

## 5.06 Rationalizing Denominators

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

There is a tradition in mathematics of eliminating the radicals from the denominators (or numerators) of fractions. The process is called **rationalizing the denominator (or numerator) of the fraction**. This may be done to simplify the radical expression or to make calculation of the expression easier, especially in days when calculators were not available. For example, knowing the value of  $\sqrt{2}$  to be approximately **1.414**, to calculate  $\frac{20}{\sqrt{2}}$  without a calculator would require **long division** of 20

divided by 1.414. It is much easier to multiply numerator and denominator by  $\sqrt{2}$ ,

$$\frac{20}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{20\sqrt{2}}{2} = 10\sqrt{2}.$$

As you can see, it is easier to multiply **10(1.414)**, than to divide  $\frac{20}{1.414}$ .

### MONOMIAL DENOMINATORS

When rationalizing a **monomial** square root denominator, you need to multiply numerator and denominator by "something" that makes the denominator result in a perfect square. The next examples and exercises in this section illustrate the process.

**EXAMPLE 1.** Rationalize the denominator for  $\frac{20}{\sqrt{10}}$ . Calculate the decimal value.

**Solution:** Multiply the numerator and denominator by  $\sqrt{10}$ .

$$\frac{20}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$\frac{20\sqrt{10}}{10}$$

$$2\sqrt{10}$$

Calculate the decimal value to nearest hundredth:  $\frac{20}{\sqrt{10}} = 6.32$ ;  $2\sqrt{10} = 6.32$ .

**EXERCISES.**

Rationalize the denominators. Calculate the values to the nearest hundredth.

1.  $\frac{6}{\sqrt{2}} \cdot \text{—}$

2.  $\frac{20}{\sqrt{5}}$

3.  $\frac{24}{\sqrt{3}}$

4.  $\frac{40}{\sqrt{10}}$

5.  $\frac{6}{\sqrt{3}}$

6.  $\frac{33}{\sqrt{11}}$

7.  $\frac{34}{\sqrt{17}}$

8.  $\frac{30}{\sqrt{6}}$

9.  $\frac{46}{\sqrt{2}}$

10.  $\frac{98}{\sqrt{7}}$

11.  $\frac{7}{\sqrt{2}}$

12.  $\frac{7}{\sqrt{3}}$

13.  $\frac{12}{\sqrt{15}}$

14.  $\frac{15}{\sqrt{6}}$

15.  $\frac{10}{\sqrt{6}}$

16.  $\frac{6}{\sqrt{10}}$

In the next examples and exercises, there are essentially three steps:

- STEP 1:** Simplify the Radical.  
**STEP 2:** Rationalize the Denominator.  
**STEP 3:** Reduce the Fraction.

**EXAMPLE 2.** Simplify the radical and rationalize the denominator  $\frac{12}{\sqrt{45}}$ .

**Solution:** **STEP 1:** Simplify the radical:  $\frac{12}{\sqrt{9}\sqrt{5}}$   
 $\frac{12}{3\sqrt{5}}$

**STEP 2:** Rationalize the denominator:  $\frac{12}{3\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$   
 $\frac{12\sqrt{5}}{3 \cdot 5}$

**STEP 3:** Reduce the fraction:  $\frac{4\sqrt{5}}{5}$

Calculator values:  $\frac{12}{\sqrt{45}} = 1.79$ ;  $\frac{4\sqrt{5}}{5} = 1.79$ .

**EXERCISES.** Simplify the radicals, reduce the fractions, and calculate the values.

17.  $\frac{6}{\sqrt{18}}$

18.  $\frac{40}{\sqrt{20}}$

19.  $\frac{30}{\sqrt{45}}$

$\frac{6}{3\sqrt{2}}$

$\frac{6}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$\frac{6 \cdot \sqrt{2}}{3 \cdot 2} = \underline{\hspace{2cm}}$

$\underline{\hspace{2cm}}$

20.  $\frac{12}{\sqrt{18}}$

21.  $\frac{12}{\sqrt{20}}$

22.  $\frac{12}{\sqrt{45}}$

23.  $\frac{10}{\sqrt{75}}$

24.  $\frac{8}{\sqrt{80}}$

25.  $\frac{12}{\sqrt{80}}$

26.  $\frac{6}{\sqrt{24}}$

27.  $\frac{18}{\sqrt{72}}$

28.  $\frac{15}{\sqrt{72}}$

Sometimes the radical contains the entire fraction, like  $\sqrt{\frac{12}{5}}$ . In these cases, the quotient property for square roots applies, and you take separate square roots of the numerator and denominator.

**Product Property of Square Roots:**  $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ ,  $a > 0$  and  $b > 0$ .  
**Quotient Property of Square Roots:**  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ ,  $a > 0$  and  $b > 0$ .

**EXAMPLE 3.** Simplify the radical and rationalize the denominator  $\sqrt{\frac{64}{81}}$ .

**Solution:**  $\frac{\sqrt{64}}{\sqrt{81}}$  which is  $\frac{8}{9}$ .

**EXERCISES.** Simplify the radicals using the quotient property of square roots.

29.  $\sqrt{\frac{9}{25}}$

30.  $\sqrt{\frac{81}{4}}$

31.  $\sqrt{\frac{121}{144}}$

32.  $\sqrt{\frac{49}{16}}$

33.  $\sqrt{\frac{32}{81}}$

34.  $\sqrt{\frac{72}{169}}$

**EXAMPLE 4.** Simplify the radical and rationalize the denominator  $\sqrt{\frac{12}{5}}$ .

**Solution:** STEP 1: Simplify the radical:  $\sqrt{\frac{12}{5}}$  means  $\frac{\sqrt{12}}{\sqrt{5}}$

$$\frac{2\sqrt{3}}{\sqrt{5}}$$

STEP 2: Rationalize the denominator:  $\frac{2\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$

$$\frac{2\sqrt{15}}{\sqrt{25}}$$

STEP 3: Reduce the fraction (if possible):  $\frac{2\sqrt{15}}{5}$

Calculator values:  $\sqrt{\frac{12}{5}} = 1.55$ ;  $\frac{2\sqrt{15}}{5} = 1.55$ .

**EXERCISES.**

**Rationalize the denominators and simplify the radical expressions. Calculate the values.**

35.  $\sqrt{\frac{1}{2}}$

36.  $\sqrt{\frac{2}{3}}$

37.  $\sqrt{\frac{18}{5}}$

38.  $\sqrt{\frac{3}{8}}$

39.  $\sqrt{\frac{8}{27}}$

40.  $\sqrt{\frac{7}{12}}$

**EXAMPLE 5.** Simplify the radical and rationalize the denominator  $\frac{6x^3}{\sqrt{48x}}$ .

**Solution:**

STEP 1: Simplify the radical:  $\frac{6x^3}{\sqrt{48x}}$  means  $\frac{6x^3}{\sqrt{16} \sqrt{3x}}$

$$\frac{6x^3}{4\sqrt{3x}}$$

STEP 2: Rationalize the denominator:  $\frac{6x^3}{4\sqrt{3x}} \cdot \frac{\sqrt{3x}}{\sqrt{3x}}$

$$\frac{6x^3 \cdot \sqrt{3x}}{4 \cdot 3x}$$

$$\frac{6x^3 \cdot \sqrt{3x}}{12x}$$

STEP 3: Reduce the fraction (if possible):  $\frac{x^2 \sqrt{3x}}{2}$

$$41. \frac{6}{\sqrt{24x}}$$

$$\frac{6}{\sqrt{4} \sqrt{6x}}$$

$$\frac{6}{2\sqrt{6x}} \cdot \frac{\sqrt{\quad}}{\sqrt{\quad}}$$

$$\frac{6 \cdot \sqrt{\quad}}{2 \cdot (6x)}$$

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$$42. \frac{6x}{\sqrt{12x^3}}$$

$$\frac{6x}{\sqrt{4x^2} \sqrt{3x}}$$

$$43. \frac{30x}{\sqrt{18x^5}}$$

$$\frac{30x}{\sqrt{9x^4} \sqrt{2x}}$$

$$44. \frac{20}{\sqrt{12x}}$$

$$45. \frac{45}{\sqrt{12x^2}}$$

$$46. \frac{40x^3}{\sqrt{20x^3}}$$

$$47. \frac{18x^5}{\sqrt{72x^7}}$$

$$48. \frac{21x^5}{\sqrt{98x^6}}$$

## BINOMIAL DENOMINATORS

When the denominator of the fraction involves **binomial** radical expressions, such as  $\frac{17}{6 - \sqrt{2}}$ , a special procedure is used. Multiplying the numerator and denominator by  $6 + \sqrt{2}$  will eliminate the radicals from the denominator. For the fraction  $\frac{6}{\sqrt{6} + \sqrt{2}}$ , multiply numerator and denominator by  $\sqrt{6} - \sqrt{2}$ . In general, whatever the binomial denominator may be, you multiply the numerator and denominator by the same quantity as the denominator but with the opposite sign in the middle. This is called the **conjugate** of the denominator.

**EXAMPLE 6.** Rationalize the denominator and simplify  $\frac{17}{6 - \sqrt{2}}$ .

**Solution:** Multiply numerator and denominator by  $6 + \sqrt{2}$ :  $\frac{17}{(6 - \sqrt{2})} \cdot \frac{(6 + \sqrt{2})}{(6 + \sqrt{2})}$

Leave the numerator factored, multiply denominator:  $\frac{17(6 + \sqrt{2})}{36 + 6\sqrt{2} - 6\sqrt{2} - 2}$

Notice that the middle term subtracts out:  $\frac{17(6 + \sqrt{2})}{34}$

Reduce the fraction:  $\frac{6 + \sqrt{2}}{2}$



**EXERCISES.**

In each of the following exercises, rationalize the denominators and reduce each fraction to lowest terms.

$$49. \frac{12}{\sqrt{6}-\sqrt{2}} \cdot \frac{(\sqrt{6}+\sqrt{2})}{(\sqrt{6}+\sqrt{2})}$$
$$\frac{12 \cdot (\sqrt{6}+\sqrt{2})}{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})}$$
$$\frac{12 \cdot (\sqrt{6}+\sqrt{2})}{6-2}$$

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$$50. \frac{6}{\sqrt{6}+\sqrt{2}} \cdot \frac{(\sqrt{6}-\sqrt{2})}{(\sqrt{6}-\sqrt{2})}$$
$$\frac{6 \cdot (\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$$

$$51. \frac{12}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$52. \frac{12}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$53. \frac{15}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}}$$

$$54. \frac{15}{3+\sqrt{3}} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}}$$

$$55. \frac{15}{2\sqrt{6} - 3\sqrt{2}}$$

$$56. \frac{6}{3\sqrt{2} + 4\sqrt{3}}$$

**EXAMPLE 7.** Rationalize the denominator and simplify  $\frac{\sqrt{27}}{\sqrt{6} + \sqrt{3}}$ .

**Solution:** First, it may help to simplify the numerator:  $\frac{3\sqrt{3}}{\sqrt{6} + \sqrt{3}}$

Multiply numerator and denominator by  $\sqrt{6} - \sqrt{3}$ :  $\frac{3\sqrt{3}}{(\sqrt{6} + \sqrt{3})} \cdot \frac{(\sqrt{6} - \sqrt{3})}{(\sqrt{6} - \sqrt{3})}$

Multiply numerator and denominator:  $\frac{3\sqrt{18} - 3\sqrt{9}}{\sqrt{36} - \sqrt{18} + \sqrt{18} - \sqrt{9}}$

Simplify the radicals (the middle term subtracts out):  $\frac{3\sqrt{9}\sqrt{2} - 3 \cdot 3}{6 - 3}$

Continue to simplify:  $\frac{9\sqrt{2} - 9}{3}$

Factor the common factor which is 9:  $\frac{9(\sqrt{2} - 1)}{3}$

Reduce the fraction:  $3(\sqrt{2} - 1)$  or  $3\sqrt{2} - 3$ .

**EXERCISES.** Rationalize the denominators and simplify the radical expressions.

57.  $\frac{\sqrt{27}}{\sqrt{6} - \sqrt{3}}$

58.  $\frac{\sqrt{24}}{\sqrt{6} + \sqrt{2}}$

59.  $\frac{\sqrt{20}}{6 - \sqrt{6}}$

60.  $\frac{\sqrt{12}}{6 + \sqrt{6}}$

61.  $\frac{\sqrt{12}}{6\sqrt{2} + \sqrt{6}}$

62.  $\frac{\sqrt{27}}{2\sqrt{6} - 3\sqrt{3}}$

**EXAMPLE 8.** Rationalize the denominator and simplify  $\frac{3+\sqrt{6}}{3-\sqrt{3}}$ .

**Solution:** Multiply numerator and denominator by  $3+\sqrt{3}$ :  $\frac{(3+\sqrt{6})}{(3-\sqrt{3})} \cdot \frac{(3+\sqrt{3})}{(3+\sqrt{3})}$

Multiply numerator and denominator:  $\frac{9+3\sqrt{3}+3\sqrt{6}+\sqrt{18}}{9-3}$

Simplify the radical:  $\frac{9+3\sqrt{3}+3\sqrt{6}+3\sqrt{2}}{6}$

Factor the numerator:  $\frac{3(3+\sqrt{3}+\sqrt{6}+\sqrt{2})}{6}$

Reduce the fraction:  $\frac{3+\sqrt{3}+\sqrt{6}+\sqrt{2}}{2}$

**EXERCISES.** Rationalize the denominators and simplify the radical expressions.

63.  $\frac{3-\sqrt{6}}{3+\sqrt{3}}$

64.  $\frac{3-\sqrt{3}}{3-\sqrt{6}}$

65.  $\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}$

66.  $\frac{\sqrt{6}-\sqrt{3}}{\sqrt{6}+\sqrt{3}}$

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1.  $3\sqrt{2}$ , 4.24; 2.  $4\sqrt{5}$ , 8.94; 3.  $8\sqrt{3}$ , 13.86; 4.  $4\sqrt{10}$ , 12.65; 5.  $2\sqrt{3}$ , 3.46;
6.  $3\sqrt{11}$ , 9.95; 7.  $2\sqrt{17}$ , 8.25; 8.  $5\sqrt{6}$ , 12.25; 9.  $23\sqrt{2}$ , 32.53; 10.  $14\sqrt{7}$ , 37.04;
11.  $\frac{7\sqrt{2}}{2}$ , 4.95; 12.  $\frac{7\sqrt{3}}{3}$ , 4.04; 13.  $\frac{4\sqrt{15}}{5}$ , 3.10; 14.  $\frac{5\sqrt{6}}{2}$ , 6.12; 15.  $\frac{5\sqrt{6}}{3}$ , 4.08;
16.  $\frac{3\sqrt{10}}{5}$ , 1.90; 17.  $\sqrt{2}$ , 1.41; 18.  $4\sqrt{5}$ , 8.94; 19.  $2\sqrt{5}$ , 4.47; 20.  $2\sqrt{2}$ , 2.83;
21.  $\frac{6\sqrt{5}}{5}$ , 2.68; 22.  $\frac{4\sqrt{5}}{5}$ , 1.79; 23.  $\frac{2\sqrt{3}}{3}$ , 1.15; 24.  $\frac{2\sqrt{5}}{5}$ , 0.89; 25.  $\frac{3\sqrt{5}}{5}$ , 1.34;
26.  $\frac{\sqrt{6}}{2}$ , 1.22; 27.  $\frac{3\sqrt{2}}{2}$ , 2.12; 28.  $\frac{5\sqrt{2}}{4}$ , 1.77; 29.  $\frac{3}{5}$ ; 30.  $\frac{9}{2}$ ; 31.  $\frac{11}{12}$ ; 32.  $\frac{7}{4}$ ;
33.  $\frac{4\sqrt{2}}{9}$ ; 34.  $\frac{6\sqrt{2}}{13}$ ; 35.  $\frac{\sqrt{2}}{2}$ , 0.71; 36.  $\frac{\sqrt{6}}{3}$ , 0.82; 37.  $\frac{3\sqrt{10}}{5}$ , 1.90; 38.  $\frac{\sqrt{6}}{4}$ , 0.61;
39.  $\frac{2\sqrt{6}}{9}$ , 0.54; 40.  $\frac{\sqrt{21}}{6}$ , 0.76; 41.  $\frac{\sqrt{6x}}{2x}$ ; 42.  $\frac{\sqrt{3x}}{x}$ ; 43.  $\frac{5\sqrt{2x}}{x^2}$ ; 44.  $\frac{10\sqrt{3x}}{3x}$ ;
45.  $\frac{15\sqrt{3}}{2x}$ ; 46.  $4x\sqrt{5x}$ ; 47.  $\frac{3x\sqrt{2x}}{2}$ ; 48.  $\frac{3x^2\sqrt{2}}{2}$ ; 49.  $3(\sqrt{6}+\sqrt{2})$ ; 50.  $\frac{3(\sqrt{6}-\sqrt{2})}{2}$ ;
51.  $6(2-\sqrt{2})$ ; 52.  $6(2+\sqrt{2})$ ; 53.  $\frac{5(3+\sqrt{3})}{2}$ ; 54.  $\frac{5(3-\sqrt{3})}{2}$ ; 55.  $\frac{5(2\sqrt{6}+3\sqrt{2})}{2}$ ;
56.  $\frac{3\sqrt{2}-4\sqrt{3}}{-5}$  or  $\frac{4\sqrt{3}-3\sqrt{2}}{5}$ ; 57.  $3(\sqrt{2}+1)$ ; 58.  $3-\sqrt{3}$ ; 59.  $\frac{6\sqrt{5}+\sqrt{30}}{15}$ ; 60.  $\frac{2\sqrt{3}-\sqrt{2}}{5}$ ;
61.  $\frac{2\sqrt{6}-\sqrt{2}}{11}$ ; 62.  $-3(2\sqrt{2}+3)$ ; 63.  $\frac{3-\sqrt{3}-\sqrt{6}+\sqrt{2}}{2}$ ; 64.  $3+\sqrt{6}-\sqrt{3}-\sqrt{2}$ ; 65.  $3+2\sqrt{2}$ ;
66.  $3-2\sqrt{2}$ .

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