1.08 *Equation Solving*

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In previous sections, you have been learning how to **simplify expressions.** For example, $5x + 2$ and $2x + 5y - 7$ are expressions. On the other hand, an **equation** must have an **equal sign**, a left side, and a right side, such as $5x + 2 = 17$ or $2x + 5y - 7 = 0$. Which of the following are equations?

a) $5x - 6 = 3x + 2$	Yes
b) $8y - 4 + 4(2y - 5)$	No
c) $8y - 4 + 4(2y - 5) = 2$	___
d) $2x + 3(4x + 6) = 0$	___
e) $b = 4$	___
f) $x + 2 = 2 + x$	___
g) $4 = 3 + 1$	___
h) $4 = 6 - 1$	___
i) $3x + 5y = 8$	___
j) $y = 3x + 2$	___

ANSWER: All except #b) are equations. Notice that #f) is true for all values of x. Since #f) and #g) are always true equations, they are called **identities**. Also notice that #h) is an equation which is not true. This is a **false equation**, but nevertheless, it is an equation. It is possible for an equation to have more than one variable, like #i) and #j).

To **solve an equation** means to find all values of the variable that make the equation true. Consider the following equations:

1. $\mathbf{5x + 2 = 17}$

2. $\mathbf{2 + x = x + 2}$

3. $\mathbf{x = x + 2}$

4. $\mathbf{x^2 + 3 = 4x}$

In equation #1, $5x + 2 = 17$, if $x = 3$, the equation reads $5(3) + 2 = 17$, which is true. It turns out that for any other value of x, the equation is false. Therefore, the solution to the equation is $x = 3$.

In equation #2, $2 + x = x + 2$, since any value of x will make the equation true, it is called an identity, and there are infinitely many solutions. In equation #3, $x = x + 2$, there are no values of x that will make the equation true. (Like, can you think of a number, add 2, and still have the same number?) This means that the equation has "no solution." It is called a contradiction. Sometimes when there is no solution, we write the Greek letter "Ø " (phi, as in "Phi Theta Kappa" Honor Society), which represents the empty set, or no solution. This is *not* saying the solution is 0 (zero)!

Finally, in equation #4, $x^2 + 3 = 4x$, there is an x^2 term. As it turns out, this x^2 term allows the possibility of two solutions. This type of equation, because of the x^2 term, is called a quadratic equation, and it will be discussed in a later topic in algebra. It turns out that both $x = 1$ and $x = 3$ are solutions, since it is true that $1^2 + 3 = 4(1)$ and $3^2 + 3 = 4(3)$.

Once a solution has been found, you can always check that solution by substituting into the original equation to see if it actually works. It is important to remember that you always are asked to simplify expressions and to solve equations. You can't solve expressions.

The examples given thus far represent only a few of the many different types of equations that you will ultimately learn to solve. And as you probably could guess, different methods are used to solve different types of equations. The equations to be solved in this section are called linear equations. Linear equations involve only one variable at a time, and the variables are not raised to powers. These equations will be solved using the identity and inverse number properties and the addition and multiplication properties of equations. However, do not let these formalities scare you--it will be as easy as unwrapping presents.

Solving an equation is like "un-wrapping presents!"

In my younger years, I decided to buy a very nice gift for my wife! Since it was a very special gift, I had the gift wrapped, I hid the gift in the house, but I told her that it was hidden. What do you think she did the next time I left the house? Is the order of events significant? She had to first find (or unhide the gift), and then unwrap the gift. Notice that my wife had to undo my steps, and in the reverse order that I did them. If she didn’t like the gift, then she would have to take it back (i.e., unbuy!) the gift. This story illustrates the principle of opposites from everyday life: wrapping and unwrapping, hiding and finding.

In math we have the same principal of opposites or inverse operations. For example, subtraction (or addition of the negative of a number) is the opposite of addition. Addition is the opposite operation for subtraction. Division (or multiplication by the reciprocal or the inverse of a number) is the opposite of multiplication. Multiplication is the opposite operation for division.

The process of equation solving, like the process of finding and unwrapping the gift, is a series of unwrapping operations. Consider the following equation in Example 1.

Example 1. **Solve for *x*:** $3x + 4 = 34.$

In order to solve for x, you must "undo" everything that has been done to the x, and in reverse order. In order to keep the equation the same or in "balance", you must be sure do the same thing to both sides of the equation. Notice what was done to this x in the equation. First, x was multiplied by 3, and then 4 was added. In order to find x, you must undo these operations in the reverse order, just like finding and unwrapping the present. To undo the 4 that was added you must subtract 4, then to undo the multiplication by 3 you must divide by 3. Whatever you do to one side of the equation, you must do also to the other side of the equation. [This is called the **"golden rule" of equations: "Do unto one side, as ye do unto the other!"]**

Example 1 Solution:

$$\begin{array}{rcl} 3x + 4 &=& 34 \\ -4 & & -4 \\ \hline 3x &=& 30 \end{array}$$

First, undo the + 4 by

Next, undo the multiplication by 3 by dividing both sides by 3.

$$\frac{3x}{3} = \frac{30}{3}$$

$$x = 10$$

Check: $3(10) + 4 = 34$

Each of the following will be solved using the principles of opposites as in the previous example.

Example 2. **Solve for *x*:** $4x + 10 = 30$

Solution: $4x + 10 = 30$ First, undo the "+ 10", with "-10" to each side of the equation.

$$\begin{array}{rcl} 4x + 10 & = & 30 \\ \underline{-10} & & \underline{-10} \\ 4x & = & 20 \end{array}$$

Second, divide both sides by 4.

$$\frac{4x}{4} = \frac{20}{4}$$

$$x = 5$$

Example 3. **Solve for *x*:** $3x - 17 = 7$

Solution: $3x - 17 = 7$ First, undo the "- 17", with a "+17".

$$\begin{array}{rcl} 3x - 17 & = & 7 \\ \underline{+17} & & \underline{+17} \\ 3x & = & 24 \end{array}$$

Second, divide both sides by 3.

$$\frac{3x}{3} = \frac{24}{3}$$

$$x = 8$$

Example 4. **Solve for *x*:** $13x - 10 = 16$

Solution: $13x - 10 = 16$ First, undo the "- 10", with a "+10" to each side.

$$\begin{array}{rcl} 13x - 10 & = & 16 \\ \underline{+10} & & \underline{+10} \\ 13x & = & 26 \end{array}$$

Second, divide both sides by 13.

$$\frac{13x}{13} = \frac{26}{13}$$

$$x = 2$$

Example 5. **Solve for x:** $9x + 32 = -49$

Solution: $9x + 32 = -49$ First, undo the "+ 32", with a "-32".

$$\begin{array}{rcl} 9x + 32 & = & -49 \\ \underline{-32} & & \underline{-32} \\ 9x & = & -81 \end{array}$$

Second, divide both sides by 9.

$$\frac{9x}{9} = \frac{-81}{9}$$

$$x = -9$$

Example 6. **Solve for *x*:** $7x - 10 = -52$

Solution: $7x - 10 = -52$ First, undo the "- 10", with a "+10" to each side.

$$\begin{array}{rcl} 7x - 10 & = & -52 \\ \underline{+10} & & \underline{+10} \\ 7x & = & -42 \end{array}$$

Second, divide both sides by 7.

$$\frac{7x}{7} = \frac{-42}{7}$$

$$x = -6$$

EXERCISES. In 1 – 13, solve the equations for x.

1. $3x + 4 = 34$ 2. $5x + 12 = 47$ 3. $4x + 10 = 30$

4. $3x - 8 = 7$ 5. $13x - 10 = 16$ 6. $7x - 12 = 44$

7. $5x + 13 = 68$ 8. $8x + 34 = 2$ 9. $6x + 32 = 20$

10. $6x + 32 = -22$ 11. $9x + 32 = -49$

12. $5x - 22 = -7$ 13. $7x - 10 = -52$

Example 7. Solve for x: $-3x = 6$

In this equation $-3x = 6$, what is it that has been done to x in order to get 6? Answer: Multiplication by -3. What would you have to do in order to "undo" the multiplication by -3? Answer: Divide by -3. The number -3 is said to be the coefficient of x. In this equation $-3x = 6$, you would then want to divide both sides of the equation by the coefficient of x, which is -3. Do you see that if you divide $-3x$ by -3, you will have just $1x$? The result looks like this:

Solution: $-3x = 6$ Divide both sides by -3.

$$\frac{-3x}{-3} = \frac{6}{-3}$$

$$x = -2$$

Example 8. Solve for x: $-7x = -56$

Solution: $-7x = -56$ Divide both sides by -7.

$$\frac{-7x}{-7} = \frac{-56}{-7}$$

$$x = 8$$

Example 9. **Solve for x:** $-x = 6$

Given the equation $-x = 6$. What is the coefficient of x? Remember that -x means the same as $-1x$, so the coefficient of x is -1. To solve the equation $-x = 6$ (which really means $-1x = 6$) you must divide both sides of the equation by -1:

Solution: $-x = 6$ Divide both sides by –1.

$$\frac{-x}{-1} = \frac{6}{-1}$$

$$x = -6$$

Example 10. **Solve for x:** $-x = -12$

Solution: $-x = -12$ Divide both sides by –1.

$$\frac{-x}{-1} = \frac{-12}{-1}$$

$$x = 12$$

EXERCISES. **In 14 – 23, solve the equations for x.**

14. $-5x = 25$ **15.** $-3x = 30$ **16.** $-8x = 32$

17. $-7x = -21$ **18.** $-6x = -36$ **19.** $-12x = -60$

20. $-x = 5$ **21.** $-x = 9$ **22.** $-x = -5$ **23.** $-x = -9$

Solving More Complicated Equations

Sometimes there are variable terms on both sides of the equation. For example, consider the equation $4x = 2x + 12$. Notice that in this equation there are three terms. Two of these terms (the $4x$ and the $2x$) contain variables, and the other term, the "12", has no variable--it's just a number term. It would be nice to get all the variable terms together on the same side of the equation. You may accomplish this by adding $-2x$ to both sides of the equation, as follows in the next example.

Example 11. **Solve for x:** $4x = 2x + 12$

Solution:

$$\begin{array}{l} 4x = 2x + 12 \\ \underline{-2x \quad -2x \quad\quad} \\ 2x = 12 \\ x = 6 \end{array}$$

If there are variable terms and number terms on both sides of the equation, such as

$4x - 12 = 2x + 6,$

get all variable terms on one side, and the **non-variable or number terms on the other side**.

Example 12. **Solve for x:** $4x - 12 = 2x + 6$

Solution:

$$\begin{array}{lll} 4x - 12 = 2x + 6 & & \text{Add -2x to both sides.} \\ \underline{-2x \qquad -2x \quad} & & \\ 2x - 12 = \quad 6 & & \text{Add +12 to both sides.} \\ \underline{\quad + 12 \qquad +12} & & \\ 2x \quad = \quad 18 & & \text{Divide by 2.} \\ x \quad = \quad 9 & & \end{array}$$

Example 13. **Solve for x:** $6x + 10 = -2x - 46$

Solution:

$$\begin{array}{lll} 6x + 10 = -2x - 46 & & \text{Add } +2x \text{ to both sides.} \\ \underline{+2x \qquad +2x \quad} & & \\ 8x + 10 = \quad - 46 & & \text{Add -10 to both sides.} \\ \underline{\quad - 10 \qquad - 10} & & \\ 8x \quad = \quad - 56 & & \text{Divide by 8.} \\ x \quad = \quad - 7 & & \end{array}$$

Example 14. **Solve for x:** $-8x + 24 = -2x - 48$

Solution:

$$\begin{array}{lll} -8x + 24 = -2x - 48 & & \text{Add 2x to both sides.} \\ \underline{+2x \qquad +2x} & & \\ -6x + 24 \ = \quad - 48 & & \text{Add -24 to both sides.} \\ \underline{\quad - 24 \qquad - 24} & & \\ -6x \quad = \ - 72 & & \text{Divide by -6.} \\ x = 12 & & \end{array}$$

EXERCISES. **In 24 – 29, solve the equations for x.**

24.
$$\begin{array}{r}4x = 2x + 12\\ \underline{-2x \quad -2x \qquad}\\ 2x = \underline{\qquad}\\ x = \underline{\qquad}\end{array}$$

25. $4x - 12 = 2x + 6$

26. $6x + 10 = 2x + 50$

27. $6x + 10 = -2x - 46$

28. $-6x + 10 = -2x + 50$

29. $-8x + 24 = -2x - 30$

Sometimes the left and/or right side of an equation can be simplified by removing parentheses (distributive property) and combining like terms on a given side. Consider the following examples:

Example 15. **Solve for x:** $4(x - 1) - 2(x + 3) = 8(5 - x)$

Solution: $4(x - 1) - 2(x + 3) = 8(5 - x)$

$$\begin{array}{rcll}4x - 4 - 2x - 6 &=& 40 - 8x & \text{(Distributive property)}\\ 2x - 10 &=& 40 - 8x & \text{(Combine like terms)}\\ \underline{+8x \qquad\quad} && \underline{+\ 8x} & \text{(Add +8}x\text{ to each side)}\\ 10x - 10 &=& 40 & \text{(Add +10 to each side)}\\ \underline{+\ 10} && \underline{+10} & \\ 10x &=& 50 & \text{(Divide both sides by 10)}\\ x &=& 5 &\end{array}$$

EXERCISES. **In 30 – 36, solve the equations for x.**

30. $4(x - 1) - 2(x + 3) = 8(5 - x)$

$$\begin{array}{rcll}4x - 4 - 2x - 6 &=& 40 - 8x & \text{(Distributive property)}\\ 2x - 10 &=& 40 - 8x & \text{(Combine like terms)}\\ \underline{+8x \qquad\quad} && \underline{+\ 8x} & \text{(Add +8}x\text{ to each side)}\\ &&& \text{(Add +10 to each side)}\\ &&& \\ &&& \text{(Divide both sides by 10)}\end{array}$$

The steps required to solve an equation like the previous example may be summarized as follows:

Strategy Summary: Equation Solving

Step 1: **If there are parentheses in the problem, eliminate them by use of the distributive property.**

Step 2: **Combine like terms (if possible) on each side of the equal sign.**

Step 3: **Using the "principle of opposites," get all variable terms to one side of the equation.**

Step 4: **Using the "principle of opposites," get all number terms to the other side of the equation.**

Step 5: **Divide both sides of the equation by the coefficient of the variable--that is, the number times the variable. (Or multiply both sides times the reciprocal of the coefficient.) If the coefficient is positive, divide by a positive number. If the coefficient is negative, divide by a negative number. The coefficient of the variable MUST be a positive one (+1) when you are finished.**

31. $8(x+2) - 7x = 3(x-2) + 2$

32. $-3(2 - x) + 2(3x + 5) = 31$

33. $4(2-3x) + 4(2x-3) = 4(x+1)$

34. $3(x - 6) - 5(x - 10) = 24$

35. $3x - 5(2x - 6) = 9(2 - x)$

36. $8(2x - 5) - 5(2 - x) = -4x$

Example 16. **Solve for x:** $-3(2 - x) - 2(3x + 5) = 38 - x$

Solution:

Step 1:	$-3(2 - x) - 2(3x + 5) = 38 - x$	Distributive property
Step 2:	$-6 + 3x - 6x - 10 = 38 - x$	Combine like terms
Step 3:	$-3x - 16 = 38 - x$	Variables to one side
	$\underline{+x \qquad\qquad +x}$	
Step 4:	$-2x - 16 = 38$	Number terms to other side
	$\underline{+16 \qquad +16}$	
Step 5:	$-2x = 54$	Divide by -2
	$x = -27$	

Example 17. **Solve for x:** $7 - (x + 4) = 8$ [Notice this is not $-7(x+4)$]

Solution: It may be helpful to re-write the equation

$7 - 1(x + 4) = 8$	Distributive property
$7 - x - 4 = 8$	Combine like terms
$3 - x = 8$	Add -3 to each side
$-x = 5$	Divide each side by -1
$x = -5$	

If an equation involves an x^2 term, it is called a quadratic equation, and it can't be solved in this section. However, if the x^2 term subtracts itself out, then you have a linear equation that can be easily solved.

Example 18. **Solve for x:** $x(x + 3) = x^2 - 5x - 16$

Solution:

$x(x + 3) = x^2 - 5x - 16$	Distributive property
$x^2 + 3x = x^2 - 5x - 16$	Add $-x^2$ to each side
$3x = -5x - 16$	Add $5x$ to each side
$\underline{+5x \qquad +5x}$	
$8x = -16$	Divide by 8
$x = -2$	

Sometimes the Solution, Like Real Life, Doesn't Come Out Even!

The variable in an equation can be any symbol or letter--it is not always *x*. Also, the answers in real life don't always come out even, as in the first examples of this section on equation solving. When expressing fractions, such as $-\frac{3}{5}$, remember that $-\frac{3}{5}$, $\frac{-3}{5}$, and $\frac{3}{-5}$ are all equivalent. The tradition in math is to avoid negative denominators, so the first two forms are preferred over the last. Remember that fractions such as $\frac{12}{5}$ in which the numerator is larger than the denominator are called improper fractions. Such answers can also be written as mixed fractions, in this case $2\frac{2}{5}$. In higher math, the improper fraction is much preferred. Mixed fractions are seldom used in algebra. Nevertheless, either form is usually acceptable. Of course, be careful to reduce the fraction completely. For example, $\frac{12}{8}$ is an improper fraction but it should be reduced. Notice that both numerator and denominator are divisible by 4. Dividing numerator and denominator by 4 gives $\frac{3}{2}$. You may also reduce the fraction $\frac{12}{8}$ by dividing the 8 into 12 to obtain $1\frac{4}{8}$. This reduces to $1\frac{1}{2}$ which is equivalent to $\frac{3}{2}$, so the result is the same!

Example 19. **Solve for y:** $7y = 30 - 5y$

Solution:

$7y = 30 - 5y$	Add +5y to each side.
$\underline{+5y \qquad + 5y}$	
$12y = 30$	Divide by 12.
$y = \frac{30}{12}$	Divide numerator and denominator by 6.
$y = \frac{5}{2}$ or $2\frac{1}{2}$ or 2.5	

Example 20. **Solve for p:** $p + 8 - 13(p - 4) = -2(p + 2) + 8p$

Solution:

$p + 8 - 13(p - 4) = -2(p + 2) + 8p$	Distributive property.
$p + 8 - 13p + 52 = -2p - 4 + 8p$	Combine like terms.
$-12p + 60 = 6p - 4$	Add -6p to each side.
$\underline{-6p \qquad -6p}$	
$-18p + 60 = \quad -4$	Add -60 to each side.
$\underline{-60 \qquad -60}$	
$-18p = -64$	Divide each side by -18.
$p = \frac{-64}{-18}$ or $\frac{32}{9}$	Divide numerator and denominator by 2.

EXERCISES. In 37 – 46, solve the equations for x.

37. $9 - 2(x + 4) = 17$

38. $7 - 5(2x - 8) = -13$

39. $9 + 2(x - 4) = -17$

40. $7 + 5(2x - 8) = -13$

41. $7 - (x + 4) = 8$

42. $4 - (x + 7) = 8$

43. $7 - (x - 4) = 8$

44. $7 - (4 - x) = 8$

45. $x(x + 3) = x^2 - 5x - 16$

46. $x(x - 3) = x^2 + 3x - 18$

In 47 - 58, solve for the variable. Reduce all fractions completely.

47. $7y = 15 - 3y$

48. $8d + 4 = 2d$

49. $3c - 5c = 9 + 4c$

50. $4z - (z - 8) = 0$

51. $5 - 3(f - 4) = 13$

52. $s - (3 - s) = 5 - (2s + 6)$

53. $p + 5 - 3(p - 4) = 2(p + 2)$

54. $b - 3 -(2b + 3) = 3b$

55. $b - 3 - (2b + 3) = 3b - 6$

56. $3(7 - 2q) = 14 - 8 (q - 1)$

57. $j(j + 3) = 4 - j(2 - j)$

58. $w(w + 2) - 15 = w(w - 2)$

Sometimes in solving equations, all the variables subtract out. For example, $\mathbf{4x = 4x + 8}$. If you add $\mathbf{-4x}$ to each side, the result is $\mathbf{0 = 8}$, or some such ridiculous result. Since 0 never can equal 8, you conclude that this equation has **NO SOLUTION**. This type of equation is called a **CONTRADICTION**.

Sometimes, as with the example **$4(x + 3) = 4 + 4(2 + x)$**, all the variables subtract out, and the result is **$12 = 12$**, which is a **TRUE** statement. Since 12 always equals 12, the equation is always true. The solution is "**all values of x.**"

EXERCISES: **Solve the equations for x. Identify which are contradictions, identities, or conditional equations.**

59. $4(x + 3) = 6(x - 2) - 2x$

60. $4(x + 3) = 6(x + 2) - 2x$

61. $4(x + 3) = 6(x - 2) - x$

62. $6(x + 3) - 4(5 - 2x) = 12x$

63. $6(x + 3) - 3(6 - 2x) = 12x$

64. $x(x - 6) = 4 - x(2 - x)$

ANSWERS TO EXERCISES

1. 10; **2.** 7; **3.** 5; **4.** 5; **5.** 2; **6.** 8; **7.** 11; **8.** -4; **9.** –2; **10.** -9;

11. -9; **12.** 3; **13.** -6; **14.** -5; **15.** -10; **16.** -4; **17.** 3; **18.** 6; **19.** 5; **20.** -5;

21. -9; **22.** 5; **23.** 9; **24.** 6; **25.** 9; **26.** 10; **27.** -7; **28.** -10; **29.** 9; **30.** 5;

31. 10; **32.** 3; **33.** -1; **34.** 4; **35.** -6; **36.** 2; **37.** -8; **38.** 6; **39.** -9; **40.** 2;

41. -5; **42.** -11; **43.** 3; **44.** 5; **45.** -2; **46.** 3; **47.** 3/2; **48.** –2/3; **49.** -3/2 or -1.5;

50. -8/3; **51.** 4/3; **52.** 1/2; **53.** 13/4; **54.** -3/2; **55.** 0; **56.** 1/2; **57.** 4/5; **58.** 15/4;

59. Contradiction, No Solution; **60.** Identity, All reals; **61.** Conditional Equation, X=24;

62. Conditional Equation, X=1; **63.** Identity, All reals; **64.** Contradiction, No Solution.