3.05 *Fractional Equations*

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It is often necessary to solve equations that involve fractions. The first step here, as before when simplifying fractional exercises is to find the least common denominator (**LCD**). The second step when solving equations is different--you may multiply both sides of the equation by the **LCD**, and in doing so, you eliminate all denominators from the problem. By contrast, in working with fractional expressions (i.e., NOT equations!) in the previous section, it was necessary to carry the common denominator all the way through the problem. It is important to emphasize that the methods of this section are valid only with equations. It is not correct to say "multiply through by the **LCD**." Rather, you should emphasize the equation involved by saying, "multiply both sides of the equation by the **LCD**," as long as you don't multiply both sides of the equation by zero. **YOU ARE NEVER ALLOWED TO MULTIPLY BOTH SIDES OF AN EQUATION BY ZERO!**

There is one small "fly in the ointment"--at times you will be multiplying both sides of the equation by variables. This may not appear to be a problem, but it could be! If you multiply both sides of an equation by a variable, you really do not know what you are multiplying by--that is, not until you finish the problem and find out what "x" (or whatever the variable may be) equals. The "small fly" is this: what if, without realizing it, you multiplied both sides of the equation by zero? It could happen, and multiplying both sides of an equation by zero is not allowed! Therefore, any time you multiply both sides of an equation by a variable, you must check the answer(s) to make sure that no denominators ever equal zero. If any answer that you get ever makes a denominator equal zero, then this answer must be rejected. The answer that was obtained was not a legal answer. Like evidence that is illegally obtained and cannot be allowed in a court of law, such answers must be thrown out. If no other solutions can be found, then there is no solution for the problem. In this case, you should answer "No Solution," or the empty set, which is written either { } or ϕ .

PRINCIPLE

Whenever an equation is solved by multiplying both sides of that equation by a variable, the solution must be checked to be sure no denominators equal zero!

Before beginning the exercises, one more principle will be useful in this section. This is the **definition of equality of fractions.** Two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, are equal if and only if **$a \bullet d = b \bullet c$,** assuming no denominators are zero.

DEFINITION of EQUALITY of FRACTIONS

$$\frac{a}{b} = \frac{c}{d} \qquad \textbf{means that} \qquad a \bullet d = b \bullet c$$

CAUTION: Denominators CANNOT equal *ZERO* !

You can check out this definition to see that it makes sense by reducing a few fractions, as follows.

EXAMPLE 1. Reduce the fraction $\frac{4}{8}$ and check the answer using the above definition.

Solution: $\frac{4}{8} = \frac{1}{2}$ since **$4 \cdot 2 = 8 \cdot 1$** (which is true because 8 = 8).

EXAMPLE 2. Reduce the fraction $\frac{6}{9}$ and check the answer using the above definition.

Solution: $\frac{6}{9} = \frac{2}{3}$ because **$6 \cdot 3 = 9 \cdot 2$** (which is true because 18 = 18).

EXAMPLE 3. Reduce the fraction $\frac{3}{27}$ and check the answer using the above definition.

Solution: $\frac{3}{27} = \frac{1}{9}$ because **$3 \cdot 9 = 27 \cdot 1$** (which is true because 27 = 27).

EXAMPLE 4. Reduce the fraction $\frac{12}{15}$ and check the answer using the above definition.

Solution: $\frac{12}{15} = \frac{4}{5}$ because **$12 \cdot 5 = 15 \cdot 4$** (which is true because 60 = 60).

The main use of the definition of equality of fractions is to solve equations in which you have a fraction equal to another fraction. Using this definition, in one step, you can completely eliminate the fractions in the problem and have a regular (that is, linear or quadratic!) equation to solve.

EXAMPLE 5. **Solve the equation $\frac{x}{x+2} = \frac{4}{5}$. Check to be sure denominators are not zero.**

Solution: By the definition of equality of fractions:

$$\mathbf{5(x) = 4(x + 2)}$$

$$5x = 4x + 8$$

$$\underline{-4x \quad -4x}$$

$$x = 8$$

Check: $\frac{8}{8+2} = \frac{8}{10} = \frac{4}{5}$

EXERCISES. **Solve the equations for x. Check to be sure denominators are not zero.**

1. $\frac{x}{x-2} = \frac{4}{5}$

2. $\frac{4}{x-2} = \frac{3}{2x-9}$

3. $\frac{6}{3x+6} = \frac{4}{x-1}$

4. $\frac{5x - 3}{2} = \frac{15x - 2}{8}$

5. $\dfrac{2y+8}{5} = \dfrac{10y+4}{15}$

6. $\dfrac{x-8}{x+4} = \dfrac{1}{4}$

7. $\dfrac{x+6}{x-1} = \dfrac{1}{8}$

8. $\dfrac{1}{x+6} = \dfrac{1}{2x-6}$

9. $\dfrac{x-1}{x+6} = \dfrac{1}{8}$

10. $\dfrac{5}{x+6} = \dfrac{4}{2x-6}$

EXAMPLE 6. **Solve the equation** $\dfrac{4}{x+1} = \dfrac{6}{x+1}$. **Check that denominators are not zero.**

Solution: By the definition of equality of fractions:

$$
\begin{aligned}
4(x+1) &= 6(x+1) \\
4x + 4 &= 6x + 6 \\
\underline{-6x \qquad} & \underline{\; -6x \qquad .} \\
-2x + 4 &= 6 \\
\underline{\qquad -4} & \underline{\qquad -4} \\
-2x &= 2 \\
x &= -1
\end{aligned}
$$

Check: $\dfrac{4}{-1+1} = \dfrac{6}{-1+1}$ Division by zero is **UNDEFINED**. **No Solution**!!

11. $\dfrac{4}{x-2}=\dfrac{6}{x-2}$

12. $\dfrac{3}{x-6}=\dfrac{5}{x-6}$

13. $\dfrac{x-4}{3}=\dfrac{5x-20}{8}$

14. $\dfrac{3}{x-4}=\dfrac{8}{5x-20}$

In 15 - 16, these look like quadratic equations, but they are not since the x^2 terms subtract out!!

15. $\dfrac{x-2}{x+4}=\dfrac{x}{x+2}$

16. $\dfrac{x+6}{x}=\dfrac{x-2}{x-4}$

In Example 7 and the following exercises, these ARE quadratic equations! You know what to do!

EXAMPLE 7. **Solve the equation** $\dfrac{4}{x}=\dfrac{x-2}{2}$**.**

Solution:

$x\,(x-2)=8$ This is a quadratic equation.

$x^2-2x-8=0$ You must set equal to zero and factor.

$(x-4)(x+2)=0$

$\boldsymbol{x=4 \quad x=-2}$

Check: If $x = 4$, $\dfrac{4}{4}=\dfrac{4-2}{2}$ If $x=2$, $\dfrac{4}{-2}=\dfrac{-2-2}{2}$

$1 = 1$ $-2 = -2$ Both answers check!

17. $\frac{4}{x} = \frac{x+2}{2}$

18. $\frac{x}{x+4} = \frac{6}{x-4}$

19. $\frac{x}{x-4} = \frac{-6}{x+4}$

20. $\frac{4}{x-2} = \frac{x}{2}$

21. $\frac{x}{8} = \frac{6}{x-8}$

22. $\frac{x}{x+12} = \frac{2}{x+4}$

23. $\frac{x}{2x+9} = \frac{-3}{x+6}$

24. $\frac{x-2}{4} = \frac{3x-6}{10}$

EXAMPLE 8. **Solve for x:** $\frac{2x}{3} + \frac{3(x-4)}{2} = \frac{1}{2}$.

Solution: Multiply both sides of the equation by the **LCD** which is **6**:

$$6 \bullet \frac{2x}{3} + 6 \bullet \frac{3(x-4)}{2} = 6 \bullet \frac{1}{2}$$

$$2\cancel{6} \bullet \frac{2x}{\cancel{3}} + 3\cancel{6} \bullet \frac{3(x-4)}{\cancel{2}} = 3\cancel{6} \bullet \frac{1}{\cancel{2}}$$

$$2 \bullet 2x + 3 \bullet 3(x-4) = 3$$

$$4x + 9x - 36 = 3$$

$$+36 \quad +36$$

$$13x = 39$$

$$x = 3$$

EXERCISES. **In each of the following equations, solve for x.**

25. $\frac{3(x-2)}{4} - \frac{x}{2} = 2$

26. $\frac{x}{3} - \frac{x+2}{2} = 1$

27. $\frac{x+3}{5} - \frac{x-4}{4} = 2$

28. $\frac{x(x-1)}{6} + \frac{x}{3} = 1$

29. $\frac{x(x-8)}{6} + \frac{x-2}{2} = \frac{4}{3}$

30. $\frac{x(x+1)}{6} - \frac{x}{3} = 1$

EXTRA CHALLENGE

EXAMPLE 9. **Solve for x**: $\frac{x}{x-1} + \frac{2}{x-5} = \frac{4}{(x-5)(x-1)}$.

Solution: Multiply each side of the equation by the **LCD** which is **$(x-5)(x-1)$**. Before you do, however, notice that x cannot equal 5 or 1, since these values of x cause one or more denominators to be zero. So, $x \neq 5$ and $x \neq 1$.

$$(x-5)(x-1) \bullet \frac{x}{x-1} + (x-5)(x-1) \bullet \frac{2}{x-5} = (x-5)(x-1) \bullet \frac{4}{(x-5)(x-1)}$$

$$(x-5)\cancel{(x-1)} \bullet \frac{x}{\cancel{x-1}} + \cancel{(x-5)}(x-1) \bullet \frac{2}{\cancel{x-5}} = \cancel{(x-5)}\cancel{(x-1)} \bullet \frac{4}{\cancel{(x-5)}\cancel{(x-1)}}$$

This looks pretty scary, but look how it simplifies--EVERY denominator divides out!

$$x(x-5) + 2(x-1) = -4$$
$$x^2 - 5x + 2x - 2 = -4$$
$$+4 \qquad +4$$
$$x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$
$$x = 2 \quad x = 1$$

However, the answer of $x = 1$ must be rejected, since $x \neq 1$. The solution is $\boldsymbol{x = 2}$.

EXERCISES. Solve for x.

31. $\frac{x}{x-5} - \frac{13}{x-2} = \frac{15}{(x-5)(x-2)}$

32. $\frac{1}{x^2 - 4x + 3} - \frac{2}{x^2 + 4x - 5} = \frac{4}{x^2 + 2x - 15}$

ANSWERS 3.05

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1. -8; **2.** 6; **3.** -5; **4.** 2; **5.** 5; **6.** 12; **7.** -7; **8.** 12;
9. 2; **10.** 9; **11.** No Sol; **12.** No Sol; **13.** 4; **14.** No Sol;
15. -1; **16.** 6; **17.** -4,2; **18.** 12,-2; **19.** -12,2; **20.** 4,-2;
21. 12, -4; **22.** -6,4; **23.** -9, -3; **24.** 2; **25.** 14;
26. -12; **27.**-8; **28.**-3, 2; **29.** 7,-2; **30.** 3, -2;
31. 10 (Reject 5); **32.** No sol (Reject 3).