1.11 *Inequalities*

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P. 101-112

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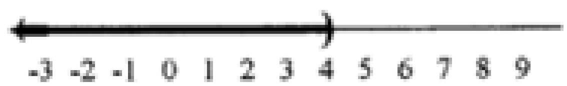
ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

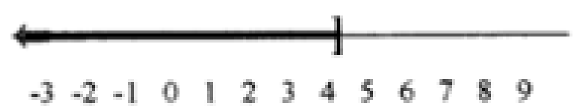
In the last few sections, you learned to solve equations and word problems that led you to solutions in the form of $x = 4$, $x = -2$, etc. However, instead of saying that the value of x is **"equal to 4,"** sometimes you may want to say that x represents a number that is **"less than 4"** (or smaller than 4). The symbol for **"x is less than 4"** is **"$x < 4$."** What this means is that x represents any number like 3, 2, 1, etc, (any whole number, fraction, or decimal) that is to the left of 4 on the number line. Notice that the number 4 is NOT included, since 4 is not less than itself. Because you can't possibly list all of the numbers (there are infinitely many of them), it is helpful to graph them on a number line, as follows. A **parentheses** opening to the left ")" is used at 4 to indicate that the 4 is not included, and then you shade the number line to the left of 4. An arrow is placed on the left end of the shading to indicate that this shading continues to infinity (forever!) to the left, as indicated below.

$x < 4$



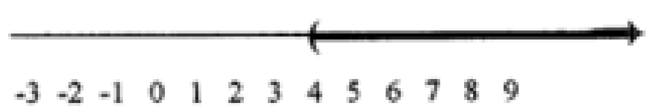
Sometimes you may want to let x represent any value that is **"less than or equal to 4."** This means that x represents any number on the number line to the left of 4, up to and including the value of 4. The symbol for **"x is less than or equal to 4"** is **"$x \leq 4$."** In this case, you want to include the value of 4, so a **bracket** "]" instead of parentheses is used on the number line at 4. Place a bracket opening to the left at 4, and then shade all values to the left of 4, with an arrow to the left as follows.

$x \leq 4$



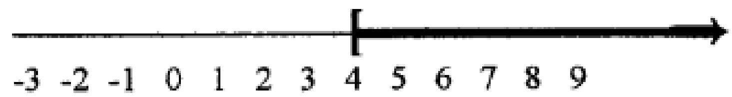
Similarly, if you want to let x represent any number to the right of 4 on the number line, but not including the 4, then you would say **"x is greater than 4,"** which is written **"$x > 4$."** On the number line, use a parentheses "(" at 4 opening to the right to indicate that the 4 is NOT included. Then shade to the right on the number line, as follows. The arrow to the right indicates that it continues to infinity on the right side.

$x > 4$



If you want to let x represent any number that is **"greater than or equal to 4,"** written **"$x \geq 4$,"** this means x represents any number to the right of 4 or including the 4 itself. On a number line, you place a bracket "[" at 4 opening to the right, and shade to the right 4 with an arrow to the right, as indicated.

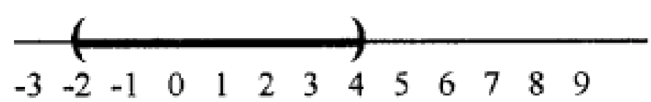
$x \geq 4$



Keep in mind that the expression **$x < 4$** means the same as **$4 > x$**, since if the value of x is smaller than 4, then 4 is larger than x.. Likewise, **$x \leq 4$** and **$4 \geq x$** are equivalent.

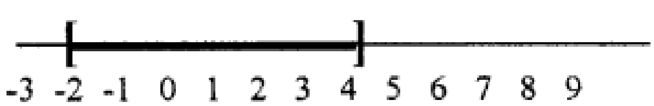
Sometimes two inequalities are placed together in what is called a **compound inequality**. For example, "**$-2 < x < 4$**" means that "**$-2 < x$** and at the same time **$x < 4$**." This means that the variable x represents a number that is **between -2 and 4**. To graph this on a number line, put a parentheses opening to the right at -2, and put a parentheses opening to the left at 4. Parentheses are used because you do NOT want to include the endpoints of -2 and 4. Shade between the endpoints. Notice that no arrows are used in this graph, since the shaded region does not continue to infinity in either direction. The graph is as follows.

$-2 < x < 4$

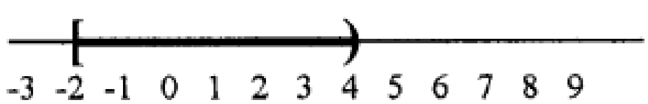


In compound inequalities, sometimes one or both endpoints are included. When the < or > symbols are used, the endpoints are NOT included, and parentheses "(" and/or ")" are used. Whenever the ≤ or ≥ symbols are used, the endpoints are included, and brackets "[" and/or "]" are used.

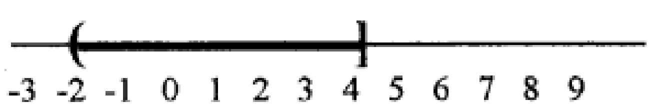
$-2 \leq x \leq 4$



$-2 \leq x < 4$



$-2 < x \leq 4$



Variable Notation **Graph on Number Line**

x < 2

x ≤ 2

x > 2

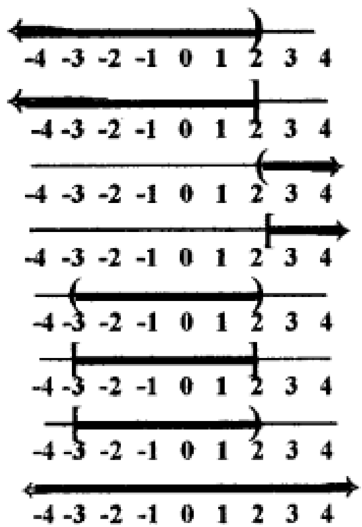
x ≥ 2

-3 < x < 2

-3 ≤ x ≤ 2

-3 ≤ x < 2

All Reals



Solving Inequalities

Inequalities, like equations, must often be solved. Remember that when you solved equations, according to the **addition rule for equations**, you were allowed to add (or subtract) the same number from each side of an equation. Also, according to the **multiplication rule for equations,** you were allowed to multiply (or divide) each side of an equation by the same non-zero number. Wouldn't it be nice if these same rules applied to inequalities as they do to equations? Well, it is almost that simple, but not quite. It will be necessary to modify the multiplication rule slightly.

Take the inequality -2 < 4, and add +2 to each side
+2 +2
0 < 6 Still true!

Use the same inequality -2 < 4, and add -2 to each side
-2 -2
-4 < 2 Still true!

Use the same inequality and multiply (or divide) both sides by +2.

-2 < 4
+2(-2)< +2(4)
-4 < 8 Still true!

Now try multiplying (or dividing) both sides by a negative, say -2.

$$-2 < 4$$

$$-2(-2) < -2(4)$$

$+4 > -8$ **NOTE: THE INEQUALITY MUST BE REVERSED!**

What was negative becomes positive, and what was positive becomes negative. The large become small, and the small become large! Everything is reversed! You can see that this is a principle that applies not only in this specific example, but in every case in which you multiply or divide both sides of an inequality by a **negative** number. The examples above verify (but of course they do **not prove!**) the following rules:

RULES FOR INEQUALITIES

Rule #1 **If any number is added or subtracted from each side of an inequality, then the inequality sign remains the same.**

Rule #2 **If each side of an inequality is multiplied or divided by a positive number, then the inequality sign remains the same.**

Rule #3 **If each side of an inequality is multiplied or divided by a negative number, then the inequality sign must be reversed.**

Note: The first two rules are just like the rules for solving equations. However, Rule #3 is a new and special rule that you need to remember. This rule applies later, again and again, in so many different areas of math. In fact, this rule is so important and useful, that it's a good idea to put a star beside it or highlight it somehow to make it stand out for future reference!!

In summary, remember that these are the same rules as the ones used to solve equations, except that **when you multiply or divide both sides of an inequality by a negative number, you must change the direction of the inequality sign**. Remember also that $x > 4$ means the same as $4 < x$. The expression $-2 < x < 6$ means that **x represents any number between -2 and 6**, and the expression $6 > x > -2$ means the same as $-2 < x < 6$, assuming of course that -2 is smaller than 6.

EXAMPLE 1. Solve $3x + 6 < 18$ and graph on a number line.

Solution:

$3x + 6 < 18$ Add - 6 to each side of the inequality (inequality remains the same).

$\underline{\quad -6 \quad -6}$

$\frac{3x}{3} < \frac{12}{3}$ Divide each side by 3 (inequality remains the same).

$x < 4$ Locate 4 on the number line, parentheses to the left, shade to left.

EXAMPLE 2. Solve $3x - 6 \leq 18$ and graph on a number line.

Solution:

$3x - 6 \leq 18$ Add + 6 to each side of the inequality (inequality remains the same).

$\underline{\quad +6 \quad +6}$

$\frac{3x}{3} \leq \frac{24}{3}$ Divide each side by 3 (inequality remains the same).

$x \leq 8$ Locate 8 on the number line, bracket to the left, shade to the left.

EXAMPLE 3. Solve $3x - 4 \geq -10$ and graph on a number line.

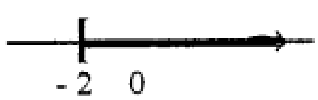
Solution:

$3x - 4 \geq -10$ Add + 4 to each side of the inequality (inequality remains the same).

$\underline{\quad +4 \quad +4}$

$\frac{3x}{3} \geq \frac{-6}{3}$ Divide each side by 3 (inequality remains the same).

$x \geq -2$ Locate - 2 on the number line, bracket to the right, shade to the right.



NOTE: **In the next examples, be careful--that all-important Rule #3 applies!!**

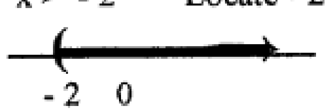
EXAMPLE 4. Solve $-3x < 6$ and graph on a number line.

Solution:

$-3x < 6$ Divide each side by - 3 **(the inequality sign must be reversed!)**

$\frac{-3x}{-3} < \frac{6}{-3}$

$x > -2$ Locate - 2 on the number line, parentheses to right, shade to right.

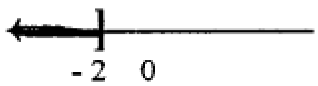


EXAMPLE 5. Solve $-3x \geq 6$ and graph on a number line.

Solution: $-3x \geq 6$ Divide each side by - 3 **(the inequality sign must be reversed!)**

$$\frac{-3x}{-3} \geq \frac{6}{-3}$$

$x \leq -2$ Locate - 2 on the number line, bracket to the left, shade to the left.

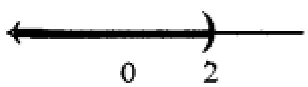


EXAMPLE 6. Solve $-3x > -6$ and graph on a number line.

Solution: $-3x > -6$ Divide each side by - 3 **(the inequality sign must be reversed!)**

$$\frac{-3x}{-3} > \frac{-6}{-3}$$

$x < 2$ Locate 2 on the number line, parenthese to the left, shade to the left.



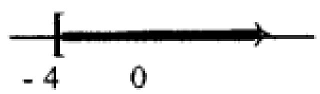
EXERCISES: **Solve each of the following inequalities and graph on a number line.**

1. $2x < 10$
2. $5x > -20$
3. $3x \geq -15$
4. $12x \leq 24$
5. $-2x < 10$
6. $-5x > -20$
7. $-3x \geq -15$
8. $-12x \leq 24$
9. $4x + 8 \geq 24$
10. $3x - 12 < 9$
11. $5x - 10 \leq -10$
12. $2x - 6 > -20$
13. $-4x + 8 \geq -24$
14. $-3x + 12 < 9$
15. $-5x - 10 \leq -10$
16. $-6x - 2 > -20$

EXAMPLE 7. Solve $7x + 12 \geq 5x + 4$ and graph on a number line.

Solution:

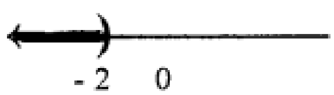
$$
\begin{array}{rcll}
7x + 12 & \geq & 5x + 4 & \text{Subtract 5x from (or add -5x to) each side.} \\
\underline{-5x} & & \underline{-5x} & \\
2x + 12 & \geq & +4 & \text{Subtract 12 from (or add - 12 to) each side.} \\
\underline{-12} & & \underline{-12} & \\
2x & \geq & -8 & \text{Divide each side by 2 } \textbf{(do NOT reverse the sign.)} \\
\frac{2x}{2} & \geq & \frac{-8}{2} & \\
x & \geq & -4 & \text{Bracket at -4 opening to the right, and shade to the right.}
\end{array}
$$



EXAMPLE 8. Solve $5x - 12 > 7x - 8$ and graph on a number line.

Solution:

$$
\begin{array}{rcll}
5x - 12 & > & 7x - 8 & \text{Subtract 7x from each side.} \\
\underline{-7x} & & \underline{-7x} & \\
-2x - 12 & > & -8 & \text{Add 12 to each side.} \\
\underline{+12} & & \underline{+12} & \\
-2x & > & 4 & \text{Divide each side by -2 } \textbf{(DO reverse the inequality sign.)} \\
\frac{-2x}{-2} & > & \frac{4}{-2} & \\
x & < & -2 & \text{Parentheses at -2 opening to the left, and shade to the left.}
\end{array}
$$



EXERCISES: **Solve each of the following inequalities and graph on a number line.**

17. $5 + 3x \leq x - 3$

18. $3x + 4 > x - 6$

19. $5 - 3x < x - 3$

20. $-3x + 15 \geq 3x - 3$

21. $5(x - 3) \geq x + 9$

22. $-3x + 4 > -4(x - 3)$

23. $x - (9 + 3x) > x - 3$

24. $-3(x - 1) \geq -2 - (3 - x)$

The rest of the examples and exercises in this section involve compound inequalities, sometimes called "double inequalities" like **$a < b < c$**. With these double inequalities, there are a couple of requirements. First, the inequality signs must point in the same direction. You are NOT allowed to write $a < b > c$ or $a > b < c$. The second requirement is best explained by describing a family with three children in which a=Alice (the youngest), b=Betty (the middle), and c=Carol (the oldest child). Now if Alice is younger than Betty, and Betty is younger than Carol, then Alice must be younger than Carol. In math symbols this is called the **transitive property: If $a < b$ and $b < c$, then $a < c$.**

The transitive property is also valid for "=", ">", "≤", and "≥" as follows:

If $a = b$ and $b = c$, then $a = c$. **If $a > b$ and $b > c$, then $a > c$.**

If $a \leq b$ and $b \leq c$, then $a \leq c$. **If $a \geq b$ and $b \geq c$, then $a \geq c$.**

EXERCISE: Identify each of the following as valid or invalid notations.

25a) $-2 < x > 6$ ____________

b) $-2 > x < 6$ ____________

c) $-2 < x < 6$ ____________

d) $-6 > x > 2$ ____________

e) $6 > x > -2$ ____________

f) $6 < x < -2$ ____________

g) $-2 < x > 6$ ____________

h) $-2 > x > 6$ ____________

i) $-8 > x > -2$ ____________

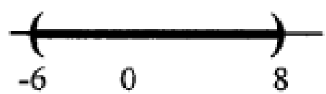
j) $-2 > x > -8$ ____________

You remember in solving equations and inequalities, you can add or subtract a number from each side of the equation. Now in solving double inequalities in the form $a < b < c$, remember that there are three parts to the inequality: the left side, the middle portion, and the right side. It's like when there are three children in the family. If you do something for one, then you must do the same thing for all of the children. Likewise, whatever you do (add, subtract, multiply, or divide) to one of the three parts of the inequality, you must do the same to each of the three parts. The following examples will illustrate:

EXAMPLE 9. Solve $-2 < x + 4 < 12$ and graph on a number line.

Solution: The objective is to isolate the variable x. This means that you must eliminate the +4 You can undo the +4 by adding - 4 to each of the three parts. It looks like this:

$$\begin{array}{ccccc} -2 & < & x+4 & < & 12 \\ \underline{-4} & & \underline{-4} & & \underline{-4} \\ -6 & < & x & < & 8 \end{array}$$

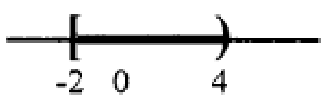


EXAMPLE 10. Solve $-6 \le 3x < 12$ and graph on a number line.

Solution: In this case, the variable x is multiplied by 3. To undo multiplication by 3, you need to divide each of the three parts by 3 as follows:

$$\frac{-6}{3} \le \frac{3x}{3} < \frac{12}{3}$$

$$-2 \le x < 4$$



EXERCISES: **Solve each of the following inequalities and graph on a number line.**

26. $-2 < x + 5 < 8$

27. $-4 \le x - 6 < 1$

28. $-5 \le x - 5 \le 0$

29. $-12 \le 6x \le 12$

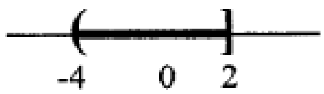
30. $-6 < 2x < 0$

31. $0 < 4x \le 20$

EXAMPLE 11. Solve $-5 < 2x + 3 \le 7$ and graph on a number line.

Solution: The variable x was multiplied by 2 and then 3 was added. These operations must be "undone" in reverse order. Therefore, first undo the + 3 with a -3, then divide by 2. Remember, whatever you do, you must do it in all three places.

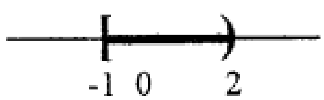
$$\begin{array}{ccccc} -5 & < & 2x+3 & \le & 7 \\ \underline{-3} & & \underline{-3} & & \underline{-3} \\ -8 & < & 2x & \le & 4 \\ \frac{-8}{2} & < & \frac{2x}{2} & \le & \frac{4}{2} \\ -4 & < & x & \le & 2 \end{array}$$



EXAMPLE 12. Solve $-1 < 3 - 2x \le 5$ and graph on a number line.

Solution: In this example, you must first "undo" the 3 with a "-3," then divide by - 2. Of course, this involves Rule #3 since you are dividing each part by a negative number. Don't forget to reverse both of the inequality signs.

$$\begin{array}{ccccc} -1 & < & 3-2x & \le & 5 \\ \underline{-3} & & \underline{-3} & & \underline{-3} \\ -4 & < & -2x & \le & 2 \\ \frac{-4}{-2} & < & \frac{-2x}{-2} & \le & \frac{2}{-2} \\ 2 & > & x & \ge & -1 \end{array} \quad \text{or} \quad -1 \le x < 2$$



32. $-7 \le 2x + 3 \le 5$

33. $-7 \le 2x - 3 \le 5$

34. $-5 < 2x + 3 < -1$

35. $-7 \le 3 - 2x < 5$

36. $-5 < 3 - 2x \le -1$

EXAMPLE 13. $-5 \le \dfrac{2x + 3}{3} \le 7$

Solution: In this example, three operations were performed on the variable x. First, the x was multiplied by 2, secondly the 3 was added, and last, the entire quantity was divided by 3. Remember, you must undo these operations in reverse order. Therefore, begin by multiplying each of the three parts times 3. Secondly, undo the "+3" with a "- 3." Then last, undo the multiplication by dividing each part by 2, as follows.

$3 \cdot -5 \le 3\left(\dfrac{2x + 3}{3}\right) \le 3 \cdot 7$ Multiply by 3, dividing out the denominator.

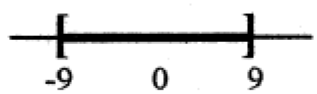
$$\begin{array}{rcccl} -15 & \le & 2x+3 & \le & 21 \\ \underline{-3} & & \underline{-3} & & \underline{-3} \\ -18 & \le & 2x & \le & 18 \end{array}$$

Undo the +3 with a - 3.

Divide each side by 2.

$$\frac{-18}{2} \le \frac{2x}{2} \le \frac{18}{2}$$

$$-9 \le x \le 9$$



EXAMPLE 14. $-1 < \dfrac{3 - 2x}{5} \le 5$

Solution: Again, three operations were performed on the variable x. First, the x was multiplied by -2, secondly the 3 was added, and last, the entire quantity was divided by 5. Remember, you must undo these operations in reverse order. Therefore, begin by multiplying each of the three parts times 5. Secondly, undo the "+3" with a "- 3." Then last, undo the multiplication by dividing each part by -2, as follows.

$5 \cdot -1 < 5\left(\dfrac{3 - 2x}{5}\right) \le 5 \cdot 5$ Multiply by 5, dividing out the denominator.

$$\begin{array}{rcccl} -5 & < & 3-2x & \le & 25 \\ \underline{-3} & & \underline{-3} & & \underline{-3} \\ -8 & < & -2x & \le & 22 \end{array}$$

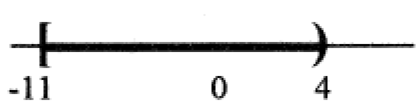
Undo the +3 with a - 3.

Divide each side by -2.
Remember Rule #3, reverse the signs!

$$\frac{-8}{-2} < \frac{-2x}{-2} \le \frac{22}{-2}$$

$$4 > x \ge -11$$

This represents all values of x between -11 and 4, including the -11.
It can also be written $-11 \le x < 4$, and graphed on the number line as follows.



37. $-7 \le \frac{2x + 3}{3} < 5$

38. $-5 < \frac{2x - 1}{3} < -1$

39. $-3 \le \frac{3 - x}{2} < 5$

40. $-1 < \frac{5 - 2x}{3} < 3$

41. $-3 \le \frac{-3x + 6}{2} \le 6$

42. $-9 < \frac{3 - x}{2} \le 6$

ANSWERS 1.11

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1. $x<5$; **2.** $x>-4$; **3.** $x\geq-5$; **4.** $x\leq2$; **5.** $x>-5$; **6.** $x<4$; **7.** $x\leq5$; **8.** $x\geq-2$; **9.** $x\geq4$; **10.** $x<7$; **11.** $x\leq0$; **12.** $x>-7$; **13.** $x\leq8$; **14.** $x>1$; **15.** $x\geq0$; **16.** $x<3$; **17.** $x\leq-4$; **18.** $x>-5$; **19.** $x>2$; **20.** $x\leq3$; **21.** $x\geq6$; **22.** $x>8$; **23.** $x<-2$; **24.** $x\leq2$; **25.** Only c, e, and j are valid; **26.** $-7<x<3$; **27.** $2\leq x<7$; **28.** $0\leq x\leq5$; **29.** $-2\leq x\leq2$; **30.** $-3<x<0$; **31.** $0<x\leq5$; **32.** $-5\leq x\leq1$; **33.** $-2\leq x\leq4$; **34.** $-4<x<-2$; **35.** $5\geq x>-1$ or $-1<x\leq5$; **36.** $4>x\geq2$ or $2\leq x<4$; **37.** $-12\leq x<6$; **38.** $-7<x<-1$; **39.** $9\geq x>-7$ or $-7<x\leq9$; **40.** $4>x>-2$ or $-2<x<4$; **41.** $4\geq x\geq-2$ or $-2\leq x\leq4$; **42.** $21>x\geq-9$ or $-9\leq x<21$.