2.07 *Factoring by Grouping/ Difference and Sum of Cubes*

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This lesson introduces the technique of factoring by grouping, as well as factoring the sum and difference of cubes. Factoring by grouping builds on the ideas that were presented in the section on factoring the common factor. While there are many different types of grouping, as you will learn in higher algebra courses, all of the grouping problems in this book involve four terms, and they work by grouping the first two terms and the second two terms together. If you do it right, **a common factor will always emerge!** Remember, the method of grouping is one of trial and error. As always, there is no substitute for practice and experience. The second half of this lesson is the sum and difference of cubes, which may be optional to your curriculum this first level of algebra. Remember, if you can learn this topic now, it will help you later.

FACTORING BY GROUPING

EXAMPLE 1. **Factor $x^3 + 2x^2 + 8x + 16$**

Solution: There are no common factors to all four terms. It is not a trinomial, and nothing discussed so far works to factor this. So, try grouping the first two terms together, and the last two terms together, and factor out the common factor within each grouping as follows: $(x^3 + 2x^2) + (8x + 16) = x^2(x + 2) + 8(x + 2)$

Notice that there is a common factor of $(x+2)$ that can be factored out:

$$= (x + 2)(x^2 + 8)$$

EXAMPLE 2. **Factor $xy - 4y + 3x - 12$**

Solution: Again, there are no common factors, and this is not a trinomial. Group the first two and the last two terms together, and factor out the common factor from each grouping:

$$(xy - 4y) + (3x - 12) = y(x - 4) + 3(x - 4)$$

Now, take out the common factor, which is $(x - 4)$:

$$= (x - 4)(y + 3)$$

EXAMPLE 3. **Factor $xy - 4y - 3x + 12$**

Solution: Group the first two and the last two terms together, and factor out the common factor from each grouping: $(xy - 4y) + (-3x + 12) = y(x - 4) + 3(-x + 4)$

This time there is no common factor. Try again, this time factoring a -3 from the last grouping. This works! $xy - 4y - 3x + 12 = y(x - 4) - 3(x - 4)$

$= (x - 4)(y - 3)$

EXAMPLE 4. Factor $xy - 4y + 3x + 12$

Solution: Group the first two and the last two terms:

$xy - 4y + 3x + 12 = y(x - 4) + 3(x + 4)$

At this point, it is important to realize that no common factor resulted. Do not try to factor out something that is not common to both groupings. In fact, there is no way to group this problem to get a common factor. This one cannot be factored by grouping. In fact, it can't be factored by any method. Remember, not all problems can be factored!

Remember that the entire process of "**grouping**" is one of trial and error, and, as you will see later, there are different types of grouping.

EXERCISES. **Factor each of the following by grouping:**

1. $xy + 7x + 4y + 28$

$= x(\qquad) + 4(\qquad)$

$= (\qquad)(\_\_ + \_\_)$

2. $2xy + 5x + 10y + 25$

3. $x^3 + 3x^2 + 9x + 27$

4. $x^3 - 3x^2 + 9x - 27$

5. $ax + bx + ay + by$

6. $ax + ac + bx + bc$

7. $ax - bx + ay - by$

8. $ax - ac + bx - bc$

9. $ax - bx - ay + by$

$= x(a - b) - y(a - b)$

$= (\quad)(\quad)$

10. $ax - ac - bx + bc$

$= \_\_\_(\quad) - \_\_\_(\quad)$

$= (\quad)(\quad)$

11. $xy - 5x - 2y + 10$

$=$

$=$

12. $x^2y + xy^2 - 5x - 5y$

$= xy(\quad) - 5(\quad)$

$=$

13. $x^3 - x^2 - 9x + 9$

$= x^2(\quad) - 9(\quad)$

$= (\quad)(\quad)$

$= (\quad)(\quad)(\quad)$

14. $x^3 - 5x^2 - 4x + 20$

$= x^2(\quad) - 4(\quad)$

$=$

$=$

15. $x^3 + 7x^2 - x - 7$

$= x^2(\quad) - 1(\quad)$

$=$

$=$

16. $x^3 - 5x^2 + 25x - 125$

$=$

$=$

Does $x^2 + 25$ factor? No!!

17. $x^3 + 5x^2 - 25x - 125$

18. $x^3 - 5x^2 - 25x + 125$

19. $x^3 - 4x^2 + 9x - 36$

20. $x^3 + 4x^2 - 9x - 36$

21. $x^3 - 4x^2 - 9x + 36$

22. $x^3 - 9x^2 - 4x + 36$

23. $x^3 - 8x^2 - x + 8$

24. $x^3 - 8x^2 + 4x - 32$

SUM AND DIFFERENCE OF CUBES (Optional−−Ask instructor!)

In this section, formulas and procedures will enable you to factor expressions in the form $x^3 - y^3$ and also $x^3 + y^3$. Recall from previous sections that $x^2 - y^2 = (x - y)(x + y)$ and that $x^2 + y^2$ cannot be factored. Begin with the multiplication problems:

$$
\begin{aligned}
(x - y)(x^2 + xy + y^2) &= x^3 + x^2y + xy^2 \\
&\quad \underline{ - x^2y - xy^2 - y^3} \\
&= x^3 - y^3 \\
(x + y)(x^2 - xy + y^2) &= x^3 - x^2y + xy^2 \\
&\quad \underline{ + x^2y - xy^2 + y^3} \\
&= x^3 + y^3.
\end{aligned}
$$

This derives the formulas, known as the **sum and difference of cubes formulas**.

SUM AND DIFFERENCE OF CUBES FORMULAS

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Translated into words, this means the sum or difference of two cubes can be factored into the product of a binomial times a trinomial. Begin by taking the cube root of the perfect cubes.

In the **difference formula**, the binomial is "**the first** **minus** **the second**". Then use this binomial to **build the trinomial** that follows: take the "**square of the first**" **plus** the "**product of the first and second**" **plus** the "**square of the second**."

The **sum formula** is similar, in that the formula begins with the binomial that is "**the first** **plus** **the second**." Then use this binomial to **build the trinomial** that follows, which except for one sign, is the same as the difference formula: take the "**square of the first**" **minus** the "**product of the first and second**" **plus** the "**square of the second**."

These formulas are really easy to remember. Notice that the $x^3 - y^3$ formula begins with $(x - y)$. The $x^3 + y^3$ formula begins with $(x + y)$. Next, notice that the trinomial factor in both formulas is the same except for one sign. This trinomial factor in each formula involves the "**square of the first**," the "**product of the two**," and the "**square of the second**." The first sign in the trinomial is the opposite of the sign of the binomial, and the last sign is always positive. Finally, you will *never* be able to factor the resulting trinomial (by ordinary trinomial methods), so you need not even try (at this level).

Before getting into the exercises, be sure to be familiar with the perfect cubes:

$$1^3 = 1;\ 2^3 = 8;\ 3^3 = 27;\ 4^3 = 64; \text{ and } 5^3 = 125.$$

Be able to recite these from memory: **1, 8, 27, 64, 125**. **Know and recognize these numbers!!**

EXERCISES. Factor completely, using the sum and difference of cubes formulas.

25. $x^3 - 8$

$= x^3 - 2^3$ [x=first; 2=second]

$= (\_\_ - \_\_)(x^2 + 2x + 2^2)$

$= (\qquad)(\qquad\qquad)$

26. $x^3 - 125$

$= (\quad)^3 - (\quad)^3$

$= (\_\_ - \_\_)(\quad + \quad + \quad)$

$= (\qquad)(\qquad\qquad)$

27. $x^3 - 64$

$= (\quad)^3 - (\quad)^3$

$= (\qquad)(\qquad\qquad)$

28. $x^3 - 27$

$= (\quad)^3 - (\quad)^3$

$= (\qquad)(\qquad\qquad)$

29. $x^3 + 8$

$= x^3 + 2^3$ [x=first; 2=second]

$= (\underline{\quad} + \underline{\quad})(x^2 - \underline{\quad\quad} + \underline{\quad\quad})$

$= (\qquad)(\qquad\qquad)$

30. $x^3 + 64$

$= (\qquad)^3 + (\qquad)^3$

$= (\quad + \quad)(\quad - \quad + \quad)$

$= (\qquad)(\qquad\qquad)$

31. $x^3 + 125$

$= (\qquad)^3 + (\qquad)^3$

$= (\qquad)(\qquad\qquad)$

32. $x^3 + 27$

$= (\qquad)^3 + (\qquad)^3$

$= (\qquad)(\qquad\qquad)$

33. $8x^3 - 125$

$= (2x)^3 - 5^3$ [$2x$=first; 5=second]

$= (\quad - \quad)[(2x)^2 + (2x)(5) + 5^2]$

$= (\qquad)(\qquad\qquad)$

34. $27x^3 - 8y^3$

$= (\qquad)^3 - (\qquad)^3$

$= (\quad - \quad)[\quad + \quad + \quad]$

$= (\qquad)(\qquad\qquad)$

35. $64x^3 + 125$

$= (\qquad)^3 + (\qquad)^3$

$= (\qquad)[\qquad\qquad]$

$= (\qquad)(\qquad\qquad)$

36. $27x^3 + 8y^3$

$= (\qquad)^3 + (\qquad)^3$

$= (\qquad)[\qquad\qquad]$

$= (\qquad)(\qquad\qquad)$

37. $8x^3 - 27y^3$

38. $125y^3 - 8x^3$

39. $8x^3 + 1$

40. $125y^3 - 1$

In the next exercises, don't forget the common factor first.

41. $16x^4 - 54x$

$= 2x(8x^3 - 27)$

$= 2x[(2x)^3 - (3)^3]$

$= 2x(\_\_\_ - \_\_\_)(\_\_\_\_ + \_\_\_\_ + \_\_\_\_)$

42. $3x^3 - 24y^3$

$=$

$=$

$=$

43. $5x^4 + 40x$

44. $10x^5y + 80x^2y^4$

45. $3x^5y^5 - 81x^2y^2$

46. $16x^2y^2 + 250x^2y^5$

ANSWERS 2.07

p. 174-180: **(NOTE: Factors may be given in <u>any</u> order!)**

1. $(y+7)(x+4)$; **2.** $(2y+5)(x+5)$; **3.** $(x+3)(x^2+9)$; **4.** $(x-3)(x^2+9)$; **5.** $(a+b)(x+y)$;

6. $(x+c)(a+b)$; **7.** $(a-b)(x+y)$; **8.** $(x-c)(a+b)$; **9.** $(a-b)(x-y)$; **10.** $(x-c)(a-b)$;

11. $(y-5)(x-2)$; **12.** $(x+y)(xy-5)$; **13.** $(x-1)(x-3)(x+3)$; **14.** $(x-5)(x-2)(x+2)$;

15. $(x+7)(x-1)(x+1)$; **16.** $(x-5)(x^2+25)$; **17.** $(x+5)^2(x-5)$; **18.** $(x-5)^2(x+5)$; **19.** $(x-4)(x^2+9)$;

20. $(x+4(x-3)(x+3)$; **21.** $(x-4)(x-3)(x+3)$; **22.** $(x-9)(x-2)(x+2)$; **23.** $(x-8)(x-1)(x+1)$;

24. $(x-8)(x^2+4)$; **25.** $(x-2)(x^2+2x+4)$; **26.** $(x-5)(x^2+5x+25)$; **27.** $(x-4)(x^2+4x+16)$;

28. $(x-3)(x^2+3x+9)$; **29.** $(x+2)(x^2-2x+4)$; **30.** $(x+4)(x^2-4x+16)$; **31.** $(x+5)(x^2-5x+25)$;

32. $(x+3)(x^2-3x+9)$; **33.** $(2x-5)(4x^2+10x+25)$; **34.** $(3x-2y)(9x^2+6xy+4y^2)$;

35. $(4x+5)(16x^2-20x+25)$; **36.** $(3x+2y)(9x^2-6xy+4y^2)$; **37.** $(2x-3y)(4x^2+6xy+9y^2)$;

38. $(5y-2x)(25y^2+10xy+4x^2)$; **39.** $(2x+1)(4x^2-2x+1)$; **40.** $(5y-1)(25y^2+5y+1)$;

41. $2x(2x-3)(4x^2+6x+9)$; **42.** $3(x-2y)(x^2+2xy+4y^2)$; **43.** $5x(x+2)(x^2-2x+4)$;

44. $10x^2y(x+2y)(x^2-2xy+4y^2)$; **45.** $3x^2y^2(xy-3)(x^2y^2+3xy+9)$; **46.** $2x^2y^2(2+5y)(4-10y+25y^2)$.