

1.04 Equations and Properties of Equations

Linear, Absolute Value, and Literal

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

Perhaps the premiere task in all of mathematics is the solving of equations. There are many, many types of equations to be solved, from simple **linear equations** (such as $2X = 6$) in a first year algebra course to **differential equations** (equations involving "derivatives") in higher mathematics courses. The solution to an equation is the set of all replacement values of the variable for which the equation is true. If an equation is true for all values of the variable, then the equation is called an **identity**. If the equation is true for some, but not all, values of the variable then the equation is called a **conditional equation**. If the equation is **never** true for any value of the variable, then the equation is called a **contradiction**, and there is **no solution**. "No solution" is frequently represented by the **empty set**, " $\{ \}$ " or the greek letter phi " ϕ ". The great majority of equations you will encounter will be conditional equations.

PROPERTIES OF EQUATIONS

Methods of solving equations are as varied as the types of equations to be solved. However varied the strategies may be, all must be executed according to and without violating several properties of equations:

- 1. REFLEXIVE PROPERTY** $a = a$. Any number is equal to itself.
- 2. SYMMETRIC PROPERTY** If $a = b$, then $b = a$. The order in which the equality is given does not matter. For example, you can say " $X=4$ " or " $4=X$ ", the meaning is the same--the value of X is 4.
- 3. TRANSITIVE PROPERTY** If $a = b$ and $b = c$, then $a = c$.
The word "trans" means "across." If you can get from point "a" to "b", and then from "b" to "c", then you can get from "a" across "b" to "c."

4. ADDITION PROPERTYIf $a = b$, then $a + c = b + c$ If $a = b$, then $a - c = b - c$.

The same number may be added (or subtracted) from both sides of an equation.

5. MULTIPLICATION PROPERTYIf $a=b$, then $ac = bc$ If $a=b$ and $c \neq 0$, then $a/c = b/c$.

Both sides of an equation may be multiplied or divided by the same non-zero number.

LINEAR EQUATIONS

A linear equation in one variable is an equation in which the highest degree of the variable is one (no variable squared, cubed, or higher terms). We usually think of an equation being "linear" as opposed to being "quadratic". If it is a linear conditional equation, there will be only one solution. This section provides an opportunity to distinguish between conditional equations, identities, and contradictions.

EXAMPLE 1. Solve for X:

$$\begin{aligned}
 5(3X-4) - X(X-5) &= X(5-X) \\
 15X - 20 - X^2 + 5X &= 5X - X^2 \\
 -X^2 + 20X - 20 &= 5X - X^2 \\
 +X^2 - 20X &\quad -20X + X^2 \\
 \hline
 -20 &= -15X \\
 \frac{-20}{-15} &= \frac{-15X}{-15}
 \end{aligned}$$

$$X = \frac{4}{3}$$

EXAMPLE 3. Solve for X:

$$\begin{aligned}
 5(3X-4) - X(X-5) &= X(20-X) - 20 \\
 15X - 20 - X^2 + 5X &= 20X - X^2 - 20 \\
 -X^2 + 20X - 20 &= -X^2 + 20X - 20
 \end{aligned}$$

IDENTITY

TRUE for ALL X

EXAMPLE 2. Solve for X:

$$\begin{aligned}
 5(3X-4) - X(X-5) &= 4X(5-X) + 3X^2 \\
 15X - 20 - X^2 + 5X &= 20X - 4X^2 + 3X^2 \\
 -X^2 + 20X - 20 &= -X^2 + 20X \\
 +X^2 - 20X &\quad +X^2 - 20X \\
 \hline
 -20 &= 0
 \end{aligned}$$

NO WAY!

CONTRADICTION - NO SOLUTION

EXAMPLE 4. Solve for X:

$$\begin{aligned}
 5(3X-4) - X(X-5) &= X(15-X) - 20 \\
 15X - 20 - X^2 + 5X &= 15X - X^2 - 20 \\
 -X^2 + 20X - 20 &= -X^2 + 15X - 20 \\
 +X^2 - 15X + 20 &\quad +X^2 - 15X + 20 \\
 \hline
 5X &= 0
 \end{aligned}$$

$$X = 0$$

EXERCISES: Solve the equations for X. Identify which are contradictions, identities, or conditional equations.

1. $4(X+3) = 6(2X-5) - 2X$ 2. $6(X+3) = 3(2X-3) + 27$

3. $6(X+3) = 3(6-2X) + 4X$ 4. $6(X+3) - 3(5-2X) = 12X$

5. $6(X+3) - 3(6-2X) = 12X$ 6. $X(X-6) = 4 - X(2-X)$

7. $X(X-2) = 4 - X(2-X)$ 8. $X(3X-8) = 12X - 3X(4-X)$

ABSOLUTE VALUE EQUATIONS

The absolute value of a number refers to the "size" of a number or the "magnitude" of a number without regard to whether the number is positive or negative. You remember that the absolute value of a number cannot be negative. The following formal definition of absolute value may at first appear to contradict this last statement.

$$\begin{aligned} \text{DEFINITION: } |X| &= X \text{ if } X \geq 0 \\ &= -X \text{ if } X < 0 \end{aligned}$$

Does it appear that in the second part of the definition $|X| = -X$, that the absolute value of X equals a "negative"? What you must remember is that in the second part of the definition, X is itself negative, that the absolute value of X is actually the negative of the negative, which is positive! This formal definition of absolute value of X confirms the fact that there are generally two cases to consider--there are two solutions to be found.

Consider the simple example, $|X| = 3$. Obviously, the solutions are $X = 3$ and $X = -3$. Likewise $|\text{Junk}| = 3$ has two solutions: $\text{Junk} = 3$ and $\text{Junk} = -3$. The next exercises illustrate a concept in which the variable, instead of being " X " or " Junk ", is " $X - 2$ " or " $3X + 2$ ":

Solve the absolute value equations:

1. $|X| = 4$
X=_____ or X=_____

2. $|X| = 6$

3. $|X| = 10$

4. $|Y| = 4$
Y=_____ or Y=_____

5. $|\$| = 6$

6. $|\text{Junk}| = 10$

7. $|X-2| = 4$
X-2=_____ or X-2=_____

8. $|X-2| = 6$

9. $|X-2| = 10$

X=_____ or X=_____

10. $|2X-2| = 4$
 $2X-2=$ or $2X-2=$

$2X=$ or $2X=$

$X=$ or $X=$

11. $|2X-2| = 6$

12. $|2X-2| = 10$

13. $|3X+2| = 4$
 $3X+2=$ or $3X+2=$

14. $|3X+2| = 6$

15. $|3X+2| = 10$

$3X=$ or $3X=$

$X=$ or $X=$

16. $|2X-7| = 5$

17. $|2X-7| = 7$

18. $|3X+6| = 6$

What if you have $|X| = -4$ or $|3X+6| = -4$? Notice that an absolute value cannot equal a negative. Therefore there is no solution. The method used in the explanation so far is not valid for absolute value equal to negatives. In such cases there is no solution. Study the following examples.

EXAMPLE 1. Solve for X:

$|2X - 5| = 3$

Solution:

$2X - 5 = 3$ or $2X - 5 = -3$

$2X = 8$ $2X = 2$

$X = 4$ $X = 1$

Check:

$X=4$ $X=1$

$|2(4)-5| = 3$ $|2(1)-5| = 3$

$|3| = 3$ $|-3| = 3$

EXAMPLE 2. Solve for X:

$|2X - 5| = -3$

Solution: **No Solution**, since

abs. value cannot

equal a negative.

Notice that in Example 2, because the absolute value equals a negative number, there are not two solutions to solve. In fact, if you try to solve two cases as in Example 2, you missed the problem completely. Whenever an absolute value of any variable equals a negative number, there is **no solution!**

EXAMPLE 3. Solve for X:

$$|2X - 5| = |X - 40|$$

Since there are two absolute values in each of these examples, it might appear that there should be two cases for each for a total of $2 \times 2 = 4$ cases to solve. The four cases are as follows:

Case I: Positive = Positive

$$(2X - 5) = (X - 40)$$

Case III: Negative = Negative

$$-(2X - 5) = -(X - 40)$$

Case II: Positive = Negative

$$(2X - 5) = -(X - 40)$$

Case IV: Negative = Positive

$$-(2X - 5) = (X - 40)$$

However, before solving all four cases, notice that Case III is actually Case I, where both sides of the equation were multiplied by -1 . Also, Case IV is the same as Case II, with both sides multiplied by -1 . Therefore, you need only solve Cases I and II.

Solution:

Case I: $2X - 5 = X - 40$

$$X = -35$$

Case II: $2X - 5 = -(X - 40)$

$$2X - 5 = -X + 40$$

$$3X = 45$$

$$X = 15$$

Check:

$$X = -35$$

$$|2(-35) - 5| = |(-35) - 40|$$

$$|-70 - 5| = |-75|$$

$$|-75| = |-75|$$

$$X = 15$$

$$|2(15) - 5| = |15 - 40|$$

$$|10 - 5| = |-5|$$

$$|5| = |-5|$$

The solution for Example 3 is $X = -35, 15$.

EXAMPLE 4. Solve for X:

$$|2X - 5| = |2X - 15|$$

Case I: $2X - 5 = 2X - 15$

$$-5 = -15$$

No Solution for Case I

Case II: $2X - 5 = -(2X - 15)$

$$2X - 5 = -2X + 15$$

$$4X = 20$$

$$X = 5$$

The solution is only $X = 5$.

SUMMARY

I. For $c \geq 0$, $|aX + b| = c$ has two cases to solve:

$$aX + b = c \quad \text{or} \quad aX + b = -c$$

[Note: If $c=0$, the two cases are the same!]

II. For $c < 0$, $|aX + b| = c$ has **No Solution!**

III. $|aX + b| = |cX + d|$ has two cases to solve:

$$aX + b = cX + d \quad \text{or} \quad aX + b = -(cX + d)$$

EXERCISES:

1. $|2X - 7| = 5$

2. $|2X - 7| = -5$

3. $|3X + 6| = -18$

4. $|3X + 6| = 18$

5. $|3X - 5| = 5$

6. $|3X - 5| = 10$

$$7. |4X - 12| = 0$$

$$8. |4X + 12| = 0$$

$$9. |2X - 3| = |X + 6|$$

$$10. |2X + 3| = |4X - 9|$$

$$11. |3X - 4| = |12 - X|$$

$$12. |3X + 4| = |12 - X|$$

$$13. |2X + 4| = |12 - 2X|$$

$$14. |3X - 5| = |5 + 3X|$$

$$15. |2X - 3| = |3 - 2X|$$

$$16. |8 - X| = |8 + X|$$

LITERAL EQUATIONS

Frequently equations (formulas) to be solved are expressed in terms of several different letters. You will probably recognize some of the formulas that we use here, since many of them come from science, business, geometry, and other areas of life. Other formulas have been made up especially for practice in this section. These are also called **literal equations**, because there are so many different "litters" (joke) in them. In these equations, you will be solving for one variable (letter) in terms of all the other variables (letters) in the equation. The following steps will be helpful in solving literal equations.

- STEP 1:** Identify the "variable" terms (that is, the terms that have the variable you are solving for), and get them all on one side of the equation.
- STEP 2:** Identify all "non-variable" terms (that is, all terms that do not have the variable you are solving for), and get these terms on the other side of the equation. Remember, "non-variable" terms may have other variables or constants--just not the variable you are solving for. **(Be careful not to combine unlike terms.)**
- STEP 3:** If you have more than one variable term, you must **factor** in order to get the variable in one place.
- STEP 4:** Divide both sides of the equation by all factors that are in the product with the variable. This leaves the variable alone, and the answer may be a very strange-looking fraction. That's okay--the answers will probably be very abstract. Just do what you know is correct algebraically and have confidence in your work!

The following series of exercises are designed to lead you through the process. In each of the following, solve for X:

1. $aX = b$ (Divide both sides by a)
2. $aX + b = c$ (Subtract b)
(Divide by a)

3. $aX - b = c$

4. $aX + bX = c$ (Factor X)

(Divide by _____)

5. $aX = bX + c$ (Subt. bX)

(Factor X)

(Divide by _____)

6. $aX + b = cX + d$ (Subt cX)

(Subt b)

(Factor X)

(Divide)

7. $aX - b = c - dX$ (Add dX)

8. $aX - b = cX - d$

9. $a(X + b) = c(X + d)$

10. $a(X - b) = c(d - X)$

11. $Y = mX + b$

12. $Y - a = m(X - b)$

13. $AX + BY = C$

14. $AX - BY = C$

15. $BY - AX = C$

16. $P = 2X + 2Y$

Frequently, you are asked to solve for a variable other than X. In each of the following, solve for the variable as indicated:

17. $a = bx + c$, for c

18. $a = bx + c$, for b

19. $a = bx + c$, for x

20. $I = Prt$, for P

21. $I = Prt$, for r

22. $C = 2\pi r$, for r 23. $V = LWH$, for H 24. $V = LWH$, for W

25. $AX + BY = C$, for Y

26. $AX + BY = C$, for A

27. $A = \frac{1}{2}bh$, for h (Mult both sides of equation by 2)

(Divide both sides by b)

28. $A = \frac{1}{2}bh$, for b

29. $V = \frac{1}{3}\pi r^2 h$, for h (Mult by _____)

30. $V = \frac{4}{3}\pi r^3$, for r^3

(Divide by _____)

ANSWERS 1.04

- p. 41: 1. Conditional equation, 7; 2. Identity, all reals;
 3. Conditional equation, 0; 4. Contradiction, No solution;
 5. Identity, all reals; 6. Conditional equation, -1;
 7. Contradiction, no sol; 8. Conditional equation, \emptyset .
- p. 42-43: 1. 4, -4; 2. 6, -6; 3. 10, -10; 4. 4, -4; 5. 6, -6; 6. 10, -10;
 7. 6, -2; 8. 8, -4; 9. 12, -8; 10. 3, -1; 11. 4, -2; 12. 6, -4;
 13. $2/3, -2$; 14. $4/3, -8/3$; 15. $8/3, -4$; 16. 6, 1; 17. 7, 0;
 18. 0, -4.
- p. 45-46: 1. 6, 1; 2. No sol; 3. No sol; 4. 4, -8; 5. 0, 10/3; 6. 5, -5/3;
 7. 3; 8. -3; 9. 9, -1; 10. 6, 1; 11. 4, -4; 12. 2, -8; 13. 2;
 14. 0; 15. All reals; 16. 0.

- p. 47-50: 1. $\frac{b}{a}$ 2. $\frac{c-b}{a}$ 3. $\frac{b+c}{a}$ 4. $\frac{c}{a+b}$ 5. $\frac{c}{a-b}$
6. $\frac{d-b}{a-c}$ 7. $\frac{b+c}{a+d}$ 8. $\frac{b-d}{a-c}$ 9. $\frac{cd-ab}{a-c}$ 10. $\frac{ab+cd}{a+c}$
11. $\frac{y-b}{m}$ 12. $\frac{y-a+mb}{m}$ 13. $\frac{C-BY}{A}$ 14. $\frac{C+BY}{A}$
15. $\frac{BY-C}{A}$ 16. $\frac{P-2Y}{2}$ 17. $a-bX$ 18. $\frac{a-c}{x}$ 19. $\frac{a-c}{b}$
20. $\frac{I}{rt}$ 21. $\frac{I}{Pt}$ 22. $\frac{C}{2\pi}$ 23. $\frac{V}{LW}$ 24. $\frac{V}{LH}$ 25. $\frac{C-AX}{B}$
26. $\frac{C-BY}{X}$ 27. $\frac{2A}{b}$ 28. $\frac{2A}{h}$ 29. $\frac{3V}{\pi r^2}$ 30. $\frac{3V}{4\pi}$