\underline{\underline{(1000)}}\,dy$ (mass of thin disc)

$dm = \pi(4-y^2)\underline{(1000)}dy$ (mass of disc at height y)

$dF = g\,dm = \pi(4-y^2)(1000)\underline{(9.8)}\,dy$

(Force needed to overcome gravity acting on the mass of water in this thin circular disc at height y)

$dW = (3-y)\,dF = \pi(4-y^2)(1000)(9.8)\underline{\underline{(3-y)}}\,dy$

(Work required to pump out this thin disk of water $\underline{(3-y)}$ metres thru the outflow pipe.

Key concept: $(3-y)$ represents the distance that each circular disk of water at height y must travel to be pumped out of the outflow pipe.

check $y=-2$ (ground) $\Rightarrow 3-y = 3--2 = 5$ metres ✓

check $y=2$ (top) $\Rightarrow 3-y = 3-2 = 1$ metre ✓

step2] Set up definite integral and find work required.

$$dw = \pi(4-y^2)(1000)(9.8)(3-y)\,dy$$

$$W = \int_{y=-2}^{y=2} dW = \int_{-2}^{2} \pi(4-y^2)(1000)(9.8)(3-y)\,dy$$

$$W = \pi(1000)(9.8)\int_{-2}^{2} (4-y^2)(3-y)\,dy$$

$$W = \pi(1000)(9.8)\int_{-2}^{2} (12-4y-3y^2+y^3)\,dy$$

$$W = \pi(1000)(9.8)\left[12y - \frac{4y^2}{2} - \frac{\cancel{3}y^3}{\cancel{3}} + \frac{y^4}{4}\right]_{-2}^{2}$$

$$W = \pi(1000)(9.8)\left[24-8-8+4-(-24-\cancel{8}+\cancel{8}+4)\right]$$

$$W = \pi(1000)(9.8)[24-8-8+4+24-4]$$

$$W = \pi(1000)(9.8)[32] = 313600\pi \cong 985203.46 \text{ J}$$

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Work Integrals 7

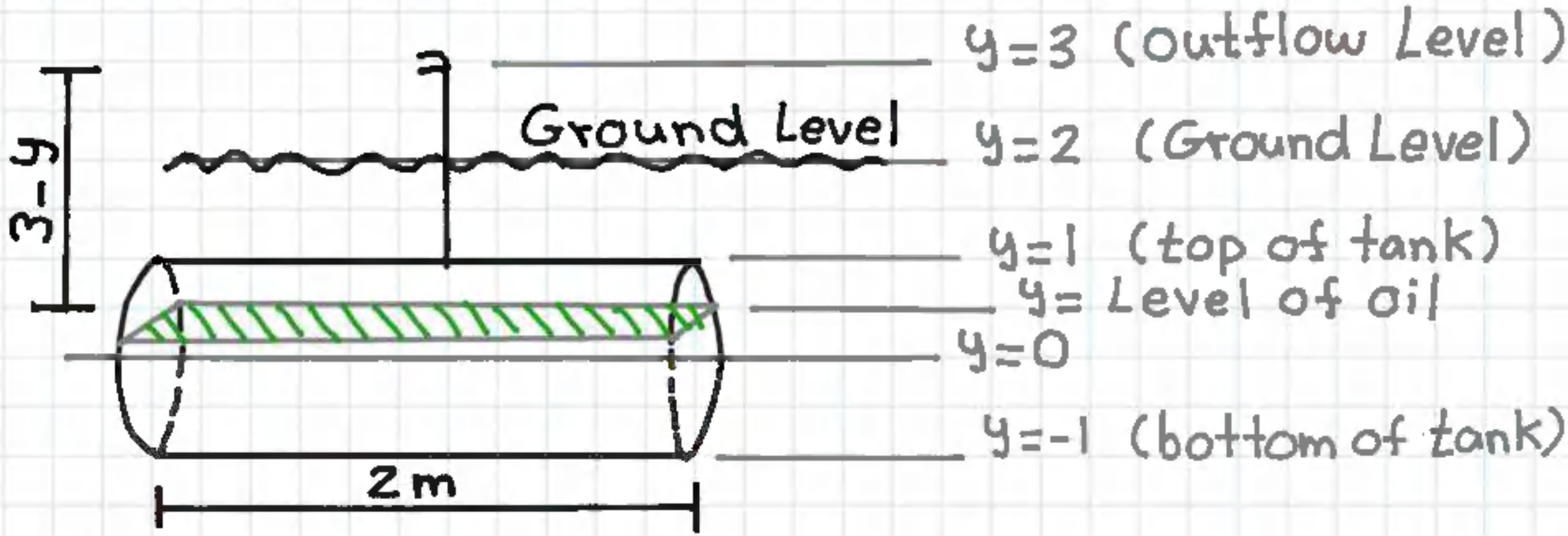
Ex] A cylindrical tank with a radius of 1 m and Length of 2 metres is full of crude oil and is buried 3 metres below the ground. (That means the bottom of the tank is 3 m below the ground)

How much work is required to pump all the oil thru an outlet pipe 1 metres above the ground.

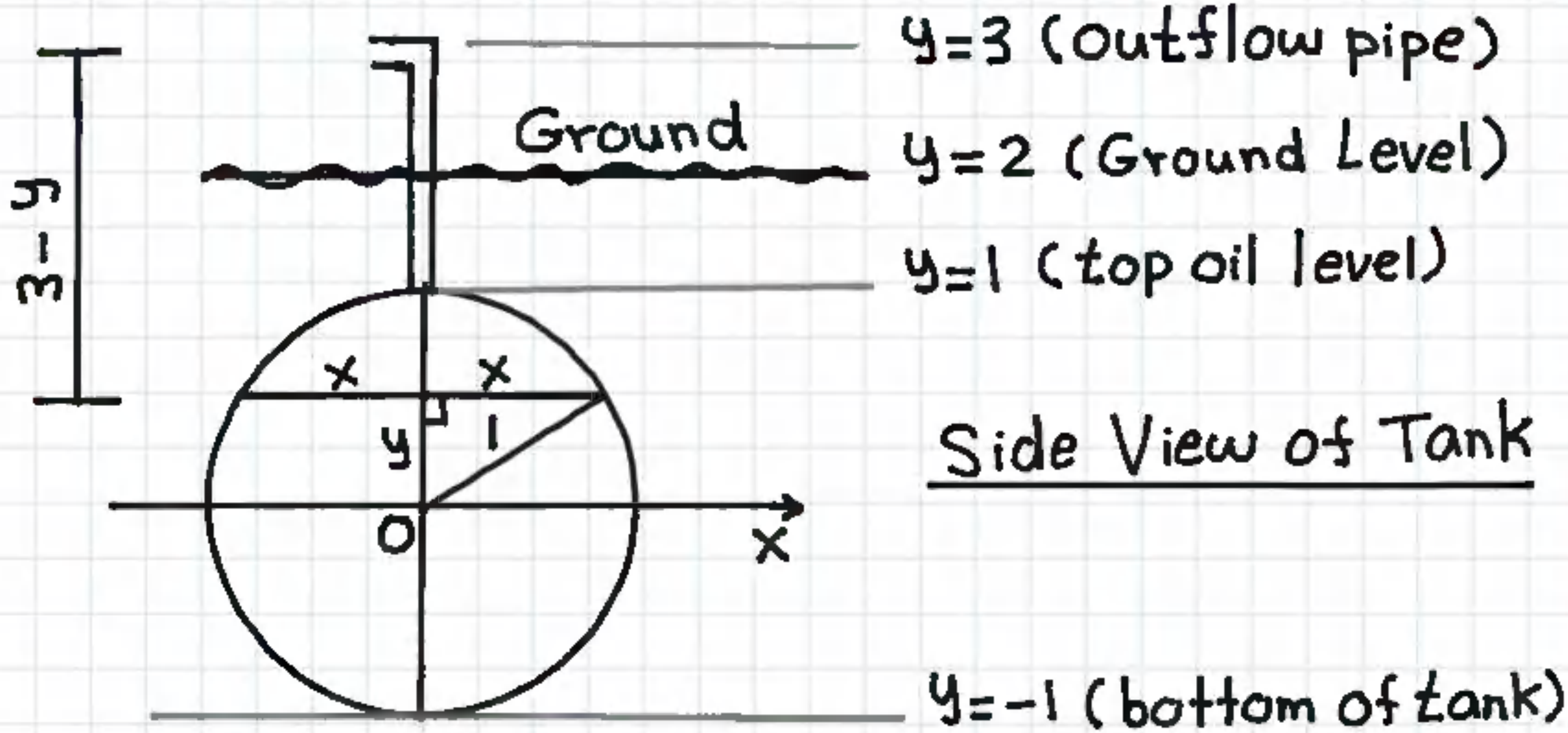
($g = 9.8\ m/s^2$; $\rho_{oil} = 900\ Kg/m^3$)

Solution: To exploit the circular symmetry of the tank, we choose $y=0$ (x axis) at the center of the circular end (along the axis of symmetry of the cylinder)

Step 1| Sketch a diagram and define variables



Key Concept: Cylindrical tank lies between $y=-1$ and $y=1$ and every rectangular cross section at height y has to be raised $\underline{3-y}$ metres to be pumped out of the outflow pipe which is 3 m above center of the tank.



Key concept: Cylindrical tank whose bottom is buried 3 metres below the ground lies between $y=-1$ m and $y=1$ m. Every horizontal cross section of oil is a rectangle with width $2X$ metres and Length 2 metres. $A(y)=(2X)(2)$

Since all horizontal cross sections perpendicular to the y axis are rectangles with width 2x metres and Length 2 metres, the area of the rectangular sectional slice is given by $A(y) = (2x)(2) = 4x$

Let's apply pythagoras to express x in terms of y.

$$x^2 + y^2 = 1^2 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \pm\sqrt{1-y^2} \Rightarrow x = \sqrt{1-y^2}$$

$$A(y) = (2x)(2) = 4x = 4\sqrt{1-y^2}$$

$dV = A(y)\,dy = 4\sqrt{1-y^2}\,dy$ (volume of rectangular slice)

$dm = \rho\,dV = 4\sqrt{1-y^2}\,\underline{\underline{(900)}}\,dy$ (mass of slice)

$$dF = g\,dm = 4\sqrt{1-y^2}\,(900)\,\underline{\underline{(9.8)}}\,dy$$

$$dF = 4\sqrt{1-y^2}\,(900)\underline{(9.8)}\,dy$$

(Force needed to overcome gravity acting on the mass of oil in this rectangular slice at height y)

$$dW = (3-y)\,dF = 4\sqrt{1-y^2}\,(900)(9.8)\underline{\underline{(3-y)}}\,dy$$

(Work required to pump out this rectangular slice of oil $\underline{(3-y)}$ metres thru the outflow pipe.

Key Concept: $\underline{(3-y)}$ represents the distance that each rectangular slice of oil at height y must travel to be pumped out of the outflow pipe.

check $y=-1$ (tank bottom) $\Rightarrow 3-y = 3--1 = 4$ metres ✓

check $y = 1$ (tank top) $\Rightarrow 3-y = 3-1 = 2$ metres ✓

step 2] set up definite integral and find work required.

$$dw = 4\sqrt{1-y^2}\,(900)(9.8)(3-y)\,dy$$

$$W = \int_{y=-1}^{y=1} dw = \int_{-1}^{1} 4\sqrt{1-y^2}\,(900)(9.8)(3-y)\,dy$$

$$W = 4(900)(9.8)\int_{-1}^{1} \sqrt{1-y^2}\,(3-y)\,dy$$

$$W = 4(900)(9.8)(3)\int_{-1}^{1} \sqrt{1-y^2}\,dy$$

$$-4(900)(9.8)\int_{-1}^{1} y\sqrt{1-y^2}\,dy$$

Let's split up the integrals and evaluate separately.

$$W = 105840 \int_{-1}^{1} \sqrt{1-y^2}\, dy - 35280 \int_{-1}^{1} y\sqrt{1-y^2}\, dy$$

We split up the integrals and evaluate separately.

$$\int_{-1}^{1} \sqrt{1-y^2}\, dy = \frac{\pi(1)^2}{2} = \frac{\pi}{2}$$

This integral is simply the area of a semicircle of radius 1 $\Rightarrow A = \frac{\pi r^2}{2} = \frac{\pi(1)^2}{2} = \pi/2$

$$\int_{-1}^{1} y\sqrt{1-y^2}\, dy = -\frac{1}{3}(1-y^2)^{3/2}\Big|_{-1}^{1}$$

$$= -\frac{1}{3}(0)^{3/2} - -\frac{1}{3}(0)^{3/2} = 0$$

U-substitution

$u = 1-y^2$

$du = -2y\, dy$

$y\, dy = du/-2$

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$$W = 105840 \underbrace{\int_{-1}^{1} \sqrt{1-y^2}\,dy}_{=\pi/2} - 35280 \underbrace{\int_{-1}^{1} y\sqrt{1-y^2}\,dy}_{=0}$$

$$W = 105840(\pi/2) - 35280(0)$$

$$W = 52920\pi - 0 = 52920\pi\ J \cong 166253.08\ J$$

Therefore it takes 52920π Joules of work to pump all the oil out of the cylindrical tank.

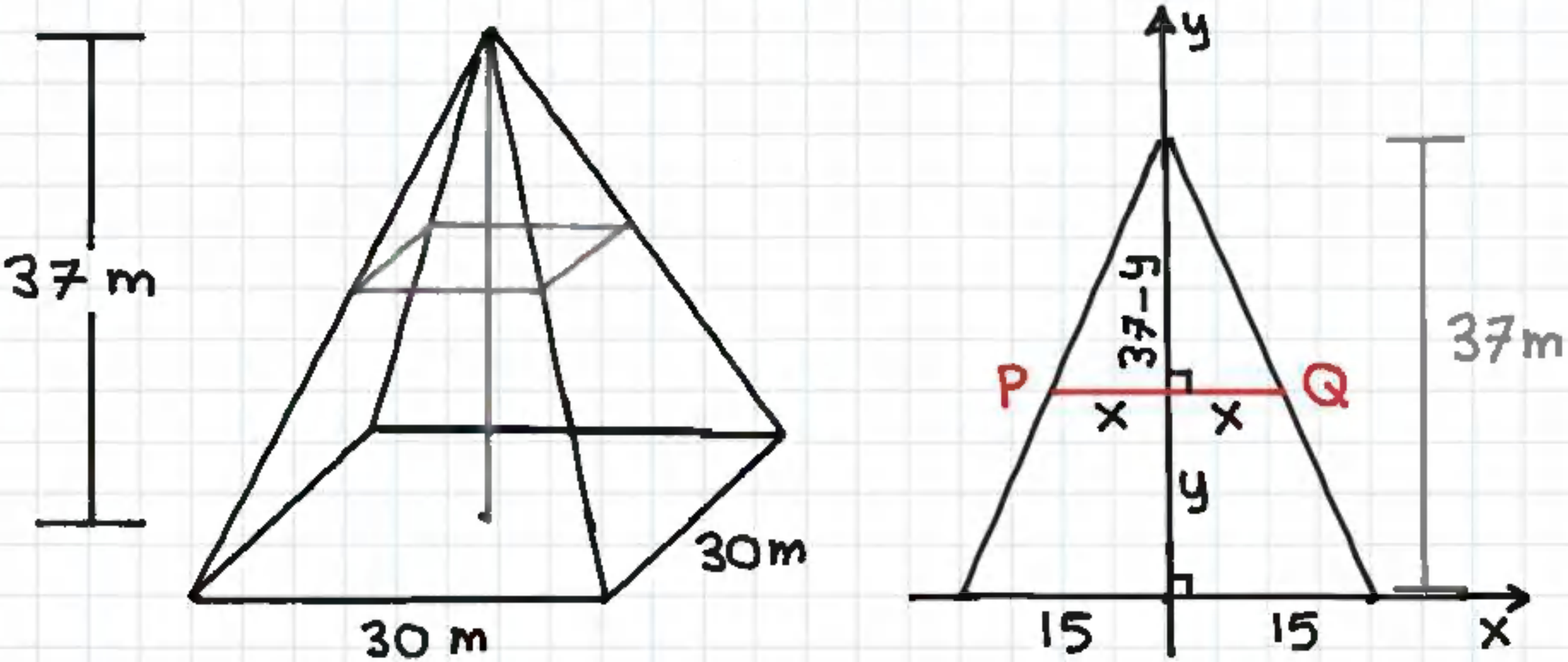
Work Integrals 8

Ex| The pyramid of Cestius built around 12 BC, in Rome, Italy was made of concrete brick slabs. This pyramid has a height of 37 metres and has a square base with edge length of 30 metres. The density of concrete brick is about 2000 kg/m^3

a) Estimate the total work done in lifting concrete slabs from ground level to their final position in the pyramid. (Find the total work done)

b) Assuming the pyramid took 330 days to construct and that each laborer did 200 Joules/hr of work and worked 12 hours per day for 330 days, how many laborers were required to buid the pyramid.

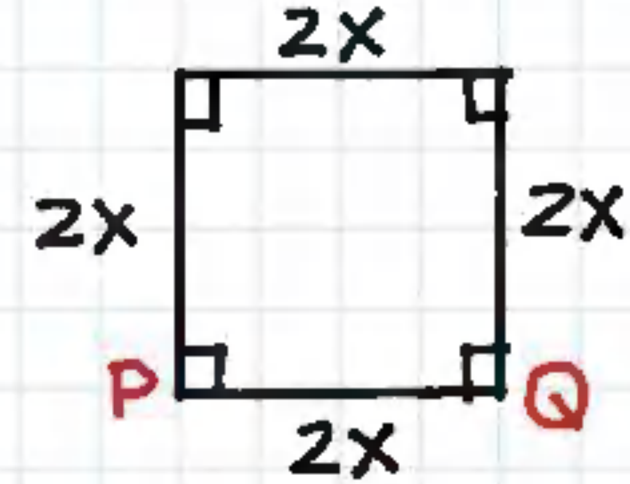
a) step 1| Sketch a diagram and define variables



$\rho_{rock} = 2000\,kg/m^3$; $g = 9.8\,m/s^2$; height $= 37\,m$

square base edge Length $= 30\,m$

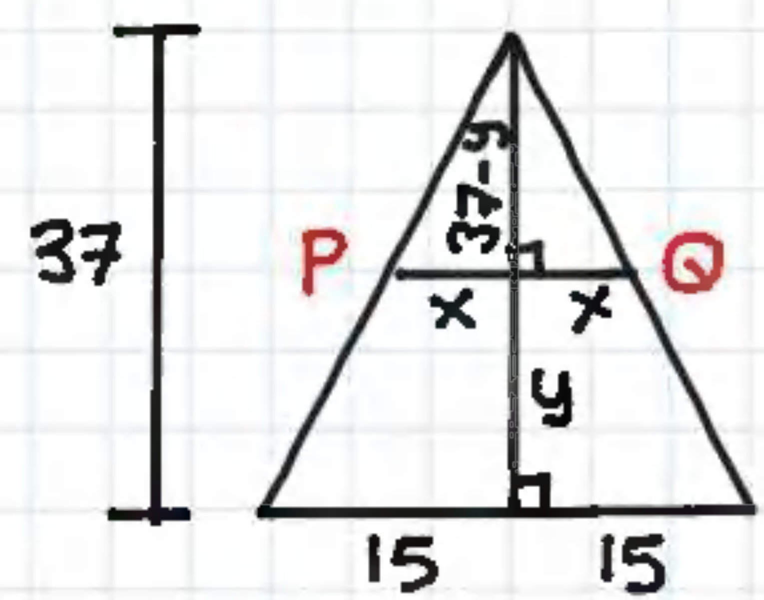
Since the pyramid has a square base, all horizontal cross-sections at height y are squares with edge length $2x$ with area $=(2x)(2x)$



$$A(y) = (2x)(2x) = 4x^2$$

Let's apply similiar triangles to express x in terms of y

$$\frac{37-y}{x} = \frac{37}{15} \quad \text{similiar triangles}$$



$$37x = 15(37-y)$$

$$x = \frac{15}{37}(37-y)$$

$$A(y) = 4x^2 = 4\left(\frac{15}{37}\right)^2 (37-y)^2$$

$A(y) = 4\left(\frac{15}{37}\right)^2 (37-y)^2$ (cross sectional area at height y)

$dV = A(y)\ dy$

$dV = 4\left(\frac{15}{37}\right)^2 (37-y)^2\, dy$ (volume of thin slab of brick)

$dm = \rho\, dV = 4\left(\frac{15}{37}\right)^2 (37-y)^2 \underline{(2000)}\, dy$

(mass of thin slab of brick)

$dF = g\, dm = 4\left(\frac{15}{37}\right)^2 (37-y)^2 (2000)\underline{(9.8)}\, dy$

Force needed to overcome gravity acting on the mass of rock in this thin square slab of concrete.

$dW = y\, dF$ (work required to lift section of rock from ground level <u>y metres</u> to final position in pyramid.)

$$dW = 4\left(\frac{15}{37}\right)^2 (37-y)^2 (2000)(9.8)\underline{(y)}\,dy$$

step 2] Set up definite integral and find work required

$$W = \int_{y=0}^{y=37} dW = \int_0^{37} 4\left(\frac{15}{37}\right)^2 (37-y)^2 (2000)(9.8)(y)\,dy$$

$$W = 4\left(\frac{15}{37}\right)^2 (2000)(9.8) \int_0^{37} (37-y)^2\, y\, dy$$

$$W = 12885.32 \int_{37}^{0} -u^2 (37-u)\,du$$

Apply u-Subst.

$u = 37-y \quad du = -dy$

$y=0 \Rightarrow u = 37-y \Rightarrow u = 37$

$y=37 \Rightarrow u = 37-y \Rightarrow u = 0$

$dy = -du \; ; \; y = 37-u$

Recall: $\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$

$$W = 12885.32 \int_0^{37} u^2(37-u)\,du$$

$$W = 12885.32 \int_0^{37} (37u^2 - u^3)\,du$$

$$W = 12885.32 \left[\frac{37u^3}{3} - \frac{u^4}{4} \right]_0^{37}$$

$$W = 12885.32 \left[\frac{(37)^4}{3} - \frac{(37)^4}{4} - (0-0) \right]$$

$$W = 12885.32 \left[\frac{4(37)^4 - 3(37)^4}{12} \right] = \frac{12885.32\,(37)^4}{12}$$

$W \cong 2012430351$ Joules

$W \cong 2$ Billion Joules

b) How many workers were needed to build the pyramid.

Assumptions of model:

1) Each laborer did 200 Joules/hour of work.

2) Each laborer worked 12 hours per day.

3) It took 330 days to construct the pyramid.

Each laborer does $200 \times 12 \times 330 = 792000$ Joules of work and the total work required by all the laborers to build the pyramid is 2012430351 J

Based on our model assumptions:

$$n = \frac{2012430351 \text{ Joules}}{792\,000 \text{ Joules/laborer}} \cong 2541 \text{ Laborers}$$

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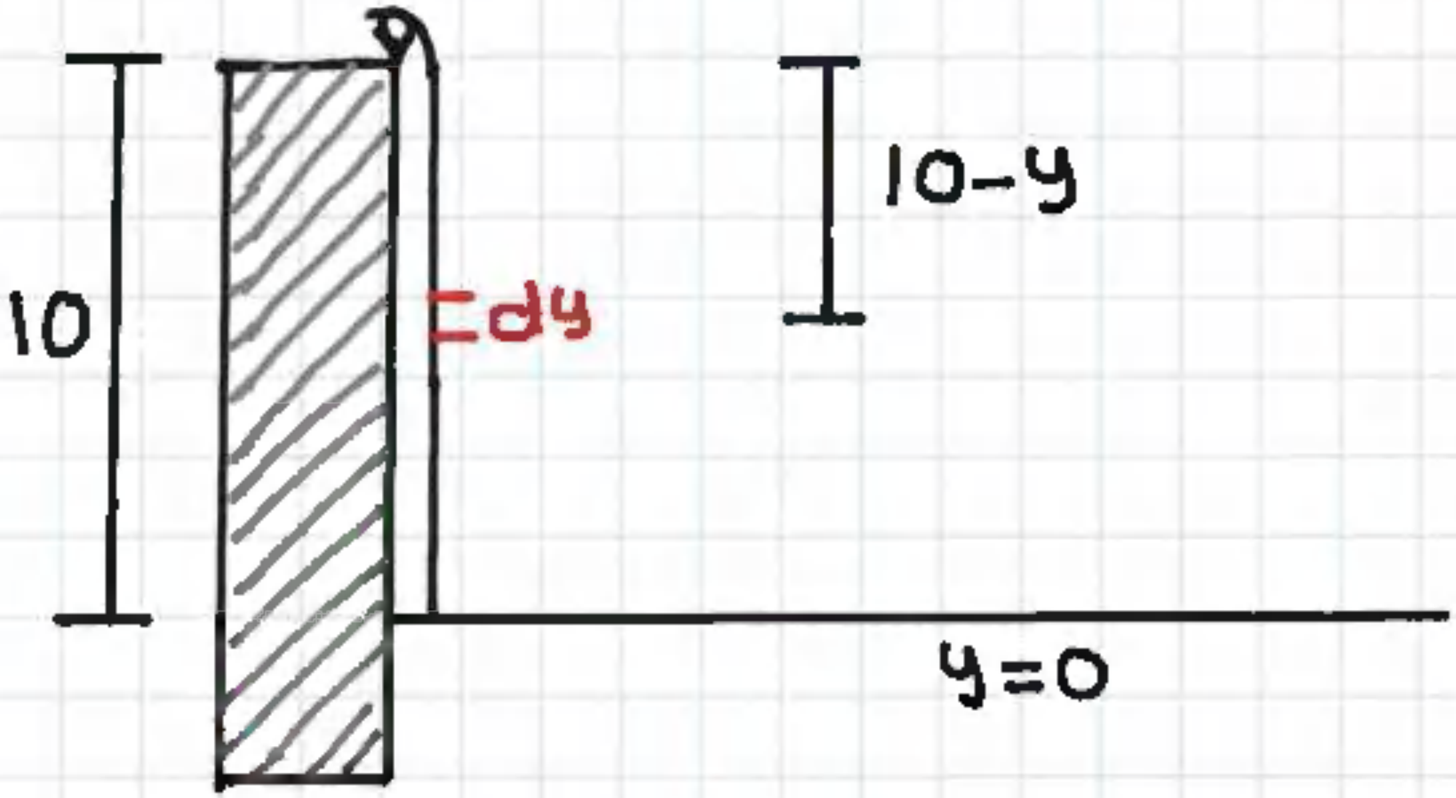
Work integrals 9

Ex] A heavy cable 10 metres long that weighs 2kg/m hangs over the edge of a building 12 metres high.

How much work is required to pull the entire 10 metre cable to the top of the building.

Given: $g = 9.8 m/s^2$

Solution: Find the mass of a small segment of the 10 metre cable and find the force acting on this small piece of cable and find the work needed to lift this cable piece to the top step1| Sketch a diagram and define variables



cable $\frac{\text{mass}}{\text{Length}} = \frac{2\text{kg}}{\text{m}}$

$g = 9.8\ m/s^2$

A piece of cable with length dy with linear density of 2kg/m has a mass of $\underline{2\,\text{kg/m} \cdot dy}$ that is:

$dm = 2\,dy$ (mass of small piece of cable)

$dm = 2\,dy$ (mass of small piece of cable)

$dF = 2\underline{(9.8)}\,dy$ (Force acting on this cable piece)

$dW = (10-y)\,dF$ (work needed to lift this small piece of cable $(10-y)$ m to the top of building)

$$dw = 2(9.8)(10-y)\,dy$$

Key concept: Work = Force × distance

However each piece of 10 metre cable is lifted different vertical distance to reach the top of the building, and hence we need to integrate.

step 2] set up definite integral and find work needed.

$$dW = 2(9.8)(10-y)\,dy$$

$$W = \int_{y=0}^{y=10} dW = \int_0^{10} 2(9.8)(10-y)\,dy$$

$$W = 2(9.8)\int_0^{10}(10-y)\,dy$$

$$W = 2(9.8)\left[\,10y - y^2/2\,\right]_0^{10} = 19.6\left[100 - 50 - (0-0)\right]$$

$$W = 19.6[50] = 980 \text{ Joules}$$

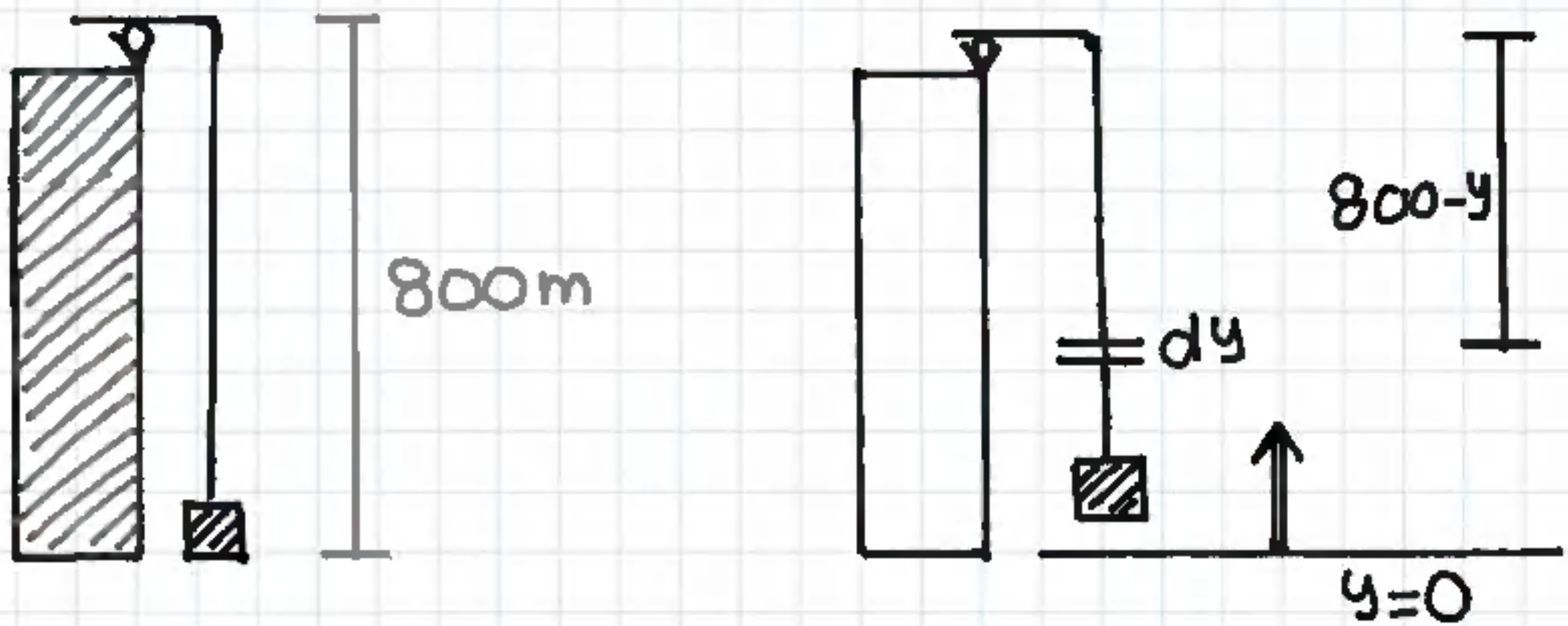
Work Integrals 10

Ex] A cable with linear density of 3 kg/m is attached to a large container filled with diamonds that has mass of 500 kg. The container is initially at the bottom of a 800-m-deep vertical mine shaft. The container is lifted up to the midpoint of the mine shaft by winding up the top half (400 m) of the cable. Find the total work required in Joules. ($g = 9.8\ m/s^2$)

Solution strategy: 3 components (Add them up)

1. Work to lift container of diamonds 400 m
2. Work done to lift top half of cable
3. Work done to lift bottom half of cable

Solution:



Cable linear density 3kg/m

$g = 9.8 \ m/s^2$

mass of container = 500 kg

Work For the container

mass = 500 kg

Force = 500×9.8 N

Work = Force × Distance = $500 \times 9.8 \times 400$ N-m

Work = 1960000 N-m = 1960000 J

Work For the top half of cable

A piece of cable of length dy at height y metres from the bottom of mine shaft has mass $\underline{3\,dy}$

$dm = 3\,dy$ (mass of small piece of cable)

$dF = 3(9.8)\,dy$ (Force acting on piece of cable)

$$dF = 3(9.8)\,dy$$

$$dW = (800-y)\,dF$$

$$dW = 3(9.8)(800-y)\,dy$$

Work required to lift a small piece of top half of cable $\underline{800-y}$ metres.

$$W = \int_{y=400}^{y=800} dW = \int_{400}^{800} 3(9.8)(800-y)\,dy$$

Total work required to lift top half of cable

$$W = 3(9.8)\int_{400}^{800} (800-y)\,dy = 29.4\left[800y - \frac{y^2}{2}\right]_{400}^{800}$$

$$W = 29.4\left[800y - \frac{y^2}{2}\right]_{400}^{800}$$

$$W = 29.4\left[800^2 - \frac{800^2}{2} - \left(800(400) - \frac{400^2}{2}\right)\right]$$

$$W = 29.4\left[320000 - 240000\right]$$

$W = 2352\,000$ Joules

Work required to lift top half of cable

Work For the bottom half of cable

Key concept: The top half of cable and bottom half of cable have to be treated separately. The reason for this is the top half of cable has to be lifted piece by piece to reach the top but the lower half of cable is lifted as a whole unit (400 metres of cable) to reach the halfway point.

$$\text{Work} = \underbrace{400\,\text{m} \times 3\,\frac{\text{kg}}{\text{m}}}_{\text{mass}} \times \underbrace{9.8\,\frac{\text{m}}{\text{s}^2}}_{g} \times \underbrace{400\,\text{m}}_{\text{distance}} = 4704000\,\text{J}$$

ofcourse we can set up a definite integral as well.

$$W = \int_0^{400} \underbrace{400 \times 3}_{\text{mass}} \times \underbrace{9.8}_{g}\, dy$$

$$W = 11760 \int_0^{400} dy = 11760\, y \Big|_0^{400}$$

$$W = 11760(400) - 0 = 4704000 \text{ Joules}$$

This is the same answer as using formula:

Work = Force × distance = mass × g × distance

$$\text{Work} = \underbrace{400\,m \times 3\,\frac{kg}{m}}_{\text{mass}} \times \underbrace{9.8\,\frac{m}{s^2}}_{g} \times \underbrace{400\,m}_{\text{distance}} = 4704000 \text{ J}$$

Now add it all up!

Total Work = 1960000 J + 2352000 J + 4704000 J

Total Work = Container + Top half + Bottom half

Total Work = 9016000 Joules

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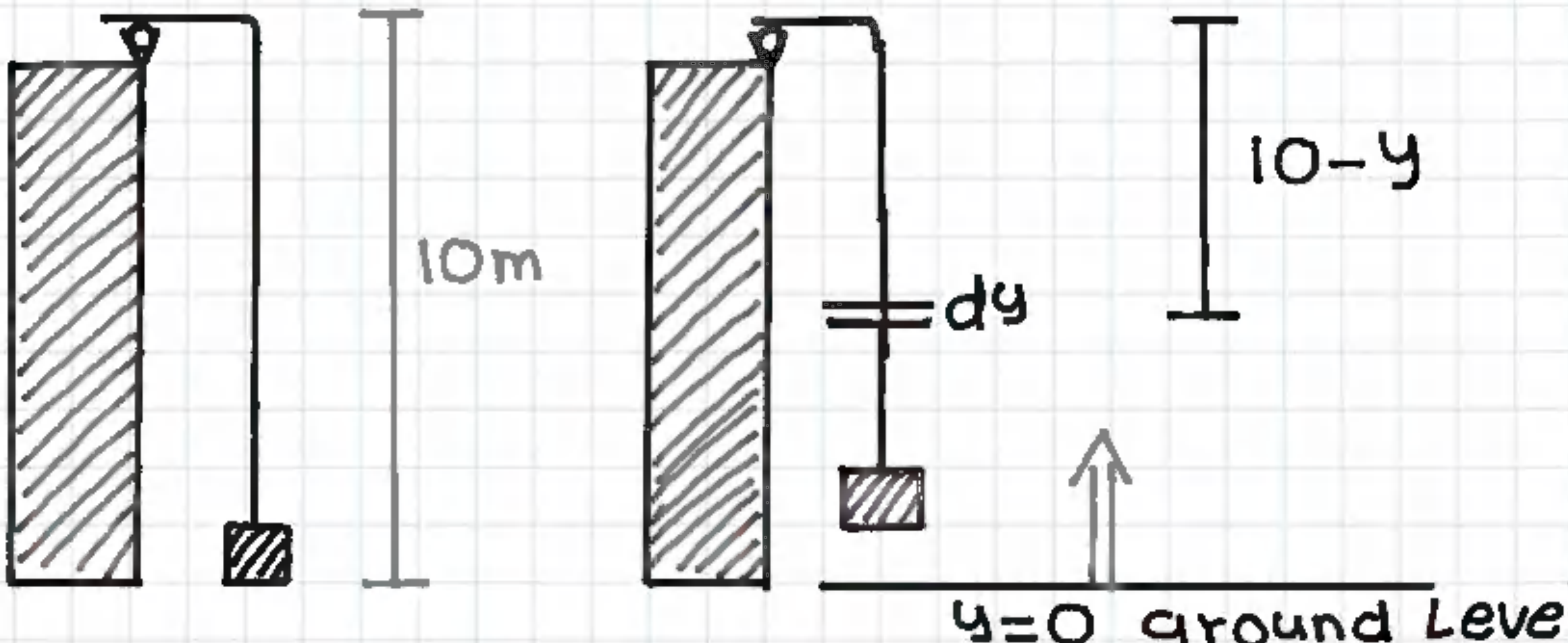
Work Integrals II

Ex1 A leaky bucket with mass 12 kg is raised from the ground to a height of 10 m, at a constant speed with a rope that has linear density of 3 kg/m. Initially the bucket contains 40 kg of water but the water leaks out of the bucket at a constant rate and finishes draining just as the bucket reaches a height of 10 m. Find the work done raising the leaky bucket to a height of 10 m.

Solution strategy: 3 components (Add them up)

1. Work done to lift empty bucket dy metres
2. Work done to lift piece of rope dy metres
3. Work done to lift leaking water dy metres

Step 1| Sketch a diagram and define variables



rope linear density 3 kg/m ; $g = 9.8\ m/s^2$

mass of empty bucket = 12 kg

mass of water at ground level = 40 kg

mass of water at top level (10m) = 0 kg

step 2] Set up definite integral and find total work

Work to lift bucket 10 metres to top

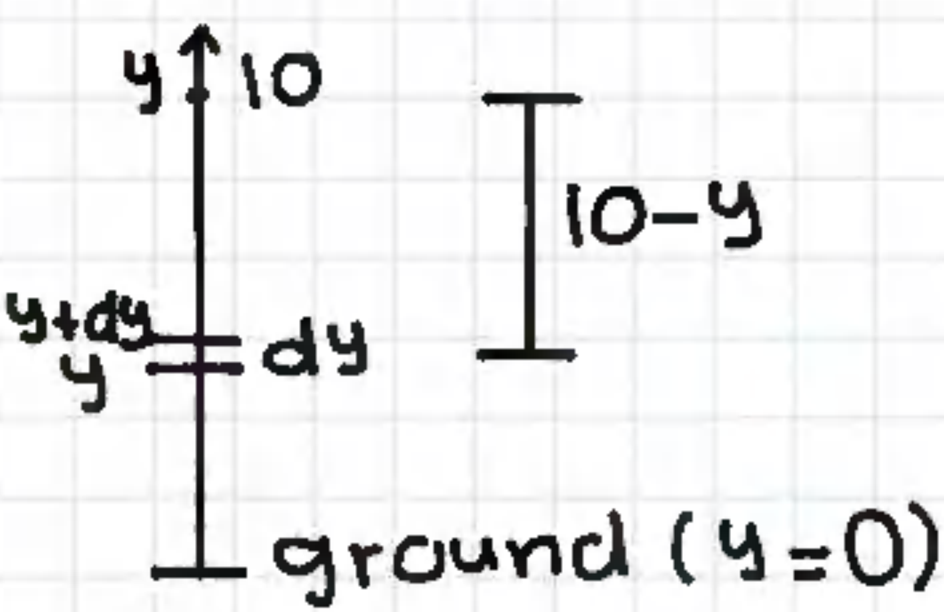
mass = 12 kg

Work = Force x Distance

Force = $12\,kg \times 9.8\,m/s^2 = 117.6\,N$

Work = $12\,kg \times 9.8\,m/s^2 \times 10\,m = 1176$ Joules

Work done to lift entire rope 10 metres to top



A piece of rope of length dy at height y metres from ground level has mass 3dy

$dm = 3\,dy$

mass of rope in $(y, y+dy)$

$dm = 3\,dy$ (mass of small piece of rope)

$dF = 3(9.8)\,dy$ (Force acting on piece of rope)

$dW = (10-y)\,dF$

$dW = 3(9.8)\underline{(10-y)}\,dy$

Work required to lift small piece of rope $\underline{10-y}$ m

$$W = \int_{y=0}^{y=10} dW = \int_0^{10} 3(9.8)(10-y)\,dy$$

Total work required to lift entire rope 10 m

$$W = 3(9.8)\left[10y - y^2/2\right]_0^{10} = 29.4[100-50-0]$$

$W = 1470$ Joules

Work done to lift 40 kg of leaking water to top

Key concept: Since 40 kg of water is leaking at a constant rate from ground level $y=0$ to top level $y=10$ m we must express Mass of water as a linear function of height y that is $M(y)$

when $y=0$ $\quad M=40$ kg

when $y=10$ $\quad M=0$ kg

$$M-M_1=\frac{\Delta M}{\Delta y}(y-y_1)$$ (Mass as linear function of y)

$$M-40=\frac{40-0}{0-10}(y-0) \Rightarrow M(y)=40-4y$$

We now have mass of water as a function of y

$$dW = \underbrace{(40-4y)}_{\text{mass}}\underbrace{(9.8)}_{g}\underbrace{dy}_{\text{distance}}$$

$$W = \int_{y=0}^{y=10} dW = \int_0^{10} (40-4y)(9.8)\,dy$$

$$W = 9.8\int_0^{10} (40-4y)\,dy = 9.8\left[40y - \frac{4y^2}{2}\right]_0^{10}$$

$$W = 9.8[400-200-0] = (9.8)(200) = 1960 \text{ Joules}$$

Work done in lifting 40 kg of leaking water 10 metres to the top level.

Now add up the 3 components of work

$$dW_{total} = dW_{bucket} + dW_{rope} + dW_{water}$$

$$dW_{total} = 12 \times 9.8\,dy + 3(9.8)(10-y)\,dy + 9.8 \times (40-4y)\,dy$$

$$W = \int_0^{10} 12 \times 9.8\,dy + \int_0^{10} 3(9.8)(10-y)\,dy + \int_0^{10} 9.8(40-4y)\,dy$$

$$W = 1176\,J + 1470\,J + 1960\,J = 4606 \text{ Joules}$$

Total Work done in lifting bucket of leaky water + rope 10 metres to top level.

Work Integrals 12

Ex] According to Newton's Law of gravitation, two bodies with masses M_1 and M_2 attract each other with a Force of:

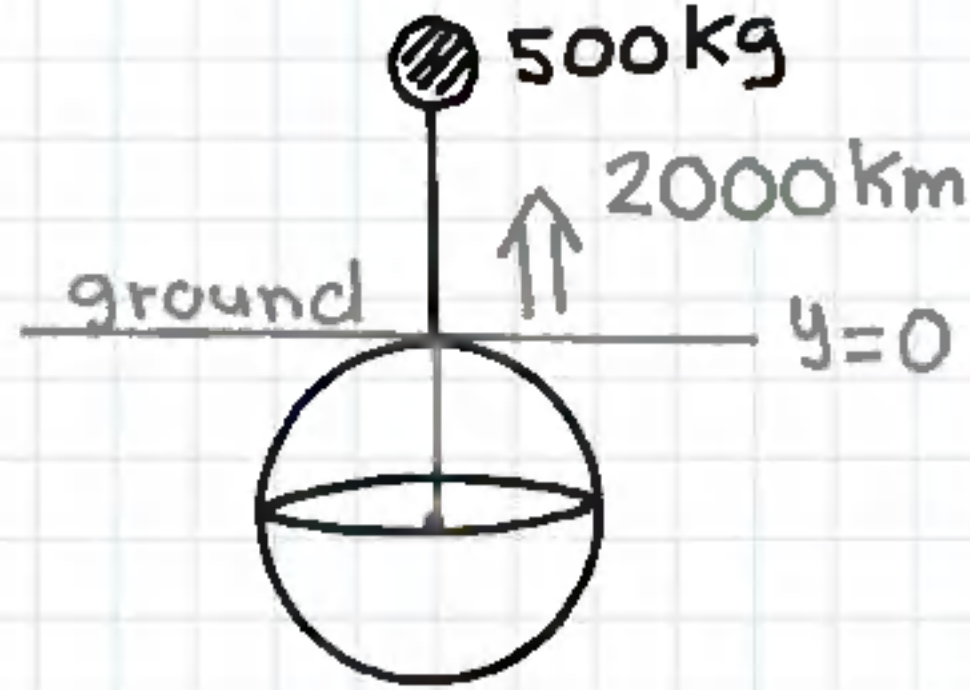
$F = G M_1 M_2 / r^2$ where r = distance between two bodies and G is gravitational constant

a) Use Newton's law of gravitation to find the work required to launch a 500 kg satellite vertically to an orbit 2000 km high.

$M_{earth} = 5.98 \times 10^{24}$ kg ; $r_{earth} = 6.37 \times 10^6$ m

$G = 6.67 \times 10^{-11}$ N·m²/kg²

b) Find the work done in raising the orbit of this 500 Kg satellite from 2000 km to 3000 km above the surface of the earth.



Solution:

Lets simplify Newton's formula

$F = GM_1M_2/r^2$ to $F = \dfrac{KM_2}{(R+y)^2}$

where $K = GM_1 = GM_{earth}$

R = radius of earth

y = distance of satellite from surface of earth

M_2 = mass of satellite

a) Find the work done in launching 500 kg satellite to orbit of 2000 km high.

solution: $F = \frac{kM_2}{(R+y)^2}$

$$dW = \frac{kM_2}{(R+y)^2}\, dy = \frac{K(500)}{(R+y)^2}\, dy$$

$$W = \int_{y=0}^{y=2000\,km} dW = \int_{0}^{2000\,000\,m} \frac{500k}{(R+y)^2}\, dy$$

u-Subst
$u = R+y$
$du = dy$

$$W = -\frac{500K}{R+y}\Bigg|_{0}^{2000\,000} = -500K\left[\frac{1}{R+2000\,000} - \frac{1}{R}\right]$$

$$W = -500k\left[\frac{1}{R+2000000} - \frac{1}{R}\right]$$

Substituting $R = 6.37 \times 10^6\ m$

$$K = GM_1 = GM_{earth} = 6.67 \times 10^{-11} \times 5.98 \times 10^{24}$$

$$K = 39.89 \times 10^{13}$$

$$W = -500(39.89 \times 10^{13})\left[\frac{1}{6.37\times 10^6 + 2000000} - \frac{1}{6.37 \times 10^6}\right]$$

$$W = -19945 \times 10^{13}\left[0.000000119 - 0.000000157\right]$$

$W = 7{,}579{,}100{,}000 \cong$ 7.6 Billion Joules b] Find the work done in lifting the 500 kg satellite from an orbit of 2000 km to 3000 km.

Solution : $F = \dfrac{kM_2}{(R+y)^2}$

Apply U-Subst
$U = R + y \quad du = dy$

$$W = \int_{2000\,000}^{3000\,000} \frac{500k}{(R+y)^2}\,dy$$

$$W = \frac{-500k}{R+y}\Bigg|_{2000\,000}^{3000\,000}$$

where $K = 39.89 \times 10^{13}$ and $R = 6.37 \times 10^{6}$ m

$$W = -\frac{500k}{R+y}\Bigg|_{2000\,000}^{3000\,000}$$

$$W = -500k\left[\frac{1}{6.37\times10^{6} + 3\times10^{6}} - \frac{1}{6.37\times10^{6} + 2\times10^{6}}\right]$$

$$W = -500(39.89\times10^{13})\left[0.000000107 - 0.000000119\right]$$

$$W = 2{,}393{,}400{,}000 \text{ J} \cong 2.4 \text{ Billion Joules}$$

Work required in lifting the 500 kg satellite from an orbit of 2000 km to 3000 km.

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Work Integrals 13

Ex] The gravitational force of planet Mars on a satellite of mass m at a distance of y metres from the surface of Mars is given by:

$$F(y) = \frac{Km}{(R+y)^2} \quad \text{where } K = 4.28 \times 10^{13}\ m^3/s^2$$

and $R = \text{Radius of Mars} = 3.39 \times 10^6\ m$

a) Find the work required to launch this satellite of mass m into outer space.

b) Find the escape velocity Ve such that the satellite has sufficient kinetic energy at firing to supply the work needed to raise

b) cont.| to raise the satellite of mass m to infinite height.

a) Find the work required to launch the satellite.

Solution: $F(y) = \dfrac{km}{(R+y)^2}$

$$dW = \frac{km}{(R+y)^2}\,dy \Rightarrow W = \int_{y=0}^{y=\infty} dW$$

$$W = \lim_{d\to\infty} \int_0^d \frac{km}{(R+y)^2}\,dy$$

Improper Integral

Apply U-Subst.
$U = R+y \quad du = dy$

$$W = \lim_{d \to \infty} \int_0^d \frac{km}{(R+y)^2}\, dy = \lim_{d \to \infty} \left. \frac{-km}{(R+y)} \right|_0^d$$

$$W = \lim_{d \to \infty} \left(\underbrace{\frac{-km}{R+d}}_{\Downarrow \atop 0 \text{ as } d \to \infty} + \frac{km}{R} \right) = \frac{km}{R} = \frac{k\ \text{mass}}{R}$$

where $K_{mars} = 4.28 \times 10^{13}\ m^3/s^2$

$R_{mars} = 3.39 \times 10^6\ m$

$$W = \frac{km}{R} = \frac{4.28 \times 10^{13} \times \text{mass}}{3.39 \times 10^6} \cong 12.63 \times 10^6 \text{ mass Joules}$$

$W \cong 12.63$ million Joules multiplied by Mass of satellite

b) Find the escape velocity.

Solution: For the satellite to overcome the gravitational pull of Mars and reach infinite height the satellite must have sufficient kinetic energy at firing to supply the work needed to raise the satellite of mass m to infinite height. ($KE = \frac{1}{2} mv^2$)

Strategy: Kinetic energy = Work done

$$\frac{1}{2} \cancel{m} v^2 = 12.63 \times 10^6 \cancel{m} \qquad \text{where } m = \text{mass}$$

$$V^2 = 2 \times 12.63 \times 10^6$$

b cont.

$V^2 = 2 \times 12.63 \times 10^6$

$V = 5025.93$ m/s

take positive square root

Let's change units to km/hr

$$V = 5025.93 \frac{\cancel{m}}{\cancel{s}} \cdot \frac{1\,km}{1000\,\cancel{m}} \times \frac{3600\,\cancel{s}}{1\,hr} = 18093.36 \frac{km}{hr}$$

Escape velocity must be at least $18093.36 \frac{km}{hr}$

$V_{escape} \geq 18093.36$ km/hr