Calculus I MT2 Review Questions Worksheet

Calculus I Review Topics for MT2

- Implicit Differentiation
- Logarithmic Differentiation
- Inverse Trig. Derivatives
- Related Rates Word Problems
- Optimization Word Problems
- Curve Sketching
- L'Hôpital's Rule

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1) Implicit Differentiation

The curve is defined by $x^2y + y^3 = 4$.

(a) Find dy/dx in terms of x,y and simplify.

(b) Find d^2y/dx^2 as a function of x,y and y'.

(c) The point $(\sqrt{3}, 1)$ lies on the curve. Compute the slope there and state whether the curve is concave up, concave down, or neither at that point.

2) Logarithmic Differentiation

Differentiate using logarithmic differentiation:

$$y = \frac{(x^2 + 1)^x(\sqrt{1 - x^2})^3}{(x - 1)^{5/2}e^{x^2}}$$

Give your final answer factored and expressed as y times a simplified bracket.

3) Inverse Trig Derivatives

(a) Find the derivative:

$$f(x) = \tan^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right), \;\; |x| < 1$$

(b) Find the derivative:

$$g(x) = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$$

4) Tangent Line Question

Find the point(s) on $y = \arcsin x$ where the tangent line has slope 3/2.

Then give the corresponding tangent line(s) in point–slope form. State the domain restriction(s) used.

5) Related Rates I

A 13 ft ladder leans against a vertical wall. The bottom slides away at 0.6 ft/s. When the bottom is 5 ft from the wall:

(a) How fast is the top sliding down? (Include sign.)

(b) At that instant, how fast is the angle θ between the ladder and the ground changing (rad/s)? (Include sign.)

(c) How fast is the area of the triangle formed by the wall, ground, and ladder changing at that instant?

6) Optimization I

A rectangular storage container with an open top is to have a volume of 12 m^3. The length of its base is three times the width. Material for the base costs \$12 per square meter. Material for the sides costs \$5 per square meter.

(a) Set up the total cost as a function of a single variable and find the dimensions for the cheapest such container.

(b) Verify that your critical point gives a global minimum and report the minimal cost.

7) Optimization II

A poster has a printed text area of 400 cm^2. It must have 2 cm side margins and 3 cm margins at the top and bottom. Let the total poster dimensions be width W and height H (in cm).

(a) Express the total area in terms of W only.

(b) Verify that your critical point gives a minimum and find the minimal total area.

8) Curve Sketching

Let

$$f(x) = x^{5/3} - 5x^{2/3}$$

(a) Find the domain, x and y intercepts (if any), horizontal/vertical asymptotes (if any).

(b) Find all critical points (including where f' does not exist but f is defined). Classify with the first derivative test.

(c) Determine intervals of concavity and any inflection points.

(d) Please provide a sketch indicating local max or local min (if any), points of inflection (if any), horizontal or vertical asymptotes (if any), x intercepts or y intercepts (if any).

9) First Derivative Test

Suppose

$$f'(x) = (x + 3)(x - 1)^2(x - 2)e^{-x^2}$$

(a) Identify all critical points and classify each (local max/min/neither).

10) Related Rates 2

A particle moves along the curve

$$y^2 - x^2 = 5$$

Its x-coordinate changes at a constant rate of 1.5 units/s. How fast is the distance from the origin changing when the particle is at (2, 3)?

11) Related Rates 3

Water is leaking out of an inverted conical tank at a rate of 8,000 cm^3/min at the same time as water is being pumped into the tank at a constant rate. The tank has height 500 cm and the diameter at the top is 300 cm. If the water level is rising at a rate of 18 cm/min when the height of the water is 150 cm, find the rate at which water is being pumped into the tank.

12) Related Rates 4

A lighthouse stands on a small island 2.5 km from the nearest point P on a straight shoreline. Its light makes 3 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1.2 km from P?

13) Related Rates 5

Suppose

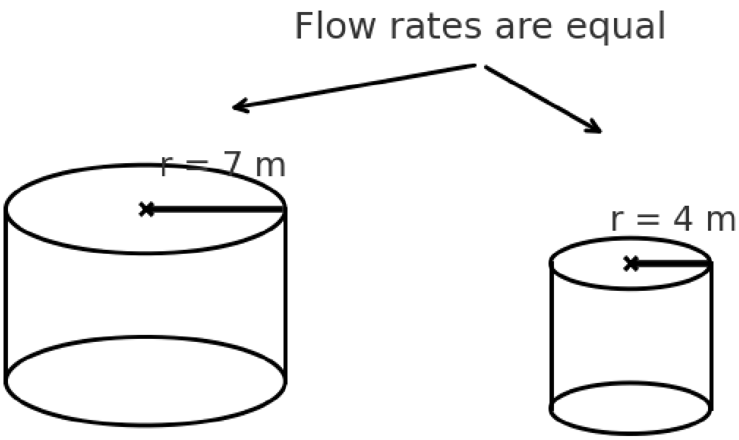
$$4x^2 + 9y^2 = 100$$

(a) If dy/dt = 1/4, find dx/dt when x = 4 and y = 2.

(b) If dx/dt = 5/4, find dy/dt when x = −1 and y = $(4\sqrt{6})/3$.

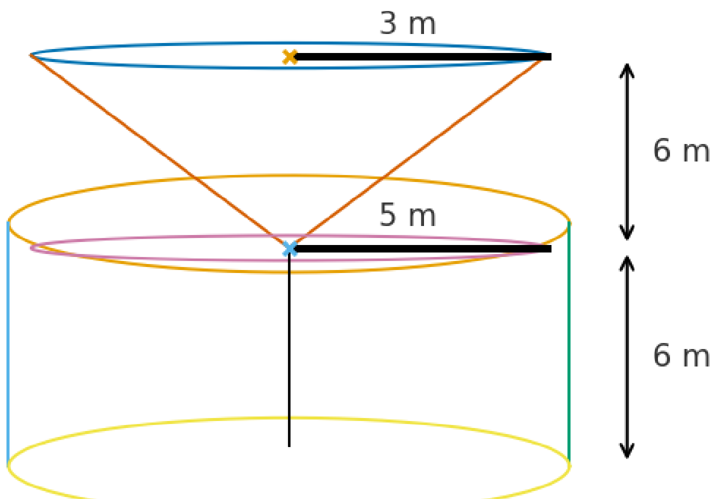
14) Related Rates 6

Two cylindrical swimming pools are being filled simultaneously at the same volumetric rate (m^3/min). The smaller pool has a radius of 4 m, and the water level rises at a rate of 0.6 m/min. The larger pool has a radius of 7 m. How fast is the water level rising in the larger pool?



15) Related Rates 7

A conical tank with an upper radius of 3 m and a height of 6 m drains into a cylindrical tank with a radius of 5 m and a height of 6 m. If the water level in the conical tank drops at a rate of 0.4 m/min, at what rate does the water level in the cylindrical tank rise when the water level in the conical tank is (i) 2 m, (ii) 4 m?



16) Optimization III

At which point(s) on the curve

$$y = 1 + 40x^3 - 3x^5$$

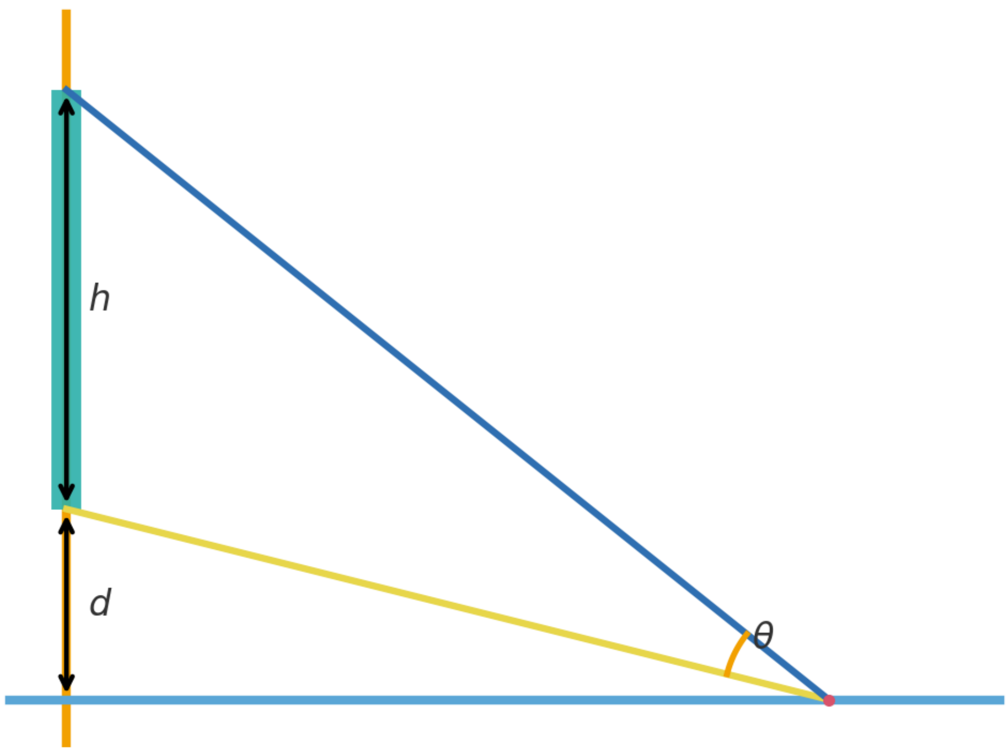
does the tangent line have the largest slope?

17) Optimization IV

What is the smallest possible area of the triangle that lies in the first quadrant and whose hypotenuse is tangent to the parabola $y = 4 - x^2$ at some point?

18) Optimization V

A painting in an art gallery has height h and its lower edge is a distance d above the observer's eye. How far from the wall should the observer stand to maximize the viewing angle θ?



19) Optimization VI

Find the point(s) on the ellipse $4x^2 + y^2 = 4$ that are farthest from the point $(1, 0)$.

20) Optimization VII

A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

21) Related Rates 8

(a) A woman walks in a straight line away from a light post that is three times as tall as she is. Her distance from the post is $s(t) = 4t - (1/2)t^2$, where s is in meters and t is in seconds. Find the rate of change of the length of her shadow after 3 seconds. Define all variables you use.

(b) What is the rate of change of the length of her shadow after she has walked a total of 10 meters?

22) Related Rates 9

Particle A travels with a constant speed of 2 units per minute along the x-axis starting at (4, 0) and moving away from the origin. Particle B travels with a constant speed of 1 unit per minute along the y-axis starting at (0, 8) and moving toward the origin. Find the rate of change of the distance between the two particles when the distance is exactly 10 units.

23) Related Rates 10

Suppose a particle travels according to

$$x\cos y + y\ln x = 0$$

If its x-coordinate is changing at a constant rate of 2/3 units per second, what is dy/dt when the particle is at $(1, \pi/2)$?

24) Optimization 8

A one metre long piece of wire is cut into two pieces. One piece is bent into a square, the other into an equilateral triangle. Where, if anywhere, should the wire be cut so that the total area enclosed in the square and triangle is minimal? Where, if anywhere, should the wire be cut so that the total area is maximal?

25) Implicit Differentiation 2

Find y" given the relation:

$$x^y = y^x$$

Hint: Log both sides first.

26) Implicit Differentiation 3

(a) Find the slope of the curve at the given point:

$$\sin^2 y + 2\cos^4 x = 1 \text{ at } \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

(b) Find an equation for the tangent line to the curve at the indicated point:

$$x^{2/3} + y^{2/3} = 4 \text{ at } \left(-3\sqrt{3}, 1\right)$$

(c) Find y" if:

$$\sqrt{x} + \sqrt{y} = 1$$

27) Implicit Differentiation 4

Find the equations of all tangent lines to

$$(x^2 + y^2)^2 = 2(x^2 - y^2)$$

that are parallel to the line y + 1 = 0. Give answers in y = mx + b.

28) Logarithmic Differentiation 2

Find the derivative of

$$(\sin x)^{\ln x}, \ x \in (0, \pi)$$

29) Inverse Trig. Derivatives 2

Evaluate:

$$\arctan(\tan(5\pi/4))$$

30) Inverse Trig. Derivatives 3

Evaluate:

$$\arcsin(\sin(7\pi/6))$$

31) Inverse Trig. Derivatives 4

Differentiate each:

$(a)\ e^{-2\arcsin(x)}$ $\qquad (b)\ \arccos(3x+2)$

$(c)\ \arctan(\arcsin x)$ $\qquad (d)\ \arctan\left(\frac{x-1}{x+1}\right)$

32) Inverse Trig. Derivatives 5

Show that

$$\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

33) L'Hôpital's Rule 1

Compute:

$$\lim_{x \to \infty} \left(\frac{2x-3}{2x+5} \right)^{2x+1}$$

34) L'Hôpital's Rule 2

Prove that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0 \text{ for any } p > 0$$

35) L'Hôpital's Rule 3

Compute:

$$\lim_{x \to 1^+} [\ln(x^7 - 1) - \ln(x^5 - 1)]$$

36) L'Hôpital's Rule 4

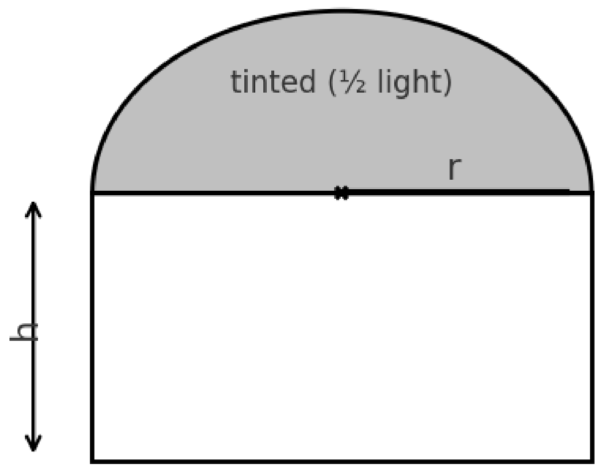
Compute:

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$$

37) Optimization 9

A window is in the form of a rectangle surmounted by a semicircle (a Norman window). The rectangular portion is made of clear glass; the semicircular portion is made of tinted glass that transmits **half** as much light per unit area as the clear glass.

If the total perimeter of the window is 10 m (including the semicircular arc), find the dimensions (radius r and rectangle height h) that admit the **most light**.



38) Related Rates 11

A person 6 ft tall walks at 5 ft/s along one edge of a straight road that is 30 ft wide. On the opposite edge of the road there is a light atop a pole 18 ft high. How fast is the **length of the person's shadow on the ground** increasing when the person is 40 ft from the point directly across the road from the pole?

39) Curve Sketching 2

A function f(x) defined on the whole real line satisfies the following conditions:

$$f(0) = 0 \qquad f(2) = 2 \qquad \lim_{x \to +\infty} f(x) = 0 \qquad f'(x) = K(2x - x^2)\,e^{-x}$$

for some positive constant K. (Read carefully: you are given the derivative of f(x), not f(x) itself.)

(a) Determine the intervals on which f is increasing and decreasing and the location of any local maximum and minimum values of f.

(b) Determine the intervals on which f is concave up or down and the x-coordinates of any inflection points of f.

(c) Determine lim as $x \to -\infty$ of f(x).

(d) Sketch the graph of y = f(x), showing any asymptotes and the information determined in parts (a) and (b).

40) Critical Value (Theoretical)

For what values of the numbers a and b does the function

$$f(x) = a\,x\,e^{bx^2}$$

have the maximum value f(2) = 1?

41) Find the absolute maximum and absolute minimum on the interval [0, 3].

$$f(x) = \frac{x}{x^2 - x + 1}$$

42) Find the absolute maximum and absolute minimum on the interval [-2, 3].

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

43) Find the absolute maximum and absolute minimum on the interval [-1, 1].

$$f(x) = \ln(x^2 + x + 1)$$

44) Absolute Extrema (parameters a, b > 0).
If a and b are positive numbers, find the maximum value of f(x) on the interval $0 \le x \le 1$.

$$f(x) = x^a(1-x)^b, \quad 0 \le x \le 1,\ a, b > 0$$

45) Related Rates 12 — Two cars approaching an intersection.

Two roads, one running east-west and the other north-south, intersect at a point A. Suppose that the car B is travelling west on the first road towards the intersection A at 75 km/h, while the car C is travelling north on the second road towards the intersection A at 100 km/h. Initially, the car B is 300 km away from A and the cars are 500 km from each other. How fast is the distance between the cars changing after 2 hours? Remember that you do not need to simplify your answer in this question.