2.09 *Quadratic Equations by Factoring*

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A great deal of effort has just been spent learning to factor algebraic e*x*pressions. The question has probably been raised more than once, "What good is factoring?" Many situations arise in math in which it is necessary to use the factored form of an e*x*pression. One such situation is solving a **quadratic equation**−−an equation in which the variable is raised to the second power. In general, the equation is in the form $ax^2+bx+c=0$. Before solving quadratic equations by the method of **factoring**, it is necessary to remember one very important fact about numbers: **If the product of two numbers is <u>zero</u>, then one of the numbers must be zero.** First notice that this is a special property of <u>*zero*</u>, and there is no other number of which this could be said. Secondly, notice that in order to use this property, it is necessary to use a **product (i.e., factored form)** of numbers. Of course, this is where the process of factoring comes in. This is called the **ZERO PRODUCT RULE**.

ZERO PRODUCT RULE

$$\text{IF} \quad x \cdot y = 0, \quad \text{THEN} \quad x=0 \quad \text{OR} \quad y=0$$

EXAMPLE 1.   **Solve $x^2 + 3x = 0$**

Solution: Factor $x(x + 3) = 0$ **This is a product equal to zero.**

$x=0$ or $x+3 = 0$ Set each factor equal to zero.
$\underline{-3 \quad -3}$ Solve for x in each case.
$x = 0; \quad x = -3$

You may wish to check these answers in the original equation.

Check: $x = 0$ $\qquad\qquad x = -3$
$0^2 + 3(0) = 0 \qquad (-3)^2 + 3(-3) = 0$
$0 \ = 0 \qquad 9 + (-9) = 0$

EXAMPLE 2.   Solve $x^2 + 3x = 4$

Solution: Notice that this equation does not equal zero. The first step is to make it equal to zero by adding -4 to each side:

$$x^2 + 3x - 4 = 0$$

Factor: $(x + 4)(x - 1) = 0$

At this point it may be helpful to think of splitting the problem in half. Think of exchanging the one "large" equation for two "small" ones, as shown below:

$(x + 4)(x - 1) = 0$ Set each factor equal to 0.

$x + 4 = 0 \quad x - 1 = 0$ Solve for x in each case.

$\underline{-4 \ -4 \quad \ +1 +1}$

$x = -4 \quad \ x = 1$

Again, you may wish to check the answers (use original equation).

SOLVING QUADRATIC (x^2) EQUATIONS
BY FACTORING

Step 1: **Must equal zero**

Step 2: **Must be in factored form**

Step 3: **Set each factor equal to zero, and solve the "small" equations.**

EXERCISE.   Solve the following quadratic equations.

1. $x^2 + 5x = 0$

$x(x + 5) = 0$

$x=0 \quad x+5=0$

$\quad \underline{-5 \ -5}$

$x = 0 \quad x = \underline{\quad}$

2. $x^2 - 6x = 0$

$\underline{\quad}(\quad) = 0$

$x = 0 \quad \underline{\quad\quad\quad}$

3. $x^2 - 8x = 0$

$\underline{\quad}(\quad) = 0$

$\underline{\quad\quad\quad\quad}$

4. $x^2 + 2x - 8 = 0$

$(\quad)(\quad) = 0$

$(\quad) = 0 \ (\quad) = 0$

$x = \underline{\quad} \quad x = \underline{\quad}$

5. $x^2 - 6x - 40 = 0$

$(\quad)(\quad) = 0$

$(\quad) = 0 \ (\quad) = 0$

$x = \underline{\quad} \quad x = \underline{\quad}$

6. $x^2 + 5x + 6 = 0$

$(\quad)(\quad) = 0$

$(\quad) = 0 \ (\quad) = 0$

$x = \underline{\quad} \quad x = \underline{\quad}$

7. $x^2 - 49 = 0$

( )( ) = 0

( ) = 0 ( ) = 0

$x =$ ____ $x =$ ____

8. $x^2 - 25 = 0$

( )( ) = 0

( ) = 0 ( ) = 0

$x =$ ____ $x =$ ____

9. $x^2 - 169 = 0$

( )( ) = 0

( ) = 0 ( ) = 0

$x =$ ____ $x =$ ____

10. $x^2 - 4x - 5 = 0$

11. $x^2 - 3x - 10 = 0$

12. $x^2 + 3x - 10 = 0$

In the next exercises, remember the first step must be to set the equation equal to zero.

13. $x^2 + 2x = 24$

$x^2 + 2x - 24 = 0$

( )( ) = 0

14. $x^2 + 5x = 14$

_____ = 0

( )( ) = 0

15. $x^2 - 10x = -21$

_____ = 0

( )( ) = 0

[Caution: In #16, when you add $-7x$ to the left side, does it matter where you "put" it? **Ans: YES!**]

16. $x^2 + 12 = 7x$

$x^2 - 7x + 12 = 0$

( )( ) = 0

17. $x^2 + 5 = -6x$

_____ = 0

( )( ) = 0

18. $x^2 - 12 = -x$

_____ = 0

( )( ) = 0

After some practice, you will probably find that it is not necessary to show as many steps in solving problems. The next two examples illustrate the way we usually show the work.

EXAMPLE 3. Solve for x: $x^2 + 21x = 100$

Solution:

$x^2 + 21x - 100 = 0$

$(x + 25)(x - 4) = 0$

$x = -25$ or $x = 4$

EXAMPLE 4. Solve for x: $(x-3)(x-4) = 2$

Solution:

$x^2 - 7x + 12 = 2$

$x^2 - 7x + 10 = 0$

$(x - 5)(x - 2) = 0$

$x = 5$ or $x = 2$

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19. $x(x + 4) = 5$

$x^2 + 4x - 5 = 0$

20. $x(x + 5) = 6$

21. $3(2x - 9) = -x^2$

22. $7x = 18 - x^2$

23. $x(6 - x) = -40$

24. $x(5 - x) = -50$

25. $2x^2 - 3x = 0$

26. $5x^2 + 4x = 0$

27. $x(2x + 7) = -5$

28. $x(3x + 1) = 2$

29. $x(5x - 1) = 6$

30. $x(2x - 9) = 5$

In the next exercises, remember to factor the common factor first.

31. $5x^2 + 20x + 15 = 0$

32. $2x^2 - 22x + 48 = 0$

$5(\qquad) = 0$

$5(\quad)(\quad) = 0$

$5 \neq 0$ _____ $= 0$ _____ $= 0$

$5 \neq 0$ $x =$ _____ $x =$ _____

Since $5 \neq 0$, there are only two solutions, $x=$____ or _____

33. $x^3 - 11x^2 + 24x = 0$

    $x(\qquad\qquad) = 0$

    $x(\qquad)(\qquad) = 0$

  $x = 0 \quad x - \quad = 0 \quad x - \quad = 0$

  $x = 0 \quad x = \quad\quad x =$

    There are three solutions:

    $x =$____ , _____, **or** _____

34. $5x^3 + 20x^2 + 15x = 0$

Compare and contrast #31−32 with #33−34. Notice that a linear equation (with x only) usually has one solution, a quadratic equation (with x^2) usually has two solutions, a cubic equation (with x^3) usually has three solutions, etc. Notice that the constant factor in #31−32 has no effect on the solutions. However, if the monomial factor has a variable (as in #33−34), it is a solution, and it therefore cannot be ignored.

Does it seem that all quadratic equations have two solutions? Consider the following:

35. $x^2 - 6x + 9 = 0$

    $(\quad)(\quad) = 0$

  $x =$ ____ $x =$ ____

(It is not necessary to write the answer twice!)

    or $x =$ ____

36. $x^2 + 10x + 25 = 0$

37. $4x^2 + 20x + 25 = 0$

    $(\quad)(\quad) = 0$

38. $4x^2 + 9 = 12x$

    _____

    $(\quad)(\quad) = 0$

Although each of the last four exercises is quadratic, each has a single solution that occurs twice. In higher math courses, we call the solutions to an equation the "**roots**" of the equation. When an answer occurs twice, as in these exercises, we call it a "double root."

It is also possible that the quadratic equation does not factor. These equations can be solved by other methods (see "Completing the Square" and "Quadratic Formula") that will be explained later. Sometimes there is no real solution. For example, $x^2 + 4 = 0$ has no real solution. It does not factor (does it?). Besides, what real number can you square, then add 4 and still end up with zero. If you square a negative number, you get a positive; if you square a positive number, you get a positive number; and if you square 0 you get 0. And in all of these cases, if you add 4 to the result, you certainly do not get 0. Therefore, there is no real solution.

EXERCISES.　　　Solve the equations:

39.　　$x^3 + 5x^2 + 4x = 0$

40. $x^3 - 8x^2 + 7x = 0$

41.　　$x^3 - 9x = 0$

42. $x^3 - 25x = 0$

43.　　$3x^2 - 75 = 0$

44. $3x^3 - 75x = 0$

45. $x^3 - 12x^2 + 36x = 0$

46. $x^3 - 9x^2 = 0$

47. $x(x + 2) = 8$

48. $x(x - 2) = 8$

49. $(x-3)(x+3) = 8x$

50. $(x-3)(x-2) = 12$

51. $x^2 = 4(3 - x)$

52. $x^2 = 4(3 + x)$

53. $2x^2 = 3 + 5x$

54. $2x^2 = 5x - 3$

55. $(x - 4)^2 = 2x$

56. $(x - 4)^2 = 32 - 2x$

57. $x(x-4) = -2x + 8$

58. $2x(x + 5) = 3x + 15$

ANSWERS 2.09

(NOTE: Answers may be given in <u>any</u> order!)

1. 0, -5; **2.** 0, 6; **3.** 0, 8; **4.** -4, 2; **5.** 10, -4; **6.** -2, -3; **7.** 7, -7; **8.** 5, -5; **9.** 13, -13;

10. 5,-1; **11.** 5,-2; **12.** -5,2; **13.** -6,4; **14.** -7, 2; **15.** 3,7; **16.** 4,3; **17.** -5,-1; **18.** -4,3;

19. -5, 1; **20.** -6, 1; **21.** -9, 3; **22.** -9, 2; **23.** 10, -4; **24.** 10, -5; **25.** 0, 3/2; **26.** 0, -4/5;

27. -5/2, -1; **28.** 2/3, -1; **29.** 6/5, -1; **30.** -1/2, 5; **31.** -3, -1; **32.** 8, 3; **33.** 0, 8, 3;

34. 0, -3, -1; **35.** 3; **36.** -5; **37.** -5/2; **38.** 3/2; **39.** 0, -4, -1; **40.** 0, 7, 1; **41.** 0, 3, -3;

42. 0, 5, -5; **43.** 5, -5; **44.** 0, 5, -5; **45.** 0, 6; **46.** 0, 9; **47.** -4, 2; **48.** 4, -2; **49.** 9, -1;

50. 6, -1; **51.** -6, 2; **52.** 6, -2; **53.** -1/2, 3; **54.** 3/2, 1; **55.** 8, 2; **56.** 8, -2; **57.** 4, -2;

58. 3/2, -5.