5.02 *Cube Roots and More*

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In the previous section, you learned that the expression $\sqrt{x}$ or $\sqrt[2]{x}$ means the **square root of** ***x***. This essentially means, "**What squared would equal *x*?**" The quantity inside the radical sign (in this case ***x***) is called the **radicand**, and the **2** (in this case) is called the **index of the radical**. Now, the expression $\sqrt[3]{x}$ is called the **cube root of x,** and it asks the question, "**What cubed would equal *x*?**" Likewise, $\sqrt[4]{x}$ means the fourth root of ***x***, $\sqrt[5]{x}$ means the fifth root of ***x***, etc. In general, $\sqrt[n]{x}$ means the **nth root of *x*,** where the **radicand** is *x*, and the **index of the radical** is **n.**

The operations of **square root, cube root, fourth root, etc.** are actually **inverse operations** for the operations of **squaring, cubing, raising to the fourth power, etc.** When taking **square roots** in the last section, it was essential to be familiar with the **perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144,** and **169.** Also, remember that the **even powers (x^2, x^4, x^6, x^8, x^{10}, etc.)** were and are perfect squares. Now, when taking a **cube root**, it is essential to be familiar with (i.e., **memorize them!!**) the **perfect cubes,** and other powers, especially the numbers **1, 8, 27, 64,** and **125.**

$2^3 = 8$	$2^4 = 16$	$2^5 = 32$
$3^3 = 27$	$3^4 = 81$	
$4^3 = 64$		
$5^3 = 125$		

And again, the list goes on. However, these are the main numbers that we use, and with which you need to be familiar. You really need to have the numbers **8, 27, 64,** and **125** in your head before you continue this lesson!!

Taking a **cube root** of a number is actually the **inverse operation of cubing**. Suppose you cubed the number 5. The answer of course is 125. Now, what would you have to do to the 125 to get back to the 5? You would take the cube root of 125, written $\sqrt[3]{125}$, which is 5.

NOTE: **Frequently the terminology "square root" and "radical" are used interchangeably. They are NOT the same. The term "radical" may be used generally to refer to a square root, cube root, etc. The term "square root" does not include cube roots, fourth roots, etc.**

EXAMPLE 1. Fill in the blanks below to find $\sqrt[3]{\mathbf{8}}$.

$\sqrt[3]{\mathbf{8}}$ = ______ because ($)^3$ = _______.

Solution: $\sqrt[3]{\mathbf{8}}$ = __2__ because (2 $)^3$ = __8__.

Therefore, the cube root of 8 or $\sqrt[3]{\mathbf{8}}$ is 2.

EXAMPLE 2. Fill in the blanks below to find $\sqrt[4]{\mathbf{81}}$.

$\sqrt[4]{\mathbf{81}}$ = ______ because ($)^4$ = _______.

Solution: $\sqrt[4]{\mathbf{81}}$ = __3__ because (3 $)^4$ = __81__.

Therefore, the fourth root of 81 or $\sqrt[4]{\mathbf{81}}$ is 3.

EXERCISES. Fill in the blanks to find the radical expressions.

1. $\sqrt[3]{\mathbf{27}}$ = _______ because ($)^3$ = 27.
2. $\sqrt[3]{\mathbf{64}}$ = _______ because ($)^3$ = 64.
3. $\sqrt[3]{\mathbf{1}}$ = _______ because ($)^3$ = ______.
4. $\sqrt[3]{\mathbf{0}}$ = _______ because ($)^3$ = ______.
5. $\sqrt[3]{\mathbf{125}}$ = _______ because ($)^3$ = ______.
6. $\sqrt[3]{\mathbf{8}}$ = _______ because ($)^3$ = ______.
7. $\sqrt[4]{\mathbf{16}}$ = _______ because ($)^4$ = ______.
8. $\sqrt[4]{\mathbf{81}}$ = _______ because ($)^4$ = ______.
9. $\sqrt[5]{\mathbf{32}}$ = _______ because ($)^5$ = ______.
10. $\sqrt[5]{\mathbf{1}}$ = _______ because ($)^5$ = ______.

The following exercises are repeated for extra practice!

11. $\sqrt[3]{64}$ 12. $\sqrt[3]{125}$ 13. $\sqrt[4]{16}$ 14. $\sqrt[3]{0}$

15. $\sqrt[3]{27}$ 16. $\sqrt[3]{8}$ 17. $\sqrt[5]{32}$ 18. $\sqrt[4]{81}$

In the next exercises, you will need to remember and use the law of exponents: $(x^m)^n = x^{mn}$. When you raise an exponent to a power, you multiply the exponents. In particular, when you cube an exponent, you multiply the exponent times 3, when you raise a power to the fourth power, you multiply the exponent times 4, when you raise to the fifth power, you multiply times 5, etc..

EXAMPLE 3. Simplify $(x^4)^3$

Solution: $(x^4)^3 = x^{12}$

EXAMPLE 4. Simplify $(x^5)^4$

Solution: $(x^5)^4 = x^{20}$

EXAMPLE 5. Complete the blanks in order to find $\sqrt[3]{x^{12}}$.

$\sqrt[3]{x^{12}}$ = _______ because ( )^3 = ________.

Solution: $\sqrt[3]{x^{12}}$ = $\underline{x^4}$ because (x^4)^3 = $\underline{x^{12}}$.

Therefore, the cube root of x^{12} (or $\sqrt[3]{x^{12}}$) is x^4 .

EXAMPLE 6. Complete the blanks in order to find $\sqrt[4]{x^{20}}$.

$\sqrt[4]{x^{20}}$ = _______ because ( )^4 = ________.

Solution: $\sqrt[4]{x^{20}}$ = $\underline{x^5}$ because (x^5)^4 = $\underline{x^{20}}$.

Therefore, the fourth root of x^{20} (or $\sqrt[4]{x^{20}}$) is x^5 .

EXERCISES. Complete the blanks in order to simplify the radicals.

19. $\sqrt[3]{x^{15}}$ = _____ because ($)^3$ = ________.

20. $\sqrt[3]{x^{30}}$ = _____ because ($)^3$ = ________.

21. $\sqrt[4]{x^{24}}$ = _____ because ($)^4$= ________.

22. $\sqrt[5]{x^{15}}$ = _____ because ($)^5$ = ________.

Simplify each of the following:

23. $\sqrt[3]{x^{6}}$ 24. $\sqrt[3]{x^{24}}$ 25. $\sqrt[4]{x^{24}}$ 26. $\sqrt[4]{x^{40}}$ 27. $\sqrt[5]{x^{40}}$ 28. $\sqrt[5]{x^{30}}$

29. $\sqrt[3]{x^{12}}$ 30. $\sqrt[5]{x^{25}}$ 31. $\sqrt[4]{x^{8}}$ 32. $\sqrt[4]{x^{16}}$ 33. $\sqrt[6]{x^{30}}$ 34. $\sqrt[10]{x^{30}}$

35. $\sqrt[3]{125x^{6}}$ 36. $\sqrt[3]{8x^{12}}$ 37. $\sqrt[3]{27x^{27}}$ 38. $\sqrt[3]{64x^{51}}$

39. $\sqrt[4]{16x^{16}}$ 40. $\sqrt[4]{81x^{12}}$ 41. $\sqrt[5]{32x^{20}}$ 42. $\sqrt[5]{32x^{60}}$

The Product Property of Square Roots from the last section can be extended to cube roots, fourth roots, and in general nth roots. When extended beyond square roots, we call this the **Product Property of Radicals**. This property can often be used to simplify radicals involving cube roots, fourth roots, etc.

Product Property of Radicals

- -

$$\sqrt{a \bullet b} = \sqrt{a} \bullet \sqrt{b}$$

$$\sqrt[3]{a \bullet b} = \sqrt[3]{a} \bullet \sqrt[3]{b}$$

$$\sqrt[n]{a \bullet b} = \sqrt[n]{a} \bullet \sqrt[n]{b}$$

As with the product property of square roots, the product property of radicals is a property of **real numbers**. Therefore, if the **index** of the radical is **even**, then the **radicands** must be positive. The same "radical two step" method still applies, except that the second step is now "cube root," "fourth root," etc. In the first step, you must **"sort"** the radical, placing the **"perfect powers"** in the first radical and the other "leftover" factors in the second radicals. In the second step, you take the appropriate root of the perfect power, and just bring down the "leftover" radical.

EXAMPLE 7. **Simplify $\sqrt[3]{24}$ and calculate its value to the nearest hundredth.**

Step 1. Find a perfect cube that is a factor of 24, and write 24 as a product. The perfect cube is 8; the "left-over" factor is 3, and the product is 8·3.

$\sqrt[3]{8} \cdot \sqrt[3]{3}$

Step 2. Take the cube root of 8, which is 2. Keep the cube root sign on the 3.

$2\sqrt[3]{3}$.

Use your calculator (see Section 1.04), $\sqrt[3]{24}$ is approximately 2.88449914, which rounds to **2.88**.

EXAMPLE 8. **Simplify $\sqrt[3]{56}$ and calculate its value to the nearest hundredth.**

Step 1. Find a perfect cube that is a factor of 56, and write as a product. The perfect cube is 8, the leftover factor is 7, and the product is 8·7.

$\mathbf{\sqrt[3]{8} \bullet \sqrt[3]{7}}$

Step 2. Take the cube root of 8, which is 2. Keep the cube root sign on the 7.

$\mathbf{2 \bullet \sqrt[3]{7}}$ or $\mathbf{2\sqrt[3]{7}}$

The decimal approximation is 3.82586236554, which rounds to **3.83**.

EXAMPLE 9. **Simplify $\sqrt[3]{250}$ and calculate its value to the nearest hundredth.**

Step 1. Find a perfect cube that is a factor of 250, and write as a product. The perfect cube is 125, the leftover factor is 2, and the product is 125·2.

$\mathbf{\sqrt[3]{125} \bullet \sqrt[3]{2}}$.

Step 2. Take the cube root of 125, which is 5. Keep the cube root sign on the 5.

$\mathbf{5\sqrt[3]{2}}$, which is approximately **6.30**.

EXERCISES. **Simplify the radicals completely. Find the values to the nearest hundredth.**

43. $\sqrt[3]{48}$

$\sqrt[3]{8} \bullet \sqrt[3]{}$

____

44. $\sqrt[3]{162}$

$\sqrt[3]{27} \bullet \sqrt[3]{}$

____

45. $\sqrt[3]{500}$

$\sqrt[3]{125} \bullet \sqrt[3]{}$

____

46. $\sqrt[3]{640}$

$\sqrt[3]{64} \bullet \sqrt[3]{}$

____

47. $\sqrt[3]{96}$

$\sqrt[3]{8} \bullet \sqrt[3]{}$

____

48. $\sqrt[3]{1250}$

$\sqrt[3]{125} \bullet \sqrt[3]{}$

____

49. $\sqrt[3]{16}$

$\sqrt[3]{} \cdot \sqrt[3]{}$

____

50. $\sqrt[3]{40}$

$\sqrt[3]{} \cdot \sqrt[3]{}$

____

51. $\sqrt[3]{72}$

$\sqrt[3]{} \cdot \sqrt[3]{}$

____

52. $\sqrt[3]{128}$

____

____

53. $\sqrt[3]{250}$

____

____

54. $\sqrt[3]{54}$

____

____

55. $\sqrt[3]{24}$

____

____

56. $\sqrt[3]{88}$

____

____

57. $\sqrt[3]{81}$

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____

58. $\sqrt[3]{80}$

____

____

59. $\sqrt[3]{192}$

____

____

60. $\sqrt[3]{375}$

____

____

61. $\sqrt[3]{32}$

____

____

62. $\sqrt[3]{56}$

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____

63. $\sqrt[3]{270}$

____

____

64. $\sqrt[3]{108}$

____

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EXTRA CHALLENGE

EXAMPLE 10. **Simplify $\sqrt[4]{320}$ and calculate its value to the nearest hundredth.**

Step 1. Find a perfect fourth power that is a factor of 320, and write as a product. The perfect fourth power is 16, the leftover factor is 20, and the product is 16·20.

$\sqrt[4]{16} \bullet \sqrt[4]{20}$.

Step 2. Take the fourth root of 16, which is 2. Keep the fourth root sign on the 20.

$2\sqrt[4]{20}$, which is approximately **4.23**.

EXERCISES. **Simplify the following radicals.**

65. $\sqrt[4]{80}$

$\sqrt[4]{16} \bullet \sqrt[4]{\quad}$

____

66. $\sqrt[4]{405}$

$\sqrt[4]{81} \bullet \sqrt[4]{\quad}$

____

67. $\sqrt[5]{96}$

$\sqrt[5]{32} \bullet \sqrt[5]{\quad}$

____

68. $\sqrt[4]{96}$

$\sqrt[4]{16} \bullet \sqrt[4]{\quad}$

____

69. $\sqrt[4]{32}$

$\sqrt[4]{\quad} \bullet \sqrt[4]{\quad}$

____

70. $\sqrt[4]{810}$

$\sqrt[4]{\quad} \bullet \sqrt[4]{\quad}$

____

71. $\sqrt[5]{64}$

72. $\sqrt[5]{128}$

73. $\sqrt[4]{160}$

74. $\sqrt[4]{162}$

75. $\sqrt[5]{320}$

76. $\sqrt[4]{48}$

EXAMPLE 11. **Simplify $\sqrt[3]{108x^6y^{14}}$.**

Step 1. Find all perfect cube factors of $108x^6y^{14}$, and write as a product. The perfect cube is 27, with a leftover factor of 4. Also, x^6 and y^{12} are perfect cubes, leaving y^2 as leftover. The product is $27x^6y^{12} \cdot 4y^2$.

$$\sqrt[3]{27x^6y^{12}} \bullet \sqrt[3]{4y^2} .$$

Step 2. Take the cube root of the first part, and keep the cube root sign on the second.

$$3x^2y^4\sqrt[3]{4y^2} .$$

EXERCISES. **Simplify the radicals completely. Find the values to the nearest hundredth.**

77. $\sqrt[3]{56x^6y^9}$

$\sqrt[3]{8x^6y^9} \bullet \sqrt[3]{\quad}$

____

78. $\sqrt[3]{72x^9y^{15}}$

$\sqrt[3]{\quad} \bullet \sqrt[3]{\quad}$

____

79. $\sqrt[3]{250x^3y^7}$

80. $\sqrt[3]{80x^6y^{10}}$

81. $\sqrt[3]{162x^5y^7}$

82. $\sqrt[3]{54x^{10}y^{20}}$

83. $\sqrt[3]{40x^8y^{12}}$

84. $\sqrt[3]{375x^{27}y^{100}}$

85. $\sqrt[4]{x^{10}y^{12}}$

$\sqrt[4]{} \bullet \sqrt[4]{}$

____

86. $\sqrt[5]{x^{10}y^{12}}$

$\sqrt[5]{} \bullet \sqrt[5]{}$

____

87. $\sqrt[4]{80x^{8}y^{12}}$

88. $\sqrt[4]{405x^{8}y^{10}}$

89. $\sqrt[5]{128x^{6}y^{12}}$

90. $\sqrt[5]{96x^{8}y^{25}}$

91. $\sqrt[4]{48x^{9}y^{13}}$

92. $\sqrt[4]{162x^{14}y^{30}}$

93. $\sqrt[5]{320x^{9}y^{32}}$

94. $\sqrt[4]{160x^{40}y^{90}}$

ANSWERS 5.02

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1. 3; 2. 4; 3. 1; 4. 0; 5. 5; 6. 2; 7. 2; 8. 3; 9. 2; 10. 1; 11. 4; 12. 5; 13. 2; 14. 0; 15. 3; 16. 2; 17. 2; 18. 3; 19. x^5; 20. x^{10}; 21. x^6; 22. x^3; 23. x^2; 24. x^8; 25. x^6; 26. x^{10}; 27. x^8; 28. x^6; 29. x^4; 30. x^5; 31. x^2; 32. x^4; 33. x^5; 34. x^3; 35. $5x^2$; 36. $2x^4$; 37. $3x^9$; 38. $4x^{17}$;

39. $2x^4$; 40. $3x^3$; 41. $2x^4$; 42. $2x^{12}$; 43. $2\sqrt[3]{6}$; 44. $3\sqrt[3]{6}$; 45. $5\sqrt[3]{4}$; 46. $4\sqrt[3]{10}$;

47. $2\sqrt[3]{12}$; 48. $5\sqrt[3]{10}$; 49. $2\sqrt[3]{2}$; 50. $2\sqrt[3]{5}$; 51. $2\sqrt[3]{9}$; 52. $4\sqrt[3]{2}$; 53. $5\sqrt[3]{2}$;

54. $3\sqrt[3]{2}$; 55. $2\sqrt[3]{3}$; 56. $2\sqrt[3]{11}$; 57. $3\sqrt[3]{3}$; 58. $2\sqrt[3]{10}$; 59. $4\sqrt[3]{3}$; 60. $5\sqrt[3]{3}$;

61. $2\sqrt[3]{4}$; 62. $2\sqrt[3]{7}$; 63. $3\sqrt[3]{10}$; 64. $3\sqrt[3]{4}$; 65. $2\sqrt[4]{5}$; 66. $3\sqrt[4]{5}$; 67. $2\sqrt[4]{3}$;

68. $2\sqrt[4]{6}$; 69. $2\sqrt[4]{2}$; 70. $3\sqrt[4]{10}$; 71. $2\sqrt[5]{2}$; 72. $2\sqrt[5]{4}$; 73. $2\sqrt[4]{10}$; 74. $3\sqrt[4]{2}$;

75. $2\sqrt[5]{10}$; 76. $2\sqrt[4]{3}$; 77. $2x^2y^3\sqrt[3]{7}$; 78. $2x^3y^5\sqrt[3]{9}$; 79. $5xy^2\sqrt[3]{2y}$;

80. $2x^2y^3\sqrt[3]{10y}$; 81. $3xy^2\sqrt[3]{6x^2y}$; 82. $3x^3y^6\sqrt[3]{2xy^2}$; 83. $2x^2y^4\sqrt[3]{5x^2}$;

84. $5x^4y^{11}\sqrt[3]{3y}$; 85. $x^2y^3\sqrt[4]{x^2}$; 86. $x^3y^2\sqrt[5]{y^2}$; 87. $2x^2y^3\sqrt[4]{5}$; 88. $3x^2y^3\sqrt[4]{5y^2}$;

89. $2xy^2\sqrt[5]{4xy^2}$; 90. $2xy^6\sqrt[5]{3x^3}$; 91. $2x^3y^3\sqrt[4]{3xy}$; 92. $3x^3y^7\sqrt[4]{2x^2y^2}$;

93. $2xy^6\sqrt[5]{10x^4y^2}$; 94. $2x^{10}y^{22}\sqrt[4]{10y^2}$.