2.13 *Zero and Negative Exponents*

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In the previous section, exponents were defined and explained by saying that $x^2 = x \cdot x$, $x^3 = x{\cdot}x{\cdot}x$, $x^4 = x{\cdot}x{\cdot}x{\cdot}x$, etc. In that section, you worked with positive, integral exponents. Now, what about something like x^0, x^{-1}, x^{-2}? You really can't say "x times itself 0 times," or "x times itself −1 or −2 times." In this section, the idea of a zero or negative exponent will be investigated.

Consider the following lists of positive exponents that you already know.

$2^4 = 16$	$3^4 = 81$	$4^4 = 256$
$2^3 = 8$	$3^3 = 27$	$4^3 = 64$
$2^2 = 4$	$3^2 = 9$	$4^2 = 16$
$2^1 = 2$	$3^1 = 3$	$4^1 = 4$
$2^0 =$ _____	$3^0 =$ _____	$4^0 =$ _____
$2^{-1} =$ _____	$3^{-1} =$ _____	$4^{-1} =$ _____
$2^{-2} =$ _____	$3^{-2} =$ _____	$4^{-2} =$ _____
$2^{-3} =$ _____	$3^{-3} =$ _____	$4^{-3} =$ _____

Notice that in the first column you have powers of 2, in the middle you have powers of 3, and in the last you have powers of 4. As you move down the left side of the each column, notice that the powers are decreasing: 4, 3, 2, 1, 0, −1, −2, −3.

Notice that in the first column (powers of 2), as you move down the column, from 16 to 8 to 4 to 2, each time you move down, you take half of the number. If you continue the pattern of taking half, the number sequence will be 16, 8, 4, 2, half of 2 is 1, half of 1 is 1/2, half of 1/2, is 1/4, and half of 1/4 is 1/8.

In the second column (powers of 3),notice the pattern coming down the list from 81 to 27 to 9 to 3. The pattern is taking one third of the previous number. Continue the pattern into the blank spaces: 81, 27, 9, 3, 1/3 of 3 is 1, 1/3 of 1 is 1/3, 1/3 of 1/3 is 1/9, 1/3 of 1/9 is 1/27.

In the last column, the pattern coming down the right side is taking 1/4 of the previous number. The numbers in the last column are 256, 64, 16, 4, 1, 1/4, 1/16, 1/64.

In summary, you should have completed the values as follows:

$2^4 = 16$	$3^4 = 81$	$4^4 = 256$
$2^3 = 8$	$3^3 = 27$	$4^3 = 64$
$2^2 = 4$	$3^2 = 9$	$4^2 = 16$
$2^1 = 2$	$3^1 = 3$	$4^1 = 4$
$2^0 = 1$	$3^0 = 1$	$4^0 = 1$
$2^{-1} = 1/2$	$3^{-1} = 1/3$	$4^{-1} = 1/4$
$2^{-2} = 1/4$	$3^{-2} = 1/9$	$4^{-2} = 1/16$
$2^{-3} = 1/8$	$3^{-3} = 1/27$	$4^{-3} = 1/64$

As you look at these numbers, notice that for each number that is raised to the zero power the answer is 1. Also, notice that for the numbers that are raised to negative powers, the answers are not negative, as you may have thought they would be. Rather, each number raised to a negative power is a positive fraction whose numerator is 1. Moreover, each denominator is the value of the number if it had been raised to the positive power (for example $2^3 = 8$, while $2^{-3} = 1/8$).

ZERO EXPONENTS

According to the numbers in the table of the previous page, **$2^0 = 1$, $3^0 = 1$,** and **$4^0 = 1$.** Indeed, it appears that any number raised to the zero power is 1. As further evidence of this, suppose you let *a* be any non−zero number, and find $\frac{a^3}{a^3}$. When you divide, you subtract exponents, so you get **a^0**. However, you also know that **when you divide any number by itself, you get 1**. Therefore, **$a^0 = 1$**. It is true, then, that any non−zero number raised to the zero power is 1.

If a is any non−zero number, then

$$a^0 = 1$$

EXAMPLE 1.

Explain the differences between

a) $(5x)^0$
b) $5(x^0)$
c) $5x^0$.

Solution:

a) In the expression $(5x)^0$, the entire quantity is raised to the zero power. Therefore, the answer is **1**.

b) In the expression $5(x^0)$, only the x is raised to the zero power, so the answer is 5 times 1, which is **5**.

c) In the expression $5x^0$, there are no parentheses, so the real question is **"What is raised to the zero power?"** According to the order of operations agreement, you must raise to powers before multiplication. By this rule, **ONLY the *x* is raised to the zero power**. Then this answer must be multiplied by 5, as in part b) of this question. The final answer is **5**.

EXERCISES.

Simplify each of the following.

1. x^0 = _____
2. y^0 = _____
3. a^0 = _____
4. b^0 = _____
5. $(2x)^0$ = _____
6. $(3y)^0$ = _____
7. $(5z)^0$ = _____
8. $(\text{Junk})^0$ = _____

9. $2(x^0) =$ _____ 10. $3(y^0) =$ _____ 11. $5(z^0) =$ _____ 12. $3(\text{Junk}^0) =$ _____

13. $2x^0 =$ _____ 14. $3y^0 =$ _____ 15. $5z^0 =$ _____ 16. $3\text{Junk}^0 =$ _____

17. $(7x)^0 =$ _____ 18. $7x^0 =$ _____ 19. $35x^0 =$ _____ 20. $(35x)^0 =$ _____

21. True or false.

a) "Any number raised to the zero power is 0." ____________

b) "Any number raised to the zero power is 1." ____________

c) "Any non–zero number raised to the zero power is 0." ____________

d) "Any non–zero number raised to the zero power is 1." ____________

22. Use your calculator to try to calculate 0^0. What happened?

NEGATIVE EXPONENTS

Next, look at the numbers from the table (page 216) that are raised to negative powers.

$2^{-1} = 1/2$	$3^{-1} = 1/3$	$4^{-1} = 1/4$
$2^{-2} = 1/4$	$3^{-2} = 1/9$	$4^{-2} = 1/16$
$2^{-3} = 1/8$	$3^{-3} = 1/27$	$4^{-3} = 1/64$

Notice that the negative powers never result in negative answers. Rather, the negative exponents cause the answers to be in the form of fractions each with numerator 1. To show what really happens with negative exponents, consider the example $\frac{2^4}{2^7}$. You may simplify this in two ways. The first way is to subtract exponents. So $\frac{2^4}{2^7} = 2^{4-7} = 2^{-3}$. The other way to look at this is as the fraction $\frac{2\cdot2\cdot2\cdot2}{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}$. This can be written $\frac{2\cdot2\cdot2\cdot2\cdot1\cdot1\cdot1}{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}$. This answer is $\frac{1}{2^3}$. Therefore, $2^{-3} = \frac{1}{2^3}$. This principle is true for any non–zero base number raised to any integer power. It can be summarized in the following formula.

If a is any non−zero number and n is any integer, then

$$a^{-n} = \frac{1}{a^n}$$

EXAMPLE 2. Express without negative or exponents.

a) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$ b) $2^{-6} = \frac{1}{2^6} = \frac{1}{64}$ c) $10^{-1} = \frac{1}{10^1} = \frac{1}{10}$

EXAMPLE 3. Use your calculator to find the values.

a) 4^{-2}. TI 30 Keystrokes [4] [y^x] [+/−] [2] [=] Answer: 0.0625
TI 83/84/86 Keystrokes [4] [^] [(−)] [2] [ENTER] Answer: 0.0625

Can your calculator convert this answer to 1/16?

b) 2^{-6}. TI 30 Keystrokes: [2] [y^x] [+/−] [6] [=] Answer: 0.015625
TI 83/84/86 Keystrokes [2] [^] [(−)] [6] [ENTER] Answer: 0.015625

Can your calculator convert this answer to 1/64?

c) 10^{-1}. TI 30 Keystrokes: [10] [y^x] [+/−] [1] [=] Answer: 0.1
TI 83/84/86 Keystrokes [10] [^] [(−)] [1] [ENTER] Answer: 0.1

Can your calculator convert this answer to 1/10?

EXERCISES.

Express without negative exponents. Use the definition of a^{-n}. Then, check by using your calculator to find the decimal value, and convert to a fraction.

23. 5^{-2} = ______ = ______ **24.** 8^{-1} = ______ = ______ **25.** 2^{-3} = ______ = ______ **26.** 10^{-1} = ______ = ______

27. 6^{-1} = ______ = ______ **28.** 3^{-2} = ______ = ______ **29.** 7^{-2} = ______ = ______ **30.** 10^{-3} = ______ = ______

The next exercises include variables, so you will not be able to give a numerical answer. Nevertheless, express without negative or zero exponents.

31. x^{-2} = ______ **32.** y^{-2} = ______ **33.** z^{0} = ______ **34.** y^{-3} = ______

35. x^{-1} = ______ **36.** y^{-1} = ______ **37.** x^{-3} = ______ **38.** y^{-5} = ______

39. a^{-4} = ______ **40.** b^{0} = ______ **41.** x^{-5} = ______ **42.** z^{-10} = ______

43. $(\text{Junk})^{-1}$ = ________ **44.** $(\text{Junk})^{-2}$ = ________ **45.** $(\text{Junk})^{-3}$ = ________

46. Use your calculator to try to find 0^{-1}. What happened?

EXAMPLE 4. Explain the differences between **a)** $(5x)^{-1}$

b) $5(x^{-1})$

c) $5x^{-1}$.

Solution: **a)** In the expression $(5x)^{-1}$, the entire quantity is raised to the $^{-}1$ power. Therefore, the answer is $\frac{1}{5x}$.

b) In the expression $5(x^{-1})$, only the x is raised to the $^{-}1$ power, so the answer is **5** times $\frac{1}{x}$, which is $\frac{5}{x}$.

c) In the expression $5x^{-1}$, there are no parentheses, so the real question is "What is raised to the $^{-}1$ power?" According to the order of operations agreement, you must raise to powers before multiplication. By this rule, as in part b), ONLY the x is raised to the $^{-}1$ power. Then this answer must be multiplied by **5**. The final answer (the same as in part b) is $\frac{5}{x}$.

EXAMPLE 5. **Explain the differences between** a) $(5x)^{-2}$
b) $5(x^{-2})$
c) $5x^{-2}$.

Solution: a) In the expression $(5x)^{-2}$, the entire quantity is raised to the $^{-}2$ power. Therefore, you must take $\frac{1}{(5x)^2}$, which is $\frac{1}{25x^2}$.

b) In the expression $5(x^{-2})$, only the x is raised to the $^{-}2$ power, so the answer is 5 times $\frac{1}{x^2}$, which is $\frac{5}{x^2}$.

c) In the expression $5x^{-2}$, there are no parentheses, so the question again is "What is raised to the negative power?" As before, according to the order of operations agreement, you must raise to powers before multiplication, and as in part b), since there are no parentheses, ONLY the x is raised to the power.

So, as in part b), the answer is 5 times $\frac{1}{x^2}$, which is $\frac{5}{x^2}$.

EXERCISES. **Simplify each of the following expressing without negative or zero exponents.**

47. $3x^{-1}$ **48.** $(3x)^{-1}$ **49.** $(7x)^{-1}$ **50.** $7x^{-1}$

51. $3x^{-2}$ **52.** $(3x)^{-2}$ **53.** $(7x)^{-2}$ **54.** $7x^{-2}$

55. $2x^{-3}$ **56.** $(2x)^{-3}$ **57.** $(3x)^{-3}$ **58.** $3x^{-3}$

In the following examples, sometimes you need to use the use the laws of exponents to simplify expressions. Of course, the objective is always to express without negative or zero exponents.

EXAMPLE 6. $x^{-2}x^{-3}$

Solution: $x^{-2}x^{-3}$ Remember, when you multiply, you add exponents.

$x^{(-2)+(-3)}$

x^{-5} Negative exponent means 1 over ( ).

$\frac{1}{x^5}$

EXAMPLE 7. $\frac{x^{-5}}{x^{-3}}$

Solution: $\frac{x^{-5}}{x^{-3}}$ When you divide, you subtract exponents.

$x^{(-5)-(-3)}$ Negative of a negative 3 is a positive 3.

$x^{(-5)+3}$

x^{-2} or $\frac{1}{x^2}$

EXERCISES. Use the laws of exponents to simplify. Eliminate all negative and zero exponents.

59. $2^{-4}2^{6}$

60. $2^{-3}2^{-6}$

61. $x^{-5}x^{4}$

62. $x^{-5}x^{-4}$

63. $\frac{x^{7}}{x^{-5}}$

64. $\frac{x^{6}}{x^{-4}}$

65. $\frac{x^{-4}}{x^{-2}}$

66. $\frac{x^{-12}}{x^{-8}}$

67. $\frac{x^{3}}{x^{-3}}$ 68. $\frac{x^{-3}}{x^{3}}$ 69. $\frac{x^{-12}}{x^{-4}}$ 70. $\frac{x^{12}}{x^{-8}}$

The last law of exponents for this section involves a fraction raised to a negative power.

EXAMPLE 8. What is the meaning of $\left(\frac{3}{4}\right)^{-1}$?

Solution: $\left(\frac{3}{4}\right)^{-1}$ means $\mathbf{1} \div \left(\frac{3}{4}\right)$

$= \mathbf{1} \bullet \left(\frac{4}{3}\right)$ or $\frac{4}{3}$

Conclusion: When a fraction is raised to the −1 power, you invert the fraction.

$$\left(\frac{x}{y}\right)^{-1} = \frac{y}{x} \quad \textbf{and} \quad \left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^{n}$$

EXAMPLE 9. **Simplify :** $\left(\frac{3}{4}\right)^{-2}$

Solution: You must invert the fraction and square. $\left(\frac{3}{4}\right)^{-2}$ means $\left(\frac{4}{3}\right)^{2}$

$= \frac{16}{9}$

EXERCISES. **Simplify each of the following.**

71. $\left(\frac{5}{2}\right)^{-1}$ 72. $\left(\frac{2}{3}\right)^{-1}$ 73. $\left(\frac{5}{7}\right)^{-1}$ 74. $\left(\frac{7}{5}\right)^{-1}$

75. $\left(\frac{5}{2}\right)^{-2}$

76. $\left(\frac{2}{3}\right)^{-2}$

77. $\left(\frac{5}{7}\right)^{-2}$

78. $\left(\frac{7}{5}\right)^{-2}$

79. $\left(\frac{2}{5}\right)^{-3}$

80. $\left(\frac{2}{3}\right)^{-3}$

81. $\left(\frac{2}{3}\right)^{-4}$

82. $\left(\frac{3}{2}\right)^{-4}$

REVIEW EXERCISES. **Simplify each of the following. Express without negative or zero exponents.**

83. $x^4 \bullet x^7 =$ ________

84. $\frac{x^8}{x^2} =$ ________

85. $(x^4)^7 =$ ________

86. $(x^3)^0 =$ ________

87. $\frac{x^{10}}{x^5} =$ ________

88. $x^4 \bullet x^0 =$ ________

89. $2^4 \bullet 2^6 =$ ________

90. $(2^3)^6 =$ ________

91. $\frac{2^{10}}{2^5} =$ ________

92. $\frac{x^3}{x^{-2}} =$ ________

93. $\frac{x}{x^{-2}} =$ ________

94. $\left(\frac{x^2}{y^3}\right)^4 =$ ________

95. $(x^3 y^4)^3 =$ ________

96. $\left(\frac{x^4 y^2}{z^5}\right)^2 =$ ________

97. $\left(\frac{x^2}{y^3}\right)^0 =$ ________

98. $x^{-3} =$ ________

99. $y^{-5} =$ ________

100. $2^{-3} =$ ________

101. $3^{-2} =$ ________

102. $4x^{0} =$ ________

103. $(4x)^{0} =$ ________

104. $(4x)^{-1} =$

105. $4x^{-1} =$

106. $(4x)^{-2} =$

107. $(2x)^{-4} =$

108. $2x^{-3} =$

109. $(2x^{-1})^{-2} =$

110. $(x^{4} \bullet x^{3})^{3} =$

111. $(x^{6} \bullet x^{-2})^{3} =$

112. $(x^{5} \bullet x^{-2})^{6} =$

113. $\left(\dfrac{x^{8}}{x^{2}}\right)^{5} =$

114. $\left(\dfrac{x^{4}}{x^{-2}}\right)^{7} =$

115. $\dfrac{x^{4} \bullet x^{10}}{x^{-6}} =$

116. $\dfrac{x^{-4} \bullet x^{10}}{x^{-6}} =$

117. $\dfrac{x^{4} \bullet x^{-10}}{x^{-6}} =$

118. $\dfrac{x^{-4} \bullet x^{-10}}{x^{-6}} =$

LAWS of EXPONENTS SUMMARY

		GENERALIZATION
1.	When you multiply (with the same base number), you add exponents.	$x^m \bullet x^n = x^{(m+n)}$
2.	When you divide (with the same base number), you subtract exponents.	$\frac{x^m}{x^n} = x^{(m-n)}$
3.	When you raise a power to a power, you multiply exponents.	$\left(x^m\right)^n = x^{mn}$
4.	When a product or a quotient is raised to a power, you raise each factor to the power.	$(xy)^m = x^m y^m$ $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
5.	Any non−zero number raised to the zero power is 1.	$x^0 = 1$
6.	Any number raised to a negative power is 1 divided by that number raised to the positive power.	$x^{-n} = \frac{1}{x^n}$
7.	One (1) divided by any number raised to a negative power is that number raised to the positive power.	$\frac{1}{x^{-n}} = x^n$
8.	A fraction raised to a negative power is the reciprocal of the fraction raised to the positive power.	$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

ANSWERS 2.13

p. 217-225:

1. 1; **2.** 1; **3.** 1; **4.** 1; **5.** 1; **6.** 1; **7.** 1; **8.** 1; **9.** 2; **10.** 3; **11.** 5; **12.** 3; **13.** 2;
14. 3; **15.** 5; **16.** 3; **17.** 1; **18.** 7; **19.** 35; **20.** 1; **21a)** False, **b)** False, **c)** False, **d)** True;
22. Calculator can't do it. Zero to zero power, like division by zero, is undefined;
23. $\frac{1}{25}$; **24.** $\frac{1}{8}$; **25.** $\frac{1}{8}$; **26.** $\frac{1}{10}$; **27.** $\frac{1}{6}$; **28.** $\frac{1}{9}$; **29.** $\frac{1}{49}$; **30.** $\frac{1}{1000}$;
31. $\frac{1}{x^2}$; **32.** $\frac{1}{y^2}$; **33.** 1; **34.** $\frac{1}{y^3}$; **35.** $\frac{1}{x}$; **36.** $\frac{1}{y}$; **37.** $\frac{1}{x^3}$; **38.** $\frac{1}{y^5}$;
39. $\frac{1}{a^4}$; **40.** 1; **41.** $\frac{1}{x^5}$; **42.** $\frac{1}{z^{10}}$; **43.** $\frac{1}{Junk}$; **44.** $\frac{1}{Junk^2}$; **45.** $\frac{1}{Junk^3}$;
46. Undefined; **47.** $\frac{3}{x}$; **48.** $\frac{1}{3x}$; **49.** $\frac{1}{7x}$; **50.** $\frac{7}{x}$; **51.** $\frac{3}{x^2}$; **52.** $\frac{1}{9x^2}$;
53. $\frac{1}{49x^2}$; **54.** $\frac{7}{x^2}$; **55.** $\frac{2}{x^3}$; **56.** $\frac{1}{8x^3}$; **57.** $\frac{1}{27x^3}$; **58.** $\frac{3}{x^3}$; **59.** 4;
60. $\frac{1}{2^9}$; **61.** $\frac{1}{x}$; **62.** $\frac{1}{x^9}$; **63.** x^{12}; **64.** x^{10}; **65.** $\frac{1}{x^2}$; **66.** $\frac{1}{x^4}$; **67.** x^6;
68. $\frac{1}{x^6}$; **69.** $\frac{1}{x^8}$; **70.** x^{20}; **71.** $\frac{2}{5}$; **72.** $\frac{3}{2}$; **73.** $\frac{7}{5}$; **74.** $\frac{5}{7}$; **75.** $\frac{4}{25}$; **76.** $\frac{9}{4}$;
77. $\frac{49}{25}$; **78.** $\frac{25}{49}$; **79.** $\frac{125}{8}$; **80.** $\frac{27}{8}$; **81.** $\frac{81}{16}$; **82.** $\frac{16}{81}$; **83.** x^{11}; **84.** x^6;
85. x^{28}; **86.** 1; **87.** x^5; **88.** x^4; **89.** 2^{10}; **90.** 2^{18}; **91.** 2^5 or 32; **92.** x^5; **93.** x^3;
94. $\frac{x^8}{y^{12}}$; **95.** x^9y^{12}; **96.** $\frac{x^8y^4}{z^{10}}$; **97.** 1; **98.** $\frac{1}{x^3}$; **99.** $\frac{1}{y^5}$; **100.** $\frac{1}{8}$; **101.** $\frac{1}{9}$;
102. 4; **103.** 1; **104.** $\frac{1}{4x}$; **105.** $\frac{4}{x}$; **106.** $\frac{1}{16x^2}$; **107.** $\frac{1}{16x^4}$; **108.** $\frac{2}{x^3}$;
109. $\frac{x^2}{4}$; **110.** x^{21}; **111.** x^{12}; **112.** x^{18}; **113.** x^{30}; **114.** x^{42}; **115.** x^{20};
116. x^{12}; **117.** 1; **118.** $\frac{1}{x^8}$.