2.01 *Products of Polynomials*

Dr. Robert J. Rapalje

ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

Recall from previous lessons that when algebraic expressions are added (or subtracted) they are called **terms**, while expressions that are multiplied are called **factors**. An algebraic expression that contains only **one term** is called a **monomial**. If the expression has **two terms**, it is called a **binomial**, and if there are **three terms,** it is a **trinomial.** A **polynomial** is an algebraic expression consisting of one or more terms. A polynomial may consist of numbers and variables, where the numerical part of a given term is called the **coefficient**. If there is only one variable in the polynomial, such as **x**, then it is called a **polynomial in x**. The **degree** (or **order)** of a polynomial in one variable is the highest exponent of the variable. If there is more than one variable in the polynomial, then the **degree** (or **order**) is the highest "sum of the exponents" of the variables of a given term.

Frequently polynomials can be simplified by combining like terms; sometimes they can be **factored**. Polynomials can be added, subtracted, multiplied **(expanded)**, or divided. Since addition and subtraction of polynomials is little more than combining like terms, and division of polynomials is saved for Chapter 2, this section will involve only the **multiplication (expansion) of polynomial expressions**. The next section is the factoring of polynomial expressions, followed immediately by solving quadratic equations by factoring. Notice that **polynomial expressions** are not equations, and therefore cannot be "solved." This chapter involves only polynomial expressions.

This explanation will begin with a review of products (monomial times various polynomials) from Chapter 1. Then we will do a **binomial times binomial, binomial times trinomial, and trinomial times trinomial.** The basic property that underlies these products is the **distributive property for multiplication** (products) **over addition** (two or more terms)**.**

Monomial Times a Polynomial

A monomial may be multiplied times a polynomial by simply using the **distributive property**.

EXAMPLE 1. **Monomial times monomial**

a) $(3x) \cdot (4x)$ b) $(3x) \cdot (4x^2)$ c) $(3x^2) \cdot (4x^2)$

Solution:

a) $(3x) \cdot (4x) = 12x^2$ b) $(3x) \cdot (4x^2) = 12x^3$ c) $(3x^2) \cdot (4x^2) = 12x^4$

EXAMPLE 2. **Monomial times binomial (using the distributive property!)**

a) $(3x) \cdot (4x + 3)$ b) $(-3x^2) \cdot (4x - 3)$

Solution:

a) $(3x) \cdot (4x + 3)$

$12x^2 + 9x$

b) $(-3x^2) \cdot (4x - 3)$

$-12x^3 + 9x^2$

EXAMPLE 3. **Monomial times trinomial**

a) $(6x) \cdot (5x^2 - 7x + 9)$ b) $(-6x^2) \cdot (5x^2 - 7x + 9)$

Solution:

a) $(6x) \cdot (5x^2 - 7x + 9)$

$30x^3 - 42x^2 + 54x$

b) $(-6x^2) \cdot (5x^2 - 7x + 9)$

$-30x^4 + 42x^3 - 54x^2$

EXERCISES: **Multiply the polynomials.**

1. $(2x) \cdot (7x)$
2. $(5x) \cdot (9x^2$
3. $(3x^2) \cdot (8x^3)$
4. $(3x) \cdot (4x + 9)$
5. $(7x^2) \cdot (2x - 7)$
6. $(-4x^2) \cdot (9x^2 - 5)$

7. $(2x)(3x^2 + 9x - 6)$

8. $(8x^3)(9x^2 - 6x - 8)$

9. $(-5x^3)(-3x^2 - 12x + 5)$

10. $(-9x^4)(6x^2 + 7x + 9)$

Binomial Times a Binomial

One of the most frequent products in all of mathematics is the product of two binomials. In order to multiply two binomials, one method is to make a substitution of some other variable in place of the first binomial. Then, as illustrated in the next example, the distributive property can be applied. The results of this can be summarized in what is known as the **F O I L** method, which is explained in the second part of Example 4.

EXAMPLE 4a) Consider the example: $(x + 2)(x + 3)$.

Solution: You could substitute $\mathbf{y} = (x + 2)$.

Now the example reads
$$\mathbf{y}\bullet(x+3) = x\bullet\mathbf{y} + 3\bullet\mathbf{y}$$
$$= x(x+2) + 3(x+2)$$
$$= x^2 + 2x + 3x + 6$$
$$= x^2 + 5x + 6.$$

The easier way to multiply binomials is known as the **F OI L** method.

F = First times first

O = Outer times outer These are usually (but not always!) like terms,

I = Inner times inner in which case, they are combined.

L = Last times last

EXAMPLE 4b) **Multiply** $(x + 2)(x + 3)$ **by the F OI L method.**

Solution:
$$(x+2)(x+3) = \overset{\textbf{F}}{x\bullet x} + \overset{\textbf{O}}{3\bullet x} + \overset{\textbf{I}}{2\bullet x} + \overset{\textbf{L}}{2\bullet 3}$$
$$= x^2 + \underline{3x} + \underline{2x} + 6$$
$$= x^2 + 5x + 6$$

EXAMPLE 5. **Multiply** $(x + 6)(x - 3)$ **by the F OI L method.**

$$\begin{array}{llllll} & & F & O & I & L \\ \textbf{Solution:} & (x + 6)(x - 3) = & x \cdot x & + (-3) \cdot x & + 6 \cdot x & + 6 \cdot (-3) \\ & = & x^2 & - \underline{3x} & + \underline{6x} & - 18 \\ & = & x^2 & + 3x & & - 18 \end{array}$$

EXERCISES: F O I L

11. $(x + 5)(x + 2)$ = _____ + _____ + _____ + _____

= _____ + _____ + _____

12. $(x + 3)(x + 7)$ = _____ + _____ + _____ + _____

= _____ + _____ + _____

13. $(x + 5)(x + 9)$ = _____ + _____ + _____ + _____

= _____ + _____ + _____

14. $(x + 4)(x + 8)$ = _____ + _____ + _____ + _____

= _____ + _____ + _____

15. $(x + 6)(x - 4)$ = ____________________________

= ____________________________

16. $(x - 6)(x + 4)$ = ____________________________

= ____________________________

At some point that this process becomes comfortable to you, you may wish to combine like terms and do the exercise in one step instead of two steps.

F O I L

17. $(x - 6)(x - 3)$ = ______________________________
= ___________________________

18. $(x + 7)(x - 5)$ = ______________________________
= ___________________________

19. $(x - 7)(x + 5)$ = ______________________________
= ___________________________

20. $(x - 9)(x - 6)$ = ______________________________
= ___________________________

What happens when you take the product of two binomials that are the same but with the opposite sign in the middle? Use these illustrations to demonstrate.

	EXAMPLE 6.	**EXAMPLE 7.**	**EXAMPLE 8.**
	$(x - 5)(x + 5)$	$(x - 7)(x + 7$	$(4x - 9)(4x + 9)$
Solutions:	$x^2 + 5x - 5x - 25$	$x^2 + 7x - 7x - 49$	$6x^2 + 36x - 36x - 81$
	$x^2 - 25$	$x^2 - 49$	$16x^2 - 81$

21. $(x–3)(x+3)$ = ___________________
= ___________________

22. $(x–4)(x+ 4)$ = ___________________
= ___________________

23. $(x –5)(x+5)$ = ___________________

24. $(x–6)(x + 6)$ = ___________________

25. $(x–7)(x+7)$ = ___________________

26. $(x–8)(x + 8)$ = ___________________

27. $(x–9)(x+9)$ = ___________________

28. $(x–10)(x +10)$ = _________________

29. $(3x - 4)(3x + 4) =$

30. $(5x - 3)(5x + 3) =$

31. $(5x - 3y)(5x + 3y) =$

32. $(5x - 12y)(5x + 12y) =$

EXAMPLE 9. $(3x - 7y)(2x + 4y)$

F O I L

Solution: $(3x - 7y)(2x + 4y) = 3x \bullet 2x + 3x \bullet 4y - 2x \bullet 7y - 7y \bullet 4y$

$= 6x^2 + \underline{12xy} - \underline{14xy} - 28y^2$

$= 6x^2 - 2xy - 28y^2$

EXAMPLE 10. $(3x - 7y)(4x - 9y)$

F O I L

Solution: $(3x - 7y)(4x - 9y) = 3x \bullet 4x - 3x \bullet 9y - 4x \bullet 7y + 7y \bullet 9y$

$= 12x^2 - \underline{27xy} - \underline{28xy} + 63y^2$

$= 12x^2 - 55xy + 63y^2$

EXERCISES: F O I L

33. $(3x + 4)(4x - 3) =$ ______________________________

= __________________________

34. $(3x - 4)(4x + 3) =$ ______________________________

= __________________________

35. $(3x - 4y)(4x - 3y) =$

36. $(5x + 3y)(4x + 5y) =$

37. $(5x + 3y)(4x - 5y) =$

38. $(5x - 3y)(4x - 5y) =$

How would you square a binomial? For example, what if you wanted to find $(x + 3)^2$? Remember, a quantity squared means the quantity times itself. This means that $(x + 3)^2 = (x + 3)(x + 3)$, which you already learned to do using **FOIL**.

EXAMPLE 11. $(x + 3)^2$

Solution: $(x + 3)(x + 3)$

$x^2 + 3x + 3x + 9$

$x^2 + 6x + 9$

EXAMPLE 12. $(x - 7)^2$

Solution: $(x - 7)(x - 7)$

$x^2 - 7x - 7x + 49$

$x^2 - 14x + 49$

EXERCISES:

39. $(x + 5)^2$

( )( )

____

40. $(x - 5)^2$

( )( )

____

41. $(x - 8)^2$

( )( )

____

42. $(x + 8)^2$

( )( )

____

43. $(\boldsymbol{x} + 9)^2$

44. $(\boldsymbol{x} - 12)^2$

45. $(\boldsymbol{x} - 13)^2$

46. $(\boldsymbol{x} + 13)^2$

47. When you square a binomial, such as $(\boldsymbol{x} + 9)^2$ or $(\boldsymbol{x} - 12)^2$, how can you quickly determine the middle term?

48. When you square a binomial, how can you tell the sign of the middle term?

EXTRA CHALLENGE:

49. $(5\,\boldsymbol{x} + 49)^2$

50. $(8\,\boldsymbol{x} - 42)^2$

Polynomial Times a Polynomial

Consider now the problem of multiplying a binomial times a trinomial, a trinomial times a trinomial, or in general, a polynomial times a polynomial. To multiply a binomial times a trinomial, you must multiply the **first** (of the binomial) times each term of the trinomial. Then take the **second** (of the binomial) times the trinomial. The next example illustrates the process.

EXAMPLE 13. **Multiply the binomial times the trinomial:** $(x + 2)(x^2 + 4x + 5)$

Solution: Multiply **x** times each term of the trinomial, then multiply **2** times each term.

$$(x + 2)(x^2 + 4x + 5) = x \bullet x^2 + x \bullet 4x + x \bullet 5 \qquad \textbf{First times}$$

$$\underline{+ 2 \bullet x^2 + 2 \bullet 4x + 2 \bullet 5} \qquad \textbf{Second times}$$

$$= x^3 + 4x^2 + 5x \qquad \textbf{First times}$$

$$\underline{+ 2x^2 + 8x + 10} \qquad \textbf{Second times}$$

$$= x^3 + 6x^2 + 13x + 10 \qquad \textbf{Combine like terms}$$

EXAMPLE 14. **Multiply the binomial times the trinomial:** $(2x – 3)(x^2 – 6x + 4)$

Solution: **Multiply the 2x times each term of the trinomial. Multiply –3 times each term.**

$$(2x - 3)(x^2 - 6x + 4) = 2x^3 - 12x^2 + 8x \qquad \textbf{First times}$$

$$\underline{- 3x^2 + 18x - 12} \qquad \textbf{Second times}$$

$$= 2x^3 - 15x^2 + 26x - 12 \qquad \textbf{Combine like terms}$$

EXERCISES: **Multiply the binomial times the trinomial.**

51. $(x + 3)(x^2 + 3x + 5) =$

52. $(x – 5)(x^2 + 7x + 6)$

53. $(2x - 5)(3x^2 - 4x + 6)$

54. $(4x - 3)(5x^2 - 6x - 8)$

Multiplying a trinomial times a trinomial is an easy extension to what you have just done. In the case of a trinomial times a trinomial, multiply the **first** term in the first trinomial times each term in the second trinomial. Then multiply the **second** term in the first trinomial times each term in the second trinomial. Last, multiply the **third** term in the first trinomial times each term in the second trinomial. There should be a total of nine terms. Finally, combine like terms as before. The next example illustrates this process.

EXAMPLE 15. **Multiply the trinomials: $(x^2 - 6x + 4)(x^2 + 4x + 8)$.**

Solution: $(x^2 - 6x + 4)(x^2 + 4x + 8)$

$$
\begin{array}{lrrrrrl}
= & x^4 & +4x^3 & +8x^2 & & & \textbf{First} \\
 & & -6x^3 & -24x^2 & -48x & & \textbf{Second} \\
 & & & \underline{+4x^2} & \underline{+16x} & \underline{+32} & \textbf{Third} \\
= & x^4 & -2x^3 & -12x^2 & -32x & +32 &
\end{array}
$$

EXERCISES: **Multiply the trinomials.**

55. $(x^2 + 6x - 4)(x^2 + 4x + 8)$

56. $(x^2 + 3x - 5)(x^2 - 2x - 9)$

57. $(x^2 - 5x - 8)(x^2 - x - 6)$

58. $(2x^2 - 3x - 6)(x^2 - 5x - 4)$

59. $(3x^2 - 7x + 6)(2x^2 + 6x - 4)$

60. $(3x^2 - 7x - 4)(5x^2 - 4x - 6)$

QUESTION: **How would you find the cube of a binomial? Consider:**

EXAMPLE 16. $(x + 2)^3$ **[Note: Does $(x + 2)^3 = x^3 + 2^3$??]**

Solution: **Rewrite $(x + 2)^3$ as $(x + 2)(x + 2)(x + 2)$**

$$(x + 2)(x^2 + 4x + 4)$$

$$\begin{array}{r} x^3 + 4x^2 + 4x \qquad \\ \underline{\quad + 2x^2 + 8x + 8} \\ x^3 + 6x^2 + 12x + 8 \end{array}$$

No! $(x + 2)^3 \neq x^3 + 2^3$!

EXERCISES. **Use the method of the previous example to find the cubes of the binomials.**

61. $(x + 3)^3$

62. $(x - 3)^3$

63. $(x - 5)^3$

64. $(x + 5)^3$

ANSWERS 2.01

p. 122–132:

1. $14x^2$; **2.** $45x^3$**;** **3.** $24x^5$; **4.** $12x^2+27x$; **5.** $14x^3–49x^2$; **6.** $–36x^4+20x^2$**;**
7. $6x^3+18x^2–12x$; **8.** $72x^5–48x^4–64x^3$; **9.** $15x^5+60x^4–25x^3$; **10.** $–54x^6–63x^5–81x^4$;
11. $x^2+7x+10$; **12.** $x^2+10x+21$; **13.** $x^2+14x+45$; **14.** $x^2+12x+32$; **15.** $x^2+2x–24$;
16. $x^2–2x–24$; **17.** $x^2–9x+18$; **18.** $x^2+2x–35$; **19.** $x^2–2x–35$; **20.** $x^2–15x+54$;
21. $x^2–9$; **22.** $x^2–16$; **23.** $x^2–25$; **24.** $x^2–36$; **25.** $x^2–49$; **26.** $x^2–64$; **27.** $x^2–81$;
28. $x^2–100$; **29.** $9x^2–16$; **30.** $25x^2–9$; **31.** $25x^2–9y^2$; **32.** $25x^2–144y^2$;
33. $12x^2+7x–12$; **34.** $12x^2–7x–12$; **35.** $12x^2–25xy+12y^2$; **36.** $20x^2+37xy+15y^2$;
37. $20x^2–13xy–15y^2$; **38.** $20x^2–37xy+15y^2$; **39.** $x^2+10x+25$; **40.** $x^2–10x+25$;
41. $x^2–16x+64$; **42.** $x^2+16x+64$; **43.** $x^2+18x+81$; **44.** $x^2–24x+144$; **45.** $x^2–26x+169$;
46. $x^2+26x+169$; **47.** Twice the product of the first times the second;
48. "+" gives "+", "–" gives "–"; 4**9.** $25x^2+490x+2401$; **50.** $64x^2–672x+1764$;
51. $x^3+6x^2+14x+15$; **52.** $x^3+2x^2–29x–30$; **53.** $6x^3–23x^2+32x–30$;
54. $20x^3–39x^2–14x+24$; **55.** $x^4+10x^3+28x^2+32x–32$; **56.** $x^4+x^3–20x^2–17x+45$;
57. $x^4–6x^3–9x^2+38x+48$; **58.** $2x^4–13x^3+x^2+42x+24$;
59. $6x^4+4x^3–42x^2+64x–24$; **60.** $15x^4–47x^3–10x^2+58x+24$; **61.** $x^3+9x^2+27x+27$;
62. $x^3–9x^2+27x–27$; **63.** $x^3–15x^2+75x–125$; **64.** $x^3+15x^2+75x+125$.

Dr. Robert J. Rapalje, Retired
Central Florida, USA