A New Perspective on the Cylindrical Antenna Theory

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Behailu Kibret
Monash University (Australia)

Assefa Teshome
Victoria University Melbourne

Daniel T. H. Lai
Victoria University Melbourne

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A New Perspective on the Cylindrical Antenna Theory

Behailu Kibret, *Member, IEEE,* Assefa K. Teshome, *Member, IEEE,* and Daniel T. H. Lai, *Member, IEEE*

***Abstract*—This paper presents a new perspective on the analysis of the cylindrical antenna theory. It applies the cylindrical surface waves on an infinitely-long cylindrical conductor, which are similar to the Sommerfeld axial cylindrical surface waves, to describe the conventional postulate of applying a sinusoidal current distribution on a cylindrical dipole antenna. This treatment leads to the derivation of simple expression for the current on a cylindrical dipole antenna of finite conductivity, which is in good agreement with the current obtained from the three-term approximations. Also, it proposes an expression for the current on an infinitely-long cylindrical dipole antenna of finite conductivity, which is also in good agreement to the current found from applying the Fourier transform technique. Moreover, the paper shows that the complex propagation constant used in the three-term approximation is similar to the complex propagation constant of the principal Sommerfeld wave on the surface of an infinitely-long cylindrical conductor. Therefore, a more accurate representation of the current near the feeding point is proposed based on the complex propagation constants of multiple Sommerfeld waves.**

***Index Terms*—cylindrical dipole antenna, three-term approximation, cylindrical surface wave.**

I. Introduction

THE expression of the current on cylindrical antennas has a long history with notable early contributors, such as, Pocklington [1], Hallén [2] and R. W. P. King [3]. The knowledge of the current on the antenna simplifies the calculation of antenna parameters, such as, the input impedance and far-field components. The calculation of the current is usually carried out solving the Pocklington equation describing the surface vector magnetic potential, which is related to known source fields via boundary conditions. For a dipole cylindrical antennas of finite conductivity, King approximated the solution of this equation to express the current as the sum of three sinusoidal functions [4]. On the other hand, for infinitely-long and perfectly conducting dipole antenna, Hallén [5] transformed a similar equation to the Fourier domain and solved for the current as the sum of residues and a convergent integral equation, which has a form of outgoing traveling wave. As an alternative, in this paper, the general form of the current on an infinitely-long cylindrical conductor, which is excited by a rotationally symmetric external electric field, is determined as a standing wave and travelling wave. Since the electric field on the surface of a delta-gap and center-fed dipole antenna is rotationally symmetric, the general form of the current derived for the cylindrical conductor is used in the Pocklington equation. This approach leads to the expression of the vector magnetic potential as a product of the current and a constant, for both the finite and infinitely-long dipole antennas. The definition of the magnetic vector potential simplifies the solution of the Pocklington equation to derive simple expressions for the current. The simple expressions derived are in good agreement with the current obtained from the three-term approximation and from the Fourier transform technique for the case of infinitely-long dipole antennas.

The technique used in this paper also shows that the complex propagation constant used in the three-term method is similar to the complex propagation constant of the principal wave of the Sommerfeld axial cylindrical surface waves [6]. Moreover, it is known that the analytic approximation methods available today, such as the three-term approximation [4], are not accurate at describing the current near the feeding gap. As reported in [11], for conducting cylinders the higher order of the Sommerfeld surface waves attenuates rapidly. This suggests that the current near the feeding point can be more accurately described if multiple Sommerfeld waves are used. Consequently, we propose a more accurate form of the current near the feeding point.

The paper is organized as follows. Firstly, the fields on the surface of an infinitely-long cylindrical conductor, which is excited by an incident rotationally symmetric axial electric field, are expressed. Also, from the expression of the surface electric and magnetic fields, the surface impedance per unit length of the cylinder is defined. Applying the boundary conditions of the magnetic fields, the general form of the induced axial current is defined as a traveling wave and a standing wave. In the following section, the standing wave general form of the current is applied in the standard Pocklington equation of a finite size, imperfectly conducting, and delta-gap excited cylindrical dipole antenna. From this equation, the complex propagation and coefficients of the current are defined. Following this, the traveling wave general form of the current is applied to the Pocklington equation of a delta-gap excited and infinitely-long cylindrical dipole antenna of finite conductivity, which is also used to define the complex propagation constant and the coefficient of the exponential function used to express the current. Lastly, from the comparison of the complex propagation constants to Sommerfeld poles, an expression of the current near the feeding point of a dipole antenna is proposed based on multiple Sommerfeld poles, for both types of dipole antennas.

II. Fields on An Infinitely-long Cylindrical Conductor

This section expresses the axial electric field, current density, surface impedance per unit length and the axial current on

The authors are with the College of Engineering and Science, Victoria University, Footscray Campus, VIC 3011, Australia.

an infinitely-long cylindrical conductor of finite conductivity when it is excited by a rotationally symmetric electric field on its surface. These expressions are used to describe the axial current in a finite and infinitely-long cylindrical dipole antennas that are excited by delta-gap electric field source, in the following sections.

Assuming a rotationally symmetric and time harmonic ($e^{j\omega t}$) external electric field is maintained on the surface of an infinitely-long and circular conducting cylinder of radius a, conductivity σ, relative permittivity ϵ and permeability μ_0, which extends along the z-axis of a system of cylindrical coordinates (ρ, ϕ, z), the magnetic vector potential inside $\mathbf{A}_1 = A_{1z}(\rho,z)\,\hat{\mathbf{z}} + A_{1\rho}(\rho,z)\,\hat{\boldsymbol{\rho}}$ can be written, from the Maxwell's equations in Lorenz gauge, as

$$\nabla^2\mathbf{A}_1 + k_1^2\mathbf{A}_1 = 0 \tag{1}$$

where $k_1 = \sqrt{-j\omega\mu_0(\sigma + j\omega\epsilon_0\epsilon)}$. Since all the magnetic or electric field components can be obtained from either of the two components of $\mathbf{A}_{1z}(\rho,z)$, we assumed $A_{1\rho}(\rho,z) = 0$. Therefore, expanding the vector identity $\nabla^{\mathbf{2}}\mathbf{A}_1 = \nabla\nabla\cdot\mathbf{A}_1 - \nabla\times\nabla\times\mathbf{A}_1$ in cylindrical coordinates and substituting it into (1) gives the following equation:

$$\frac{\partial^2 A_{1z}(\rho,z)}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_{1z}(\rho,z)}{\partial\rho} + k_1^2 A_{1z}(\rho,z) = 0. \tag{2}$$

The solution for (2) can be obtained by the method of separation of variables, such that $A_{1z}(\rho,z)$ can be written as

$$A_{1z}(\rho,z) = g(z)G(\rho). \tag{3}$$

Substituting (3) in (2) yields the equation

$$\frac{1}{g(z)}\frac{\partial^2 g(z)}{\partial z^2} + k_1^2 = -\frac{1}{G(\rho)}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial G(\rho)}{\partial\rho}. \tag{4}$$

It can be seen that the left side of the above equation is a function of z and the right side is a function of ρ; thus, the two sides can be equal if both of them are equal to a constant. Denoting the constant by υ_1^2 and expressing it as $\upsilon_1^2 = k_1^2 - \gamma^2$ so that

$$\frac{\partial^2 g(z)}{\partial z^2} + \gamma^2 g(z) = 0 \tag{5}$$

and

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial G(\rho)}{\partial\rho} + \upsilon_1^2 G(\rho) = 0. \tag{6}$$

The expression in (5) is the well known one dimensional wave equation that has a solution of a traveling wave in a lossy medium given as $g(z) = C_1 e^{\pm j\gamma z}$ or the standing wave representation as $g(z) = C_2(\sin(\gamma z) + C_3\cos(\gamma z))$, where C_1, C_2 and C_3 are arbitrary constants. From the wave equation of $g(z)$, γ is the complex propagation constant that can be described as $\gamma = \beta - j\alpha$, where β is the phase constant and α is the attenuation constant.

Equation (6) can be written in terms of a new independent variable $x = \upsilon_1\rho$ as

$$\frac{\partial^2 G\left(\frac{x}{\upsilon_1}\right)}{\partial x^2} + \frac{1}{x}\frac{\partial G\left(\frac{x}{\upsilon_1}\right)}{\partial x} + G\left(\frac{x}{\upsilon_1}\right) = 0, \tag{7}$$

which is a Bessel equation with known solutions. One of the solution for the expression in (7) is $G(\rho) = C_4 J_0(\upsilon_1\rho)$, where J_0 is the zeroth-order Bessel function and C_4 is an arbitrary constant. The Bessel function was chosen so that the vector potential is non-zero at the center of the cylinder. Consequently, the magnetic vector potential can be written as

$$A_{1z}(\rho,z) = G(\rho)\,g(z) = C_4 J_0(\upsilon_1\rho)\,g(z). \tag{8}$$

Assuming the induced total axial current is $I(z)$, from Ampere's law, the magnetic flux density $B_{2\phi}(a,z)$ on the surface of the cylinder can be written as

$$B_{2\phi}(a,z) = \frac{\mu_0 I(z)}{2\pi a}. \tag{9}$$

From the expression of the magnetic vector potential, the magnetic flux density inside the cylinder can be expressed as

$$B_{1\phi}(\rho,z) = -\frac{\partial A_{1z}(\rho,z)}{\partial\rho} = C_4\upsilon_1 J_1(\upsilon_1\rho)\,g(z) \tag{10}$$

where J_1 is the first-order Bessel function. Defining the arbitrary constant C_4 as

$$C_4 = \frac{\mu_0}{2\pi a\upsilon_1 J_1(\upsilon_1 a)} \tag{11}$$

and applying the magnetic field boundary conditions at the surface of the cylinder; the magnetic flux density at the surface of the cylinder can be written as

$$B_{1\phi}(a,z) = B_{2\phi}(a,z) = \frac{\mu_0 g(z)}{2\pi a} = \frac{\mu_0 I(z)}{2\pi a}, \tag{12}$$

which implies that

$$I(z) = g(z). \tag{13}$$

Equation (13) expresses the induced total axial current when the cylinder is excited by a rotationally symmetric electric field on the surface of the cylinder. Furthermore, if we are interested in a traveling wave current, it is expressed as

$$I(z) = C_1 e^{\pm j\gamma z}. \tag{14}$$

On the other hand, if we are interested in a standing wave form of the current, it can be expressed as

$$I(z) = C_2(\sin(\gamma z) + C_3\cos(\gamma z)). \tag{15}$$

The electric field inside the cylinder can be found from the vector magnetic potential using the relationship

$$\mathbf{E_1} = \frac{-j\omega}{k_1^2}\left(\nabla\nabla\cdot\mathbf{A_1} + k_1^2\mathbf{A_1}\right). \tag{16}$$

From (16), the axial component of the electric field $E_{1z}(\rho,z)$ can be derived as

$$\begin{aligned} E_{1z}(\rho,z) &= \frac{j\omega}{k_1^2}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(\frac{\partial A_{1z}(\rho,z)}{\partial\rho}\right) = -j\omega\frac{\upsilon_1^2}{k_1^2}A_{1z}(\rho,z) \\ &= \frac{I(z)\,\upsilon_1}{2\pi a(\sigma + j\omega\epsilon_0\epsilon)}\frac{J_0(\upsilon_1\rho)}{J_1(\upsilon_1 a)}. \end{aligned} \tag{17}$$

And the volume current density $J_{1z}(\rho,z)$ can be derived as

$$J_{1z}(\rho,z) = (\sigma + j\omega\epsilon_0\epsilon)E_{1z}(\rho,z) = \frac{I(z)\,\upsilon_1}{2\pi a}\frac{J_0(\upsilon_1\rho)}{J_1(\upsilon_1 a)}. \tag{18}$$

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Thus, the surface impedance per unit length of the cylinder z^i can be defined as

$$z^i = \frac{E_{1z}(a,z)}{2\pi a H_{1\phi}(a,z)} = \frac{E_{1z}(a,z)}{I(z)} = \frac{\upsilon_1}{2\pi a(\sigma + j\omega\epsilon_0\epsilon)} \frac{J_0(\upsilon_1 a)}{J_1(\upsilon_1 a)} \quad (19)$$

where $H_{1\phi}(a,z)$ is the magnetic field intensity on the surface of the cylinder.

III. Delta-gap Excited Finite Cylindrical Dipole Antenna

When we consider a finite length cylindrical dipole antenna of height $2h$, radius a, conductivity σ, relative permittivity ϵ, permeability μ_0, with axis aligned along the z-axis, and center-fed by a delta-gap electric field source located at $z = 0$, the current at the ends of the cylinder has to be zero. Consequently, we expect the current to be even symmetric with respect to $z = 0$ and form a standing wave with a general expression similar to (15). Therefore, since the current vanishes at $z = \pm h$ and evenly symmetric with respect to $z = 0$, for $0 < |z| \leq h$, it can be expressed as

$$I(z) = C_2 \left[\sin(\gamma|z|) - \sin(\gamma h) + C_3(\cos(\gamma z) - \cos(\gamma h))\right]. \quad (20)$$

The axial surface electric field at $z = 0$ is the delta-gap electric field source that is defined as

$$E_{1z}(a,z) = -V_0\delta(z) \quad (21)$$

where V_0 is the electromotive force (emf) applied on the infinitesimal gap at $z = 0$. Since the electric field source at $z = 0$ is rotationally symmetric, we expect the surface axial electric field on the rest of the cylinder is also rotationally symmetric. Therefore, from the expression in (19), the surface axial electric field can be expressed as

$$E_{1z}(a,z) = z^i I(z), \quad 0 < |z| \leq h. \quad (22)$$

As a result, the axial magnetic vector potential on the surface of the cylinder $A_{2z}(z)$ due to the surface axial electric field can be related as

$$z^i I(z) - V_0\delta(z) = \frac{-j\omega}{k_2^2}\left(\frac{\partial^2}{\partial z^2} + k_2^2\right) A_{2z}(z) \quad (23)$$

where $k_2 = \omega\sqrt{\epsilon_0\mu_0}$ is the free space wave number. Also, the vector magnetic potential $A_{2z}(a,z)$ can be related to the volume axial current density $J_{1z}(\rho,z)$ expressed in (18) as

$$A_{2z}(a,z) = \frac{\mu_0}{4\pi}\int_{-h}^{h}\int_{0}^{a} 2\pi J_{1z}(\rho',z')K(z-z')\rho' d\rho' dz' = \frac{\mu_0}{4\pi}\int_{-h}^{h} I(z')K(z-z')dz' \quad (24)$$

where $K(z-z')$ is the thin-wire approximate kernel defined as

$$K_{ap}(z-z') = \frac{e^{-jk_2\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}}. \quad (25)$$

From the above expressions, the standard Pocklington equation can be written as

$$\left(\frac{\partial^2}{\partial z^2} + k_2^2\right)\int_{-h}^{h} I(z')K(z-z')dz' = \frac{j4\pi k_2}{\eta_0}\left(I(z)z^i - V_0\delta(z)\right) \quad (26)$$

where $\eta_0 = 120\pi$ is the free space impedance.

For $|z| > 0$, from the boundary conditions of the electric field and the magnetic field on the surface of the cylinder, and from the expression of the vector magnetic potential in (8), the boundary conditions for the vector magnetic potential can be written as

$$A_{2z}(a,z) = -\frac{k_2^2}{\upsilon_2^2}\frac{\upsilon_1^2}{k_1^2}A_{1z}(a,z) = \frac{1}{j\omega}\frac{k_2^2}{\upsilon_2^2}z^i I(z) \quad (27)$$

where $\upsilon_2^2 = \gamma^2 - k_2^2$. From the expressions in (24) and (27), the following relation can be obtained

$$\int_{-h}^{h} I(z')K_{ap}(z-z')dz' = \Psi I(z), \quad 0 < |z| \leq h \quad (28)$$

where Ψ is a constant, which is defined as

$$\Psi = \frac{j4\pi k_2}{k_2^2 - \gamma^2}\frac{z^i}{\eta_0}. \quad (29)$$

The relationship in (28), implies that the thin-wire kernel can be approximated as

$$K_{ap}(z) \sim \Psi\delta(z), \quad (30)$$

which also implies that Ψ can be approximately computed as

$$\Psi \sim \int_{-h}^{h} K_{ap}(z)dz. \quad (31)$$

Therefore, the left-hand side expression in (26) can be simplified, replacing the expression of the current in (20), as

$$2\gamma\Psi C_2\delta(z) + \left(k_2^2 - \gamma^2\right)\Psi I(z) - \gamma^2\Psi C_2\left(\sin(\gamma h) + C_3\cos(\gamma h)\right). \quad (32)$$

Equating the terms in the above expression with that of the right-hand side of (26), the complex propagation constant γ can be obtained as

$$\gamma = k_2\sqrt{1 - \frac{j4\pi z^i}{k_2\Psi\eta_0}}, \quad (33)$$

which can also be determined from (29). The complex propagation constant γ (33) can be calculated by applying the iteration procedure. First, the iteration is initialized by setting $\gamma = k_2$ and calculating z^i, which is used to calculate γ iteratively until it converges. We have seen that such iteration converges after a few cycles.

Similarly, the constants C_2 and C_3 can be calculated as

$$C_2 = \frac{-j2\pi k_2}{\gamma\Psi}\frac{V_0}{\eta_0} \quad (34)$$

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and

$$C_3 = -\tan(\gamma h)\,. \tag{35}$$

Therefore, the axial current $I(z)$ can be written as

$$I(z) = \frac{j2\pi k_2}{\gamma\Psi\cos(\gamma h)}\frac{V_0}{\eta_0}\sin(\gamma(h-|z|)) \tag{36}$$

The accuracy of the expression for the current in (36) is validated by comparison with the current obtained from the three-term approximation, which is obtained by approximately solving the expression in (26) and involves an approximation of the thin-wire kernel integral shown in (28). King *et al* [4] derived the three-term expression of the axial current as

$$I_{1z}(z) = \frac{j2\pi k_2}{\gamma\Psi_{dR}\cos(\gamma h)}\frac{V_0}{\eta_0}\Big[\sin\gamma(h-|z|) + T_U(\cos\gamma z - \cos\gamma h) + T_D(\cos\tfrac{1}{2}k_2 z - \cos\tfrac{1}{2}k_2 h)\Big] \tag{37}$$

and

$$\gamma = k_2\sqrt{1-\frac{j4\pi z^i}{k_2\Psi_{dR}\eta_0}} \tag{38}$$

where the factor Ψ_{dR} and the coefficients T_U and T_D involve the numerical computations of eight integrals. The calculation of γ also applies the iteration procedure; the iteration is initialized by setting $\gamma = k_2$ to calculate the initial values of z^i and Ψ_{dR}. For very thin-wire dipole antennas, the expression in (36) is a good approximation to the three-term expression given in (37), as shown in Fig. 1. The three-term expression is valid for $0 \le k_2 h \le 5\pi/4$; thus, the values of h within this range are used for comparison. As shown in Fig. 1(a), the two expressions are in good agreement with differences start to appear as the height decreases. Fig. 1(b) shows that the contributions of the two cosine terms in the three-term equation is insignificant, which supports the agreement between the proposed expression in (36) and the three-term expression (37).

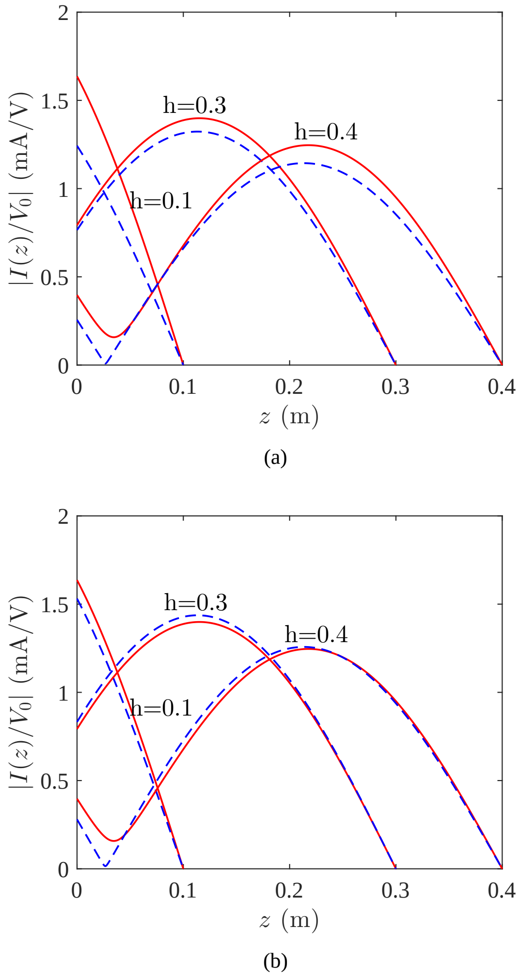

Fig. 1. Comparison of the proposed current expression with that of the three-term expression for a cylindrical dipole of radius $a = 0.001$ cm and skin-depth to radius ratio of 0.1 at 400 MHz. (a) Compares the current in (36) (broken line) to the three-term expression (37) (solid line) for the half-length h of 0.4, 0.3 and 0.1 m. (b) Compares the first sinusoidal term (broken line) and the complete three-terms (solid line) in (37).

IV. Delta-Gap Excited Infinitely-Long Cylindrical Dipole Antenna

When we consider an infinitely-long cylindrical dipole antenna of radius a, with similar dielectric properties as the finite one, that is excited with a delta-gap electric field source at $z = 0$, the general form of the current is expected to be an outgoing travelling wave similar to the one given in (14), but with a little modification to take into account the presence of infinitesimal gap at $z = 0$. Thus, the current can be defined as

$$I(z) = I_{z0}e^{-j\gamma|z|}, \quad |z| > 0, \tag{39}$$

where I_{z0} is the current near the infinitesimal gap. Similar to the finite case, the vector magnetic potential $A_{2z}(a,z)$ on the surface of the cylinder can be described as

$$A_{2z}(a,z) = \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} I(z')\,K(z-z')\,dz' \tag{40}$$

where the kernel $K(z-z')$ can either take the exact form, which is given as

$$K_{ex}(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jk_2\sqrt{(z-z')^2+4a^2\sin^2(\phi'/2)}}}{\sqrt{(z-z')^2+4a^2\sin^2(\phi'/2)}}d\phi', \tag{41}$$

which has a solution given in [7], [8] or the approximate form given in (25).

Replacing the expression of the current into the integral in (40), for $z > 0$, the vector magnetic potential can be obtained as

$$A_{2z}(a,z) = \frac{\mu_o}{4\pi}I(z)\int_{-\infty}^{\infty} K(x)\,e^{j\gamma x}dx = \frac{\mu_o}{4\pi}I(z)\,\bar{K}(\gamma) \tag{42}$$

where x is a dummy variable used for the integration and $\bar{K}(\gamma)$ is the Fourier transform of the kernel evaluated at the complex propagation constant γ. For $|\gamma| > k_2$, the Fourier-transform of the exact kernel is given in [9] as

$$\bar{K}_{ex}(\gamma) = 2I_0(\upsilon_2 a) K_0(\upsilon_2 a) \tag{43}$$

and for the approximate kernel as

$$\bar{K}_{ap}(\gamma) = 2K_0(\upsilon_2 a) \tag{44}$$

where K_0 and I_0 are the modified Bessel functions; and $\upsilon_2 = \sqrt{\gamma^2 - k_2^2}$ is defined previously in (27). Moreover, for $z < 0$, $A_{2z}(a, z)$ has the same expression as (42) since $\bar{K}(\gamma)$ is an even function. Consequently, for $|z| > 0$, applying the boundary condition of the vector magnetic potential on the surface of the cylinder given in (27), the complex propagation constant can be expressed as

$$\gamma = k_2 \sqrt{1 - \frac{j4\pi z^i}{k_2 \bar{K}(\gamma)\eta_0}}, \tag{45}$$

which can be computed applying a similar iteration procedure as the finite case. The iteration can be initialized by setting $\gamma = k_2 + \varepsilon$ to compute z^i and $\bar{K}(\gamma)$, where ε is a very small number. Then, the value of z^i and $\bar{K}(\gamma)$ is used to calculate γ. This continues until the value of γ converges. As shown in an example below, this iteration is highly convergent.

In order to express the axial current completely, we need to determine I_{z0}. The differential equation relating the vector magnetic potential and the axial electric field on the surface of the cylinder, which is given in (23), can be solved by applying Green's function $F(z)$ satisfying the relation

$$\left(\frac{\partial^2}{\partial z^2} + k_2^2\right) F(z) = 2k_2 \delta(z). \tag{46}$$

Thus, the vector magnetic potential can be expressed as

$$A_{2z}(a, z) = -j\frac{k_2}{2\omega}\Big[C_4 \cos(k_2 z) + V_0 \sin(k_2 |z|) - z^i \int_{-\infty}^{\infty} I(q) F(z - q)\, dq\Big] \tag{47}$$

where C_4 is an arbitrary constant and q is a dummy variable used for integration. By choosing the Green's function $F(z) = 2\sin(k2z)u(z)$ with the unit step function $u(z)$, the $A_{2z}(a, z)$ can be expressed for $z > 0$. Also, taking the limit of (47) as z approaches zero from the right side, C_4 can obtained as

$$C_4 = \frac{2k_2 z^i I_{z0}}{\gamma^2 - k_2^2}. \tag{48}$$

Replacing the expression of $A_{2z}(a, z)$ in (42) in (47), taking the Laplace transform on both sides, and evaluating it at the Laplace domain variable $s = 0$, it can be simplified as

$$\frac{V_0}{k_2} - \frac{2z^i I_{z0}}{j\gamma k_2} = \frac{\eta_0 \bar{K}(\gamma) I_{z0}}{2\pi\gamma}, \tag{49}$$

which can be used to obtain I_{z0} as

$$I_{z0} = \frac{2\pi k_2}{\gamma \bar{K}(\gamma)} \frac{V_0}{\eta_0}. \tag{50}$$

Then, the current on an infinitely-long cylindrical dipole antenna can be expressed as

$$I(z) = \frac{2\pi k_2}{\gamma \bar{K}(\gamma)} \frac{V_0}{\eta_0} e^{-j\gamma|z|}. \tag{51}$$

In order to validate the accuracy of the axial current expression in (51), we compared it to the current obtained from the Fourier transform method. The Pocklington equation for the infinite case can be written as

$$\left(\frac{\partial^2}{\partial z^2} + k_2^2\right) \int_{-\infty}^{\infty} I(z') K(z - z')\, dz' = \frac{j4\pi k_2}{\eta_0}\left(I(z) z^i - V_0 \delta(z)\right). \tag{52}$$

Taking the Fourier transform of both sides, the above equation can be written as

$$\left(k_2^2 - \xi^2\right) \bar{I}(\xi) \bar{K}(\xi) = \frac{j4\pi k_2}{\eta_0}\left(\bar{I}(\xi) z^i - V_0\right) \tag{53}$$

where $\bar{I}(\xi)$ is the Fourier transform of the current. Taking the Inverse-Fourier transform of the above expression, the current can be obtained as

$$I(z) = \frac{j4\pi k_2}{2\pi}\frac{V_0}{\eta_0} \int_{-\infty}^{\infty} \frac{e^{-j\xi|z|}}{Z(\xi)} d\xi \tag{54}$$

where

$$Z(\xi) = \left(\xi^2 - k_2^2\right) \bar{K}(\xi) + j4\pi k_2 \frac{z^i}{\eta_0}. \tag{55}$$

Following the technique proposed by Hallén [10, p. 451], the integral in (54) is simplified to a form suitable for computation as the sum of a convergent branch cut integral and a sum of residues

$$I(z) = \sum_n I_n(z) + I_{bc}(z) \tag{56}$$

where

$$I_n(z) = 4\pi k_2 \frac{V_0}{\eta_0} \frac{e^{-j\xi_n|z|}}{Z'(\xi_n)} \tag{57}$$

where $Z' = \partial Z/\partial \xi$ and ξ_n is the n^{th} root of Z when z^i in (19) is defined with $\upsilon_1 = \sqrt{k_1^2 - \xi^2}$. Also, the branch cut integral, taking the exact kernel, can be written as

$$I_{bc}(z) = \frac{2k_2^2}{j\pi}\frac{V_0}{\eta_0} \int_0^{\infty} \Bigg\{ dt \frac{t}{\sqrt{t^2 - 1}} \times \Bigg[\frac{e^{-k_2\sqrt{t^2-1}|z|}}{-k_2^2 t^2 J_0(ak_2 t) H_0^{(1)}(ak_2 t) + 4\omega\epsilon_0 z^i} - \frac{e^{-k_2\sqrt{t^2-1}|z|}}{k_2^2 t^2 J_0(ak_2 t) H_0^{(2)}(ak_2 t) + 4\omega\epsilon_0 z^i} \Bigg] \Bigg\} \tag{58}$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions.

The integrand in (54) has poles at ξ_n, which are sometimes called Sommerfeld poles. Note that the surface field components described in section II, which are directly proportional to the axial current, are similar to the axial cylindrical surface waves discussed by Sommerfeld. According to Stratton [11, p. 530], the function (55) has multiple roots that represent the propagation constants of multiple Sommerfeld waves. Also, for conducting cylinders, the higher order waves attenuates rapidly; therefore, it is sufficient to keep the principal wave,

TABLE I
COMPARISON THE PROPAGATION CONSTANT γ TO ξ_1 GIVEN IN [12] FOR THE FREQUENCY OF 300 MHz AND RADIUS $a = 0.01$ cm. δ_s/a IS THE SKIN-DEPTH TO RADIUS RATIO.

δ_s/a	γ	ξ_1 [12]
1	$6.3564 - j0.3762$	$6.3580 - j0.3765$
0.5	$6.3556 - j0.1122$	$6.3572 - j0.1122$
0.25	$6.3247 - j0.0488$	$6.3262 - j0.0488$

which is represented by the first pole ξ_1. From solving the first root ξ_1 of (55), it can be seen that ξ_1 is equal to the propagation constant γ we derived in (45). Therefore, the current can be written as

$$I\left(z\right) = I_{res}\left(z\right) + I_{bc}\left(z\right) \tag{59}$$

where

$$I_{res}\left(z\right) = 4\pi k_2 \frac{V_0}{\eta_0}\frac{e^{-j\xi_1|z|}}{Z'\left(\xi_1\right)} = 4\pi k_2 \frac{V_0}{\eta_0}\frac{e^{-j\gamma|z|}}{Z'\left(\gamma\right)}. \tag{60}$$

Table I shows a comparison of the propagation constant γ we calculated using (45) to the first root ξ_1 of (55) that is given in [12], which describes the admittance of infinitely-long cylindrical dipole antenna with magnetic frill excitation. The iteration used to calculate γ converged at the fourth cycle.

Figure 2 shows the comparison of the current calculated using (51) to the one obtained by applying the Fourier transform technique in (59), for an infinitely-long cylindrical dipole of radius $a = 0.01$ cm at 400 MHz. Different conductivities of the dipole are compared by taking different skin-depth to radius ratio $b = \delta_s/a$; the conductivity can be obtained as $\sigma = 2/(\omega\mu_0\delta_s^2)$. As shown in Fig. 2(a), the two expressions tends to be different near the gap or $z = 0$ due to the contribution of the branch cut integral for small z. But they are in good agreement for larger values of z. Moreover, it also shows that the two expressions are in excellent agreement for lower conductivities. From looking at the expression of I_{res} in (60) and the current in (51), it might look like our derived current approximates the residual current I_{res}. But, from the comparison of these expressions in Fig. 2(b), it can be seen that (51) is more closer to the total current (59) than the residual current.

The three-term expression in (37) is derived based on the condition that $0 \leq k_2 h \leq 5\pi/4$. This implies that, for large h, the three-term expression is not accurate. The accuracy of the expression proposed for finite cylindrical dipole in (36) is assessed by comparing the current in long cylindrical dipole antennas to the current on a lossy infinite cylindrical dipole antenna. Fig. 3 compares the current on a dipole antenna of half-length $h = 10$ m computed with (36) and (37) to the current computed using (59), for $b = \delta_s/a = 1$ and radius of 0.01 cm. As expected the current from the three-term approximation has ripples due to the numerical integrals involved. But, the current from (36) and (59) are in excellent agreement suggesting that the expression in (36) is valid for electrically long lossy cylindrical dipole antennas.

From the comparison of the complex propagation constant for the case of the proposed current on finite cylindrical dipole

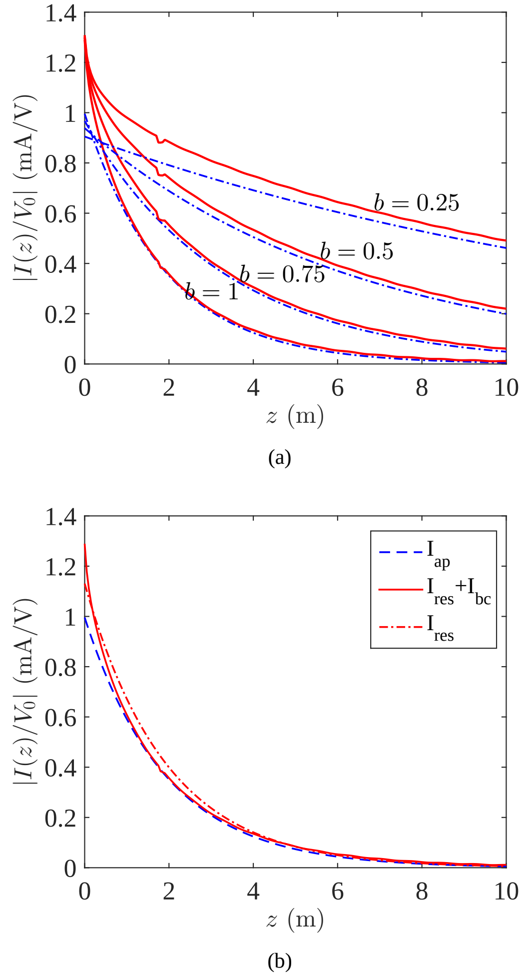

Fig. 2. The current calculated on infinitely long dipole of radius $a = 0.01$ cm and at 400 MHz, (a) Comparison of the current calculated from (51), the broken line, to from (59), the solid line, for case of different skin-depth to radius ratios represented by b (b) Comparison of the current I_{ap} from (51) to the currents in (59)

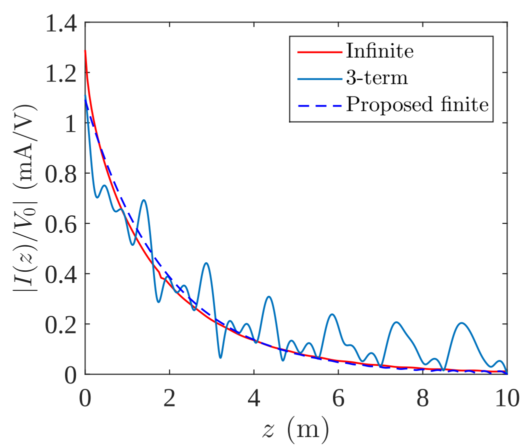

Fig. 3. Comparison of the proposed current for finite dipole (36), the three-term approximation (37) and the infinitely-long dipole (59). The finite dipole has a height $h = 10$ m, radius $a = 0.01$ cm and $b = \delta_s/a = 1$; and the infinite dipole has the same radius and conductivity.

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antenna (33), the three-term (38), and the infinitely-long dipole antenna (45) shown in Table II, it can be seen that the complex propagation constants used for the finite cylindrical dipole antennas are similar to the complex propagation constant of the principal Sommerfeld surface wave. It is known that one of the limitations of the analytical expressions of the current, such as the three-term approximations, is they are not accurate near the feeding point of the dipole antenna. One of the factors of these limitations is the use of a single propagation constant of the principal wave in the expression of the current. But, very close to the feeding point (the gap), the higher-order waves also contribute. Therefore, a better approximation of the current near the feeding point should include all the propagation constants of the multiple Sommerfeld surface waves. Thus, the current near the feeding point I_{z0} for the finite dipole antenna can be expressed as

$$I_{z0} = \lim_{z \to 0} I(z) = \frac{V_0}{\eta_0} \frac{j2\pi k_2}{\Psi} \sum_n \frac{\tan(\xi_n h)}{\xi_n} \tag{61}$$

and for the infinitely-long dipole antenna

$$I_{z0} = \lim_{z \to 0} I(z) = \frac{V_0}{\eta_0} 2\pi k_2 \sum_n \frac{1}{\xi_n \bar{K}(\xi_n)} \tag{62}$$

where ξ_n is the n^{th} Sommerfeld pole calculated as the n^{th} root of the expression $Z(\xi) = 0$ in (55). The above expressions can be applied to calculate the antenna admittance. Alternatively, the expression of $Z(\xi) = 0$ can be rewritten as

$$\xi^2 - k_2^2 \left(1 - \frac{1}{ak_1^2} P(\xi)\right) = 0 \tag{63}$$

where

$$P(\xi) = \frac{\upsilon_1 I_0(\upsilon_1 a)}{I_1(\upsilon_1 a) I_0(\upsilon_2 a) K_0(\upsilon_2 a)}, \tag{64}$$

which was derived based on the identity of the Bessel functions that $I_n(x) = (j)^{-n} J_n(jx)$. In this case, the variables υ_1 and υ_2 are defined as

$$\upsilon_1 = \sqrt{\xi^2 - k_1^2} \quad \text{and} \quad \upsilon_2 = \sqrt{\xi^2 - k_2^2}. \tag{65}$$

A method to find the multiple Sommerfeld poles is discussed in [11, p. 527]. The fields of a transverse magnetic (TM) mode discussed in [11] are similar to those discussed in section II of this paper. According to the discussion in [11], from the boundary conditions of the TM mode, the following relation can be obtained

$$\frac{k_1^2}{\upsilon_1 a} \frac{I_1(\upsilon_1 a)}{I_0(\upsilon_1 a)} = -\frac{k_2^2}{\upsilon_2 a} \frac{K_1(\upsilon_2 a)}{K_0(\upsilon_2 a)}, \tag{66}$$

which can be used to rewrite the expression in (63) as

$$I_0(u) K_1(u) u - 1 = 0 \tag{67}$$

where $u = \upsilon_2 a$ is a complex argument. The roots obtained ξ_n must satisfy (67). For example, for highly conductive cylinders, the principal root ξ_1 is close to k_2; this implies that the first root u is very small. Therefore, taking the asymptotic expressions $I_0(u) \sim 1$ and $K_1(u) \sim 1/u$, the above equation can be satisfied. Moreover, the propagation constants calculated from (33), (38) and (45) satisfy (67). For metallic conductors, the value of the higher order roots ξ_n are very large so that their contributions in (61) and (62) are negligible. But, as conductivity of the cylinder decreases, the higher order roots start to take over and with contribution of the principal wave vanishes for the case of dielectrics. For example, for an infinitely-long cylinder of radius $a = 0.01$ cm and conductivity of 100 S/m, at 300 MHz, $\xi_1 = 52.6365 - j62.4109$, which is much larger than obtained for metallic conductors.

TABLE II
COMPARISON OF γ/k_2 FOR THE CASE OF PROPOSED CURRENT ON A FINITE CYLINDRICAL DIPOLE ANTENNA, THE THREE-TERM, AND FOR THE INFINITELY-LONG DIPOLE ANTENNA, FOR THE FREQUENCY OF 300 MHz AND RADIUS $a = 0.01\ cm$. FOR THE FINITE ANTENNAS, THE HALF-LENGTH $h = 0.4\ m$ IS TAKEN.

δ_s/a	γ/k_2 (45)	γ/k_2 (33)	γ/k_2 (38)
0.1	$1.0024 - j0.0027$	$1.0038 - j0.0025$	$1.0034 - j0.0036$
0.25	$1.0062 - j0.0075$	$1.0095 - j0.0069$	$1.0083 - j0.0096$
0.5	$1.0111 - j0.0179$	$1.0177 - j0.0160$	$1.0148 - j0.0212$
1	$1.0112 - j0.0599$	$1.0306 - j0.0570$	$1.0182 - j0.0682$
1.5	$1.0114 - j0.1376$	$1.0536 - j0.1277$	$1.0242 - j0.1498$

V. REMARKS ON THE COMPUTATIONAL COMPLEXITY

The main objective of the paper is to present a new perspective of the cylindrical antenna theory by defining the general form of the current from an infinitely-long cylindrical conductor that is exposed to a rotationally symmetric wave. We also showed that such analysis leads to the derivation of simple and reasonably accurate expressions for the current in finite and infinitely-long cylindrical dipole antennas that have finite conductivities. In addition to accuracy, the derived expression reduces the computational effort to compute the current. For example, implementing the computations on Matlab, the CPU time required to compute the current on the finite dipole antenna using the three-term approximation method at 400 MHz, radius=0.001 cm, and skin-depth to radius ratio of 0.1, is 3.66 times more than that of our proposed expression. Also, for the case of the infinitely-long antenna and similar conditions, the CPU time required to implement the Fourier Transform technique is 13.02 times more than that of our proposed approach.

VI. CONCLUSION

The fields on the surface of an infinitely-long cylindrical conductor, when it is excited by a rotationally symmetric incident electric, were used to analyse imperfectly conducting cylindrical dipole antennas of finite and infinite lengths. General forms of the induced current derived from the infinitely-long conductor were used to simplify the solutions of the Pocklington equations involved. Consequently, simple expression of the current for both finite and infinite cylindrical dipole antennas were proposed. The proposed expressions for the current were validated by comparison with the current obtained from the three-term approximation method, for the case of the finite dipole antenna, and the current obtained from applying the Fourier method, for the case of the infinite one. The results

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showed that the currents are in good agreement. Moreover, from the comparison of the complex propagation constants, an expression for the current near the feeding gap was proposed based on multiple Sommerfeld poles.

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Behailu Kibret (M'11) received the B.Sc. degree in Electrical Engineering from Bahir Dar University, Bahir Dar, Ethiopia, in 2005; and the Ph. D. degree in Electrical Engineering from Victoria University, Melbourne Australia, in 2016. He is currently working as a Research Fellow in Monash University, Melbourne, Australia. His research interest includes electromagnetics, antennas and body area networks.

Assefa K. Teshome (M'11) received the B.Sc. degree in Electrical Engineering from Bahir Dar University, Bahir Dar, Ethiopia in 2003; the M. Tech. degree in Electrical Engineering from Indian Institute of Technology – Madras (IIT–Madras), Chennai, India in 2007 and the M. Eng. (research) degree in Telecommunications Engineering from the University of South Australia, Adelaide, Australia in 2013. He is currently working toward the Ph.D. degree in the College of Engineering and Science, Victoria University, Melbourne, Australia. His research interests include signal propagation and communication models for body area networks (BAN) in addition to signal processing techniques for Biomedical and Biometric applications.

Daniel T. H. Lai (M'06) received the B.Eng (Hons.) and the Ph.D. degree in electrical and computer systems from Monash University, Melbourne, Australia, in 2002 and 2006, respectively.

He was a Research Fellow in the University of Melbourne and Victoria University (2007-2010). He is currently with the College of Engineering and Science, Victoria University. He has more than 80 peer-reviewed publications and is a current reviewer for several international journals. He is also actively involved in organization of several workshops and international conferences. His research interests include new sensing and communication technologies for body area networks.