4.07 *Systems of Equations (2X2)*

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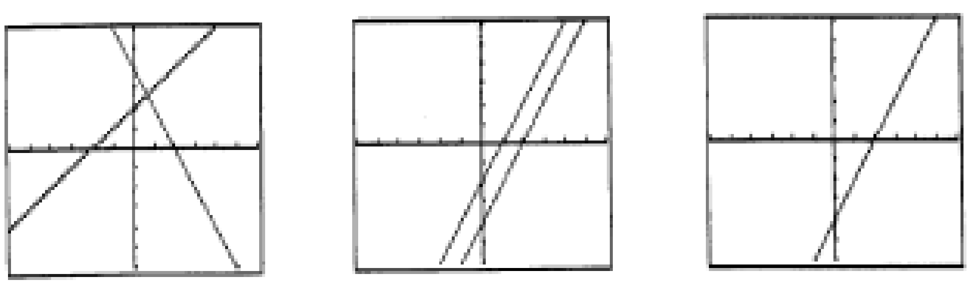
ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

If one equation with two unknowns (x and y) represents a line graph, then two equations with two unknowns must represent two lines (on the same graph, of course). For example, consider

$$\begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$$

The first equation, $2x + y = 4$, is a straight line, and it has infinitely many solutions for x and y. The second equation, $x - y = 2$, also is a straight line and also has many other solutions for x and y. Now, are there any values of x and y that satisfy both equations at the same time? Finding solutions that satisfy both equations simultaneously is the problem of this section.

Think for a moment about two straight lines on the same graph. Will there be intersection points? If so, how many points will there be? If both lines are straight lines, then they will usually intersect in one point (but never in two points!). But will they always intersect? What if the lines are parallel? In this case there will be no points of intersection. There is a third possibility. The two lines could in fact be the same line. In this third case, the solution to one equation would automatically be a solution to the other equation. There are infinitely many solutions to this case. **These three possibilities are illustrated below.**



I. UNIQUE SOLUTION	**II. PARALLEL LINES**	**III. SAME LINE**
a single point of intersection (x,y);	no intersection, no solution	solution is the entire line, infinitely many solutions

The most common methods of solving such systems include the **graphical method,** the **elimination method** (also known as the "addition" method), and the **substitution method**.

GRAPHICAL METHOD

The most obvious way to find the intersection of two straight lines is to graph the lines on the same graph, and determine if the two lines intersect in a point, are parallel, or are the same line. If they intersect at a point, you will want to find the coordinates of that point. Of course, if the equation of the line is in slope-intercept (**y=mx+b**) form, you will probably want to graph it using the slope-intercept method. If the equation of the line is in standard (**Ax + By = C**) form, you will probably want to graph it using the two intercept method.

EXAMPLE 1. Solve the system of equations

$$3x + y = 6$$
$$x - y = 6$$

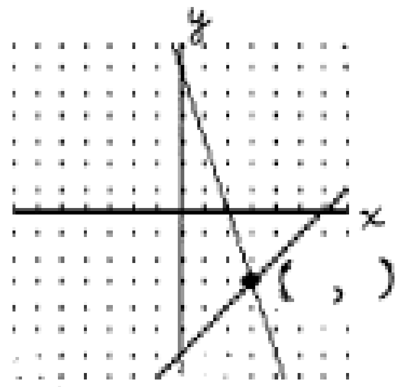
Solution: $3x + y = 6$

x	y
0	6
2	0

$x - y = 6$

x	y
0	-6
6	0

Solution: x = 3 ; y = -3 or **(3, -3)**



EXAMPLE 2. Solve the system of equations

$$3x - 3y = 18$$
$$x - y = 6$$

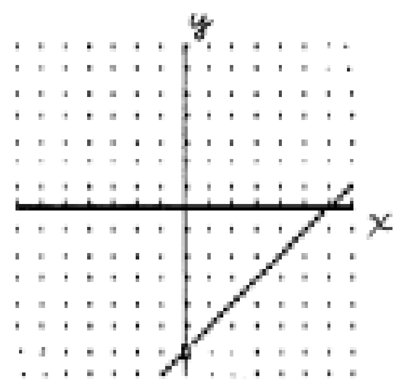
Solution: $3x - 3y = 18$

x	y
0	-6
6	0

$x - y = 6$

x	y
0	-6
6	0

Solution: Same line.
Solution is the entire line.



EXAMPLE 3. Solve the system of equations

$$x - y = 3$$
$$x - y = 6$$

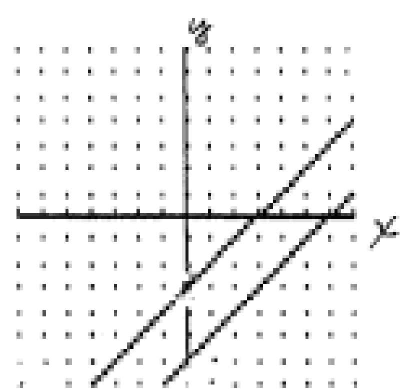
Solution: $x - y = 3$

x	y
0	-3
3	0

$x - y = 6$

x	y
0	-6
6	0

Solution: Parallel lines. No Solution.



EXERCISES. Solve the systems of equations, finding the points of intersection if the graphs intersect. If the graphs do not intersect, give the appropriate conclusion.

1.

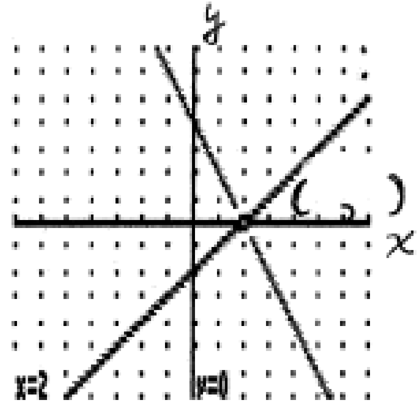
$$2x + y = 4$$
$$x - y = 2$$

Solution: $2x + y = 4$ $\qquad$ $x - y = 2$

x	y
0	4
2	0

x	y
0	-2
2	0

Solution: **x = _____; y = _____**



2.

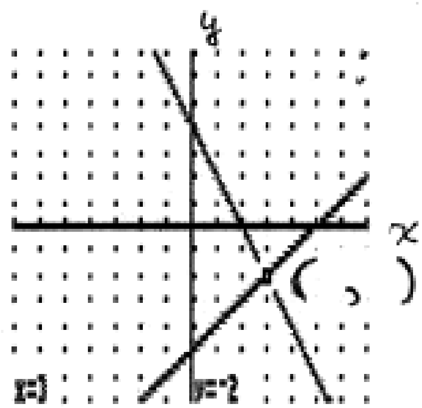
$$2x + y = 4$$
$$x - y = 5$$

Solution: $2x + y = 4$ $\qquad$ $x - y = 5$

x	y
0	4
2	0

x	y
0	-5
5	0

Solution: **x = _____; y = _____**



3.

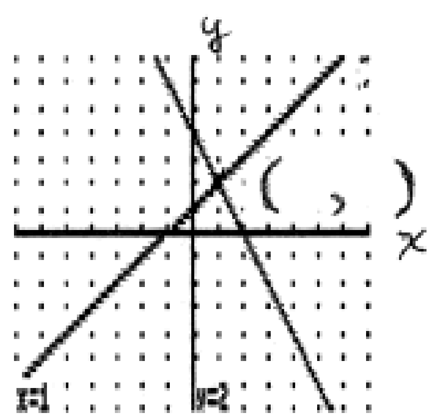
$$2x + y = 4$$
$$x - y = -1$$

Solution: $2x + y = 4$ $\qquad$ $x - y = -1$

x	y
0	4
2	0

x	y
0	1
-1	0

Solution: **x = _____; y = _____**



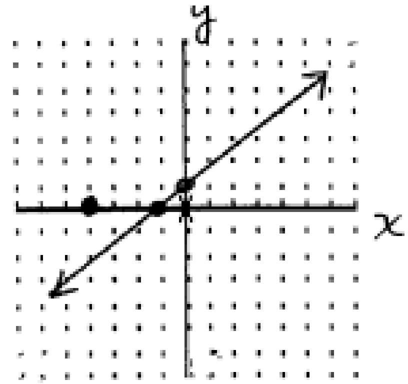
4. $x - y = -1$
$2x + y = -8$

Solution: $x - y = -1$ $2x + y = -8$

x	y
0	1
-1	0

x	y
0	-8
-4	0

Solution: ____________________



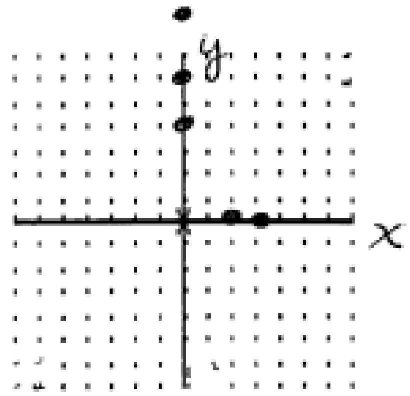
5. $2x + y = 4$
$4x + 2y = 12$

Solution: $2x + y = 4$ $4x + 2y = 12$

x	y
0	4
2	0

x	y
0	6
3	0

Solution: ____________________



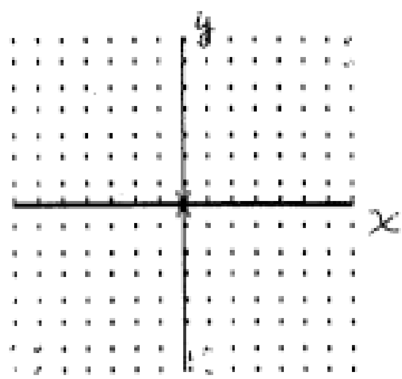
6. $2x - 2y = 10$
$x - y = 5$

Solution: $2x - 2y = 10$ $x - y = 5$

x	y
0	
	0

x	y
0	
	0

Solution: ____________________



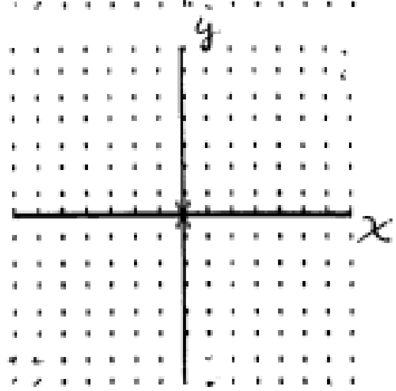
7. $2x - y = 4$
$-2x + y = -4$

Solution: $2x - y = 4$ $-2x + y = -4$

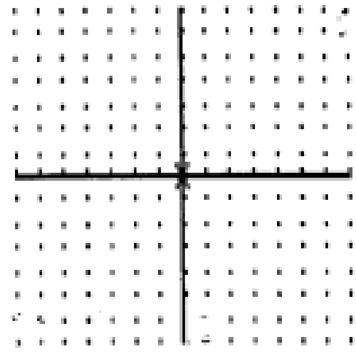
x	y

x	y

Solution: ____________________



8. $-x + y = 2$
$x - y = 6$



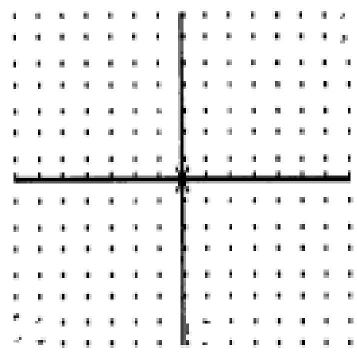
Solution: $-x + y = 2$ $x - y = 6$

x	y

x	y

Solution: ________________

9. $x - 2y = -4$
$x - y = -3$



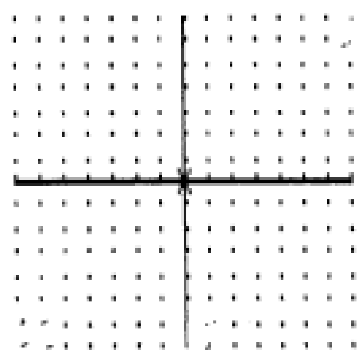
Solution: $x - 2y = -4$ $x - y = -3$

x	y

x	y

Solution: ________________

10. $x + 2y = 4$
$x - 2y = -8$



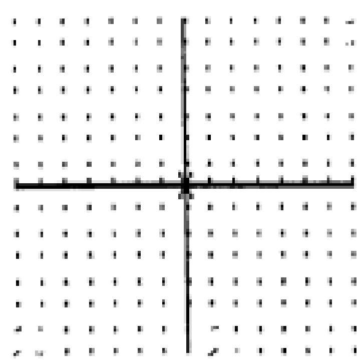
Solution: $x + 2y = 4$ $x - 2y = -8$

x	y
0	
	0

x	y
0	
	0

Solution: ________________

11. $3x + 2y = 6$
$-x - 2y = 2$



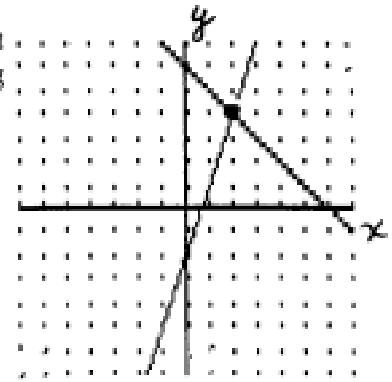
Solution: ________________

EXAMPLE 4. Solve the system of equations.

$y = 3x - 2$

$y = -x + 6$

Solution: Since both equations are in slope-intercept form, it will probably be easiest to graph each of these using the slope-intercept method of graphing.



$y = 3x - 2$	$y = -x + 6$
y-int = -2	y-int = 6
m = 3	m = -1

The solution is **x = 2, y = 4 or (2,4)**.

EXAMPLE 5. Solve the system of equations.

$$y = \frac{1}{2}x + 2$$

$$2x - y = 4$$

Solution: Since the first graph is in slope-intercept form, it will probably be easiest to use the slope-intercept method of graphing. The second equation is in standard form so use the intercepts method of graphing.

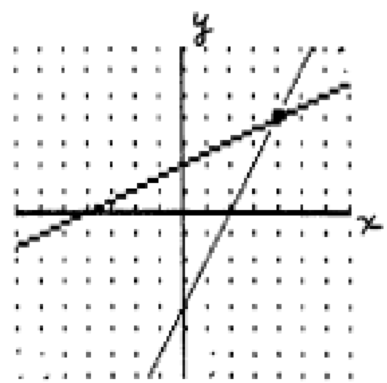
$y = \frac{1}{2}x + 2$

y-int = 2

$m = \frac{1}{2}$

$2x - y = 4$

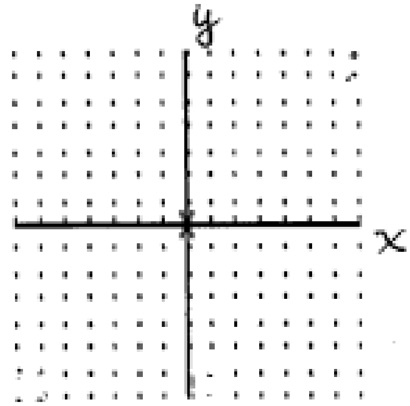
x	y
0	-4
2	0



The solution is **x = 4, y = 4 or (4,4)**.

12. $y = \frac{1}{2}x + 3$

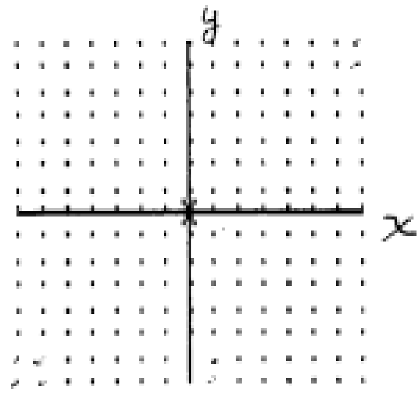
$y = 2x - 3$



Solution: ____________________

13. $y = 4 - x$

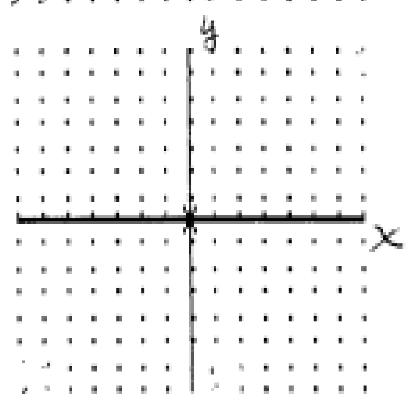
$2y - x = 2$



Solution: ____________________

14. $y = 3x - 3$

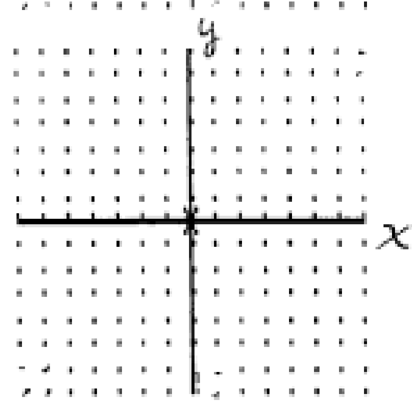
$3x + 2y = 12$



Solution: ____________________

15. $y = 3x - 3$

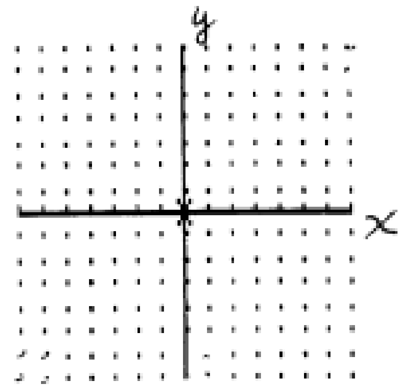
$6x - 2y = 6$



Solution: ____________________

16. $y = 3x - 3$

$6x - 2y = -6$



Solution: ____________________

There are some serious limitations to this graphing method of solving systems of equations. First, it is limited to the accuracy of the graph. In each of the exercises you just completed, you may have noticed that the answers and even the x and y-intercepts were all "rigged" to come out even. Imagine if the solution involved fractional values. The graphical solution would at best be only a guess. Second, graphing is rather time-consuming. For these reasons, the algebraic methods beginning on the next page are usually easier and certainly more accurate. The first algebraic method is known as the **elimination method**, since the objective is to **eliminate** one of the variables by adding or subtracting the equations. The second algebraic method, equally important in many cases and especially in higher math, is called the **substitution method**. The substitution method is particularly useful when one of the equations is written in the form "y = ____" or "x = _____."

ELIMINATION METHOD

The elimination method is particularly useful when both equations are written in standard form. With the elimination method, the strategy is to add (or subtract) the two equations together and in the process, cause one of the variables to subtract out. This leaves one variable in the equation, which you can solve. You may eliminate either the x or the y variable, whichever naturally subtracts out. The question of what happens if neither variable subtracts out will be addressed next.

EXAMPLE 7. Solve the system of equations by the elimination method.

$$2x + y = 4$$
$$x - y = 5$$

Solution: Notice that if the equations are added together, the y terms subtract out:

$$\begin{array}{r} 2x + y = 4 \\ \underline{x - y = 5} \\ 3x \quad = 9 \\ x \ = 3 \end{array}$$

Now, substitute this value of x into one of the equations (either will do) and solve for y.

$$\begin{array}{r} 2x + y = 4 \\ 2(3) + y = 4 \\ 6 + y = 4 \\ y = -2 \end{array}$$

Check! Always substitute the values of x and y into the <u>other</u> equation (the one you did not just use to solve for y).

$$x - y = 5$$
$$3 -(-2) = 5 \checkmark$$ The solution is **(3, -2)**.

EXAMPLE 8. Solve the system of equations by the elimination method.

$$-x + 2y = 12$$
$$x + y = -6$$

Solution: Notice that in this example, if the equations are added together, the x terms subtract out:

$$\begin{array}{r} -x + 2y = 12 \\ \underline{x + y = -6} \\ 3y = 6 \\ y = 2 \end{array}$$

Now, substitute this value of y into one of the equations (either will do) and solve for x.

$$\begin{array}{r} -x + 2y = 12 \\ -x + 2(2) = 12 \\ -x + 4 = 12 \\ -x = 8 \\ x = -8 \end{array}$$

Check! $-x + 2y = 12$

$$-(-8)+2(2) = 12 \checkmark$$ The solution is **(-8, 2)**.

EXERCISES. Solve the system of equations by the elimination method.

17. $2x - y = 12$
    $x + y = -6$

18. $2x - y = 8$
    $2x + y = 4$

19. $x + 2y = 8$
    $-x + y = 1$

20. $x - 2y = -4$
    $-x + y = 3$

21. $3x - y = 9$
    $2x + y = 16$

22. $-x + 2y = -4$
    $x + 3y = 9$

EXAMPLE 9. Solve the system of equations by the elimination method.

$$\begin{aligned} x - 2y &= -4 \\ x - y &= -3 \end{aligned}$$

Solution: In this case, notice that adding the equations does no good!

$$\begin{array}{r} x - 2y = -4 \\ \underline{x - y = -3} \\ 2x - 3y = -7 \end{array}$$

Remember, the objective is to eliminate one of the variables, so multiply both sides of one of the equations by -1 and eliminate the x.

EXAMPLE 9 **Solution (continued):**

$$-1(x - 2y = -4)$$
$$\underline{x - y = -3}$$

$$-x + 2y = 4$$
$$\underline{x - y = -3}$$

$$y = 1$$

Now substitute back into the first equation and solve for x.

$$x - 2y = -4$$
$$x - 2(1) = -4$$
$$\underline{+2 \quad +2}$$
$$x = -2$$

Check in the other equation:

$$x - y = -3$$
$$(-2) - (1) = -3$$ _ The solution is **(-2, 1)**.

EXAMPLE 10. Solve for the system of equations.

$$2x + 3y = -13$$
$$3x + y = 5$$

Solution: In order to eliminate the y, you would need a +3y and a -3y. This can be accomplished by multiplying both sides of the second eqution by -3.

$$2x + 3y = -13$$
$$-3(3x + y = 5)$$

$$2x + 3y = -13$$
$$\underline{-9x - 3y = -15}$$ This eliminates the y.
$$-7x = -28$$
$$x = 4$$ Substitute back into the second equation (it looks easier!)
$$3x + y = 5$$ and solve for y.
$$3(4) + y = 5$$
$$12 + y = 5$$ Add -12 to both sides.
$$\underline{-12 \quad -12}$$
$$y = -7$$ Be sure to use the <u>other</u> equation (the first) to check!

Check: $$2x + 3y = -13$$
$$2(4) + 3(-7) = -13$$
$$8 - 21 = -13$$ _ The solution is **(4, -7)**.

EXERCISES. Solve the system of equations by the elimination method.

23. $-x + 3y = -5$
$2x + 3y = -17$

24. $2x - 3y = -14$
$2x - y = -10$

25. $x + 2y = 8$
$5x - 6y = 8$

26. $2x - 3y = 1$
$x + y = 8$

27. $3x + 5y = 10$
$x + 2y = 1$

28. $3x - 2y = 10$
$4x - y = 15$

29. $3x + 2y = 10$
$x + 3y = 8$

30. $3x + 2y = 38$
$x + 5y = 4$

EXAMPLE 11. Solve the system of equations

$$5x + 3y = 14$$
$$9x + 4y = 7$$

Solution: In this example look for a common multiple for the x and y coefficients. The common multiple for the x-coefficients is 45, and for the y coefficients is 12. Since the numbers will be smaller, it is easier to eliminate the y terms. Multiply both sides of the first equation by 4 and multiply both sides of the second equation by -3 to eliminate the Y term.

$$4(5x + 3y = 14)$$
$$-3(9x + 4y = 7)$$

$$\begin{array}{rcl} 20x + 12y &=& 56 \\ \underline{-27x - 12y} &\underline{=}& \underline{-21} \\ -7x &=& 35 \\ x &=& -5 \end{array}$$

$5x + 3y = 14$ Substitute x into the first equation and solve y.

$5(-5) + 3y = 14$

$-25 + 3y = 14$ Add +25 to both sides

$$\begin{array}{rcl} -25 + 3y &=& 14 \\ \underline{+25} & & \underline{+25} \\ 3y &=& 39 \\ y &=& 13 \end{array}$$

Check: $9x + 4y = 7$

$9(-5) + 4(13) = 7$

$-45 + 52 = 7$ ✓ The solution is **(-5,13)**.

EXERCISES. Solve the systems of equations.

31. $5x + 3y = 6$
    $3x + 2y = 2$

32. $3x + 5y = 1$
    $-2x + 3y = 12$

33. $3x + 5y = 2$
    $2x + 3y = -4$

34. $5x + 2y = -12$
    $3x - 5y = -1$

35. $2x - 3y = -32$
    $3x - 4y = -36$

36. $12x + 5y = -24$
    $4x + 3y = 8$

What happens algebraically when the lines are parallel or the same line? Remember that when the lines are parallel, there are no common points or solutions. Also, remember that when they are the same line, there are infinitely many solutions. In these two cases, when you try to eliminate one of the variables, both variables are eliminated. The following examples illustrate this concept, where adding the equations together eliminates <u>both</u> variables. Consider Examples 12 and 13.

EXAMPLE 12

$$\begin{array}{r} x - y = 6 \\ \underline{-x + y = 2} \\ 0 = 8 \end{array}$$

EXAMPLE 13

$$\begin{array}{r} 2x - y = 4 \\ \underline{-2x + y = -4} \\ 0 = 0 \end{array}$$

Whenever eliminating one variable "by chance" results in the elimination of <u>both</u> variables, and an impossible statement such as **0 = 8**, or **0 = any non-zero number**, there is **No Solution** possible. This is the case of the two **parallel lines**. **For Example 12, there is No Solution.**

Whenever eliminating one variable "by chance" results in the elimination of <u>both</u> variables <u>and</u> the constants (number terms) as well, then the statement **0 = 0** results. This statement is always true, and this indicates that **there are many solutions**. In fact, this is the case in which the two equations represent the **same line**. **In Example 13, the solution is the entire line.**

EXERCISES.

37. $4x + 3y = 5$
 $20x + 15y = 35$

38. $4x - 3y = -6$
 $-8x + 6y = 12$

39. $4x - 7y = -28$
$4x + 7y = -28$

40. $16x - 12y = 28$
$4x - 3y = 0$

SUBSTITUTION METHOD

The substitution method is particularly useful in solving systems of equations in which one or both of the equations is expressed in the form y = _____ or x = _____. This method is really important in higher math, where the equations may no longer represent straight lines.

If the equations are both given in standard form, then it is easier to solve by the elimination method. If one or both of the equations is in slope-intercept form (or x=____ form), then the substitution method is usually easier.

EXAMPLE 14. Solve the system of equations by the Substitution Method.

$$5y - 3x = 34$$
$$x = 7 - 2y$$

Solution: Since the second equation is in the form x = _____, the substitution method is appropriate for this problem.

Rewrite the first equation:

$5y - 3(\quad x \quad) = 34$ and substitute 7-2y for x:

$5y - 3(7 - 2y) = 34$ Distribute -3

$5y - 21 + 6y = 34$ Combine like terms

$11y - 21 = 34$ Add +21

$11y = 55$

$y = 5$ The second equation (x=____) is the best place to substitute Y and solve for X.

$x = 7 - 2y$

$x = 7 - 2(5)$

$x = 7 - 10$ or $x = -3$

Check: $5Y - 3X = 34$ (You must use the <u>other</u> equation!)

$5(5) - 3(-3) = 34$

$25 + 9 = 34$

EXERCISES. Solve by the substitution method.

41. $2x + 3y = 12$

$x = 5 - y$ Substitute $(5 - y)$ for x in the first equation.

$2(\qquad) + 3y = 12$

______________ $= 12$

______________ $= 12$

$y =$ _____ Substitute this value of y into the second equation.

$x = 5 - y$

$x = 5 - (\quad)$

$x =$ _______ Remember to use the other equation to check

Check: $2x + 3y = 12$

$2(\quad) + 3(\quad) = 12$

______________ = ___

42. $y = 3x - 5$

$9x - 2y = 4$ Substitute $(3x - 5)$ for y in the second equation.

$9x - 2(\qquad) = 4$

______________ $= 4$

______________ $= 4$

$x =$ _____ Substitute this value of x into the equation first equation to find y.

$y = 3 \ x - 5$

$y = 3(\quad) - 5$

$y =$ _____ Remember to use the other equation to check

Check: $9x - 2y = 4$

$9(\quad) - 2(\quad) = 4$

_______________ = ___

43. $3x + 5y = 39$
$y = 2x$

44. $y = x - 2$
$3x + 5y = 14$

45. $3x - 5y = -10$
$y = 2x - 5$

46. $3x - 5y = 8$
$x = 3y - 4$

47. $5y - 3x = 5$
$x = 2y + 1$

48. $y = 4 - x$
$2x - y = 11$

49. $x = 5y + 24$
$y = 3x - 2$

50. $y = 4x - 25$
$x = 3y - 2$

In 51 - 52, to solve by substitution, it will be necessary to solve for one of the variables. Choose the easiest variable to solve.

51. $x - 5y = 13$
$2x + 7y = -8$

$x =$ ________

$2(\quad\quad) + 7y = -8$

52. $7x - 4y = 40$
$y - x = 2$

In #53 - 64, solve the systems of equations by the "appropriate" method. Indicate if the equations represent parallel lines or the same line.

53. $3x + 7y = 6$
$2x + 3y = -1$

54. $-3x + 7y = 4$
$2x - 3y = -6$

55. $9x - 4y = 2$
$2x + 5y = -29$

56. $50x - 9y = 1$
$7x - 2y = -8$

57. $2x - 6y = 12$
$-x + 3y = -6$

58. $x = 3y + 18$
$6y - 2x = 36$

59. $5x - 4y = 22$
$y = -4x + 5$

60. $-8x + 6y = 32$
$x = 2y + 6$

61. $17x + 8y = 4$
$32x + 18y = -16$

62. $4x - 2y = -8$
$2x - y = -4$

63. $4x - 2y = 8$
$y = 2x + 4$

64. $12y + 5x = 41$
$x = 4 - 3y$

ANSWERS 4.07

p.351 - 368:

1. (2,0); **2.** (3,-2); **3.** (1,2); **4.** (1,-2); **5.** No Sol--Parallel Lines; **6.** Same line; **7.** Same line; **8.** No Sol--Parallel Lines; **9.** (-2,1); **10.** (-2,3); **11.** (4,-3); **12.** (4,5); **13.** (2,2); **14.** (2,3); **15.** Same line; **16.** No Sol--Parallel Lines; **17.** (2,-8); **18.** (3,-2); **19.** (2,3); **20.** (-2,1); **21.** (5,6); **22.** (6,1); **23.** (-4,-3); **24.** (-4,2); **25.** (4,2); **26.** (5,3); **27.** (15,-7); **28.** (4,1); **29.** (2,2); **30.** (14,-2); **31.** (6,-8); **32.** (-3,2); **33.** (-26,16); **34.** (-2,-1); **35.** (20,24); **36.** (-7,12); **37.** No Sol--Parallel; **38.** Same line; **39.** (-7,0); **40.** No Sol--Parallel; **41.** (3,2); **42.** (-2,-11); **43.** (3,6); **44.** (3,1); **45.** (5,5); **46.** (11,5); **47.** (-15,-8); **48.** (5,-1); **49.** (-1,-5); **50.** (7,3); **51.** (3,-2); **52.** (16,18); **53.** (-5,3); **54.** (-6,-2); **55.** (-2,-5); **56.** (2,11); **57.** Same line; **58.** No Sol--Parallel; **59.** (2,-3); **60.** (-10,-8); **61.** (4,-8); **62.** Same line; **63.** No Sol--Parallel; **64.** (25,-7).