1.08 *Polynomials*

Recall from previous lessons that when algebraic expressions are added (or subtracted) they are called **terms**, while expressions that are multiplied are called **factors**. An algebraic expression that contains only **one term** is called a **monomial.** If the expression has **two terms** is called a **binomial**, and if there are **three terms** it is a **trinomial.** A **polynomial** is an algebraic expression consisting of one or more terms. A polynomial may consist of numbers and variables, where the numerical part of a given term is called the **coefficient.** If there is only one variable in the polynomial, such as **X**, then it is called a **polynomial in X.** The **degree** of a polynomial in one variable is the highest exponent of the variable. If there is more than one variable in the polynomial, then the **degree** is the highest "sum of the exponents" of the variables of a given term.

Frequently polynomials can be **simplified** by combining like terms; sometimes they can be **factored.** Polynomials can be added, subtracted, multiplied **(expanded)**, or divided. Since addition and subtraction of polynomials is little more than combining like terms, and division of polynomials is saved for Chapter 2, this section will involve only the **multiplication (expansion) of polynomial expressions.** The next section is the **factoring of polynomial expressions, followed immediately by solving quadratic equations by factoring.** Notice that **polynomial expressions** are <u>not</u> equations, and therefore can<u>not</u> be "solved." This chapter involves only **polynomial expressions.**

This explanation will begin with a review of products: **monomial times monomial, monomial times binomial, binomial times binomial, binomial times trinomial, and trinomial times trinomial.** The basic property that underlies these products is the **distributive property for multiplication** (products) **over addition** (two or more terms). Also, the law of exponents about products ("when you multiply, you add exponents") is used.

EXAMPLES:

1. **Monomial times monomial:** $4X^2 \cdot 6X^3 = 24X^5$
(by law of exponents)

2. **Monomial times binomial:** $4X^2 \cdot (6X^3 + 5X^2) = 24X^5 + 20X^4$
(by distributive property)

3. **Binomial times binomial:**
(F OI L): $(3X + 5) \cdot (4X + 7) = 12X^2 + 41X + 35$
(This is actually the distributive property applied twice!)

4. **Binomial times trinomial:** $(3X + 5) \cdot (4X^2 + 7X + 2) =$

3X times trinomial:	$12X^3 + 21X^2 + 6X$
5 times trinomial:	$20X^2 + 35X + 10$
Combine like terms:	$= 12X^3 + 41X^2 + 41X + 10$

5. **Trinomial times trinomial:** $(3X^2 + 5X + 9) \cdot (4X^2 + 7X + 2) =$

$3X^2$ times trinomial:	$12X^4 + 21X^3 + 6X^2$
5X times trinomial:	$20X^3 + 35X^2 + 10X$
9 times trinomial:	$36X^2 + 63X + 18$
Combine like terms:	$= 12X^4 + 41X^3 + 77X^2 + 73X + 18$

In the exercises that follow, these products will be followed by exercises that reach to a higher level of abstraction. Hopefully, the "one step" format will help you understand easily.

EXERCISES:

1. **(X + 4)(X + 3)**
2. **(Y + 4)(Y + 3)**
3. **[$ + 4][$ + 3]**
4. **[π + 4][π + 3]**
5. **[(Junk) + 4][(Junk) + 3]**
6. **[(Junk) + 6][(Junk) + 7]**
7. **[(Junk) + 4][(Junk) - 3]**
8. **[(Junk) - 4][(Junk) + 3]**
9. **[(Junk) - 6][(Junk) - 7]**

Consider the problem: **[(X + Y) + 4][(X + Y) + 3]**. There are two ways to find this product--it can be treated as a **"FOIL"** problem (see left below), or a **"product of trinomials"** (see right below):

[(X + Y) + 4][(X + Y) + 3]

$= (X + Y)^2 + 7(X + Y) + 12$

$= X^2 + 2XY + Y^2 + 7X + 7Y + 12.$

(X + Y + 4)(X + Y + 3)

$$\begin{array}{rrrrrr} = X^2 & + XY & +3X & & & \\ & XY & & + Y^2 & +3Y & \\ \hline & & 4X & & +4Y & +12 \end{array}$$

$= X^2 + 2XY + 7X + Y^2 + 7Y + 12.$

Of these two methods, the first is the preferred method. The second may appear easier, but it is not as efficient. Try several problems using the first method; practice other problems using the second method. Be familiar with both ways.

In 10 - 25, expand the polynomials completely:

10. $[(X + Y) - 4][(X + Y) - 3]$ **11.** $[(X + Y) - 4][(X + Y) + 7]$

12. $[(2X + Y) + 4][(2X + Y) - 7]$ **13.** $[(X - 3Y) - 8][(X - 3Y) - 6]$

14. $[(2X-3Y) + 8][(2X-3Y) - 4]$ **15.** $[(3X-2Y) - 8][(3X-2Y) + 4]$

In 16-21, remember that $[(X+Y) + 4]^2 = [(X+Y) + 4][(X+Y) + 4]$.

16. $[(X + Y) + 4]^2$ **17.** $[(X + Y) - 4]^2$

18. $[(3X - 5Y) + 4]^2$ **19.** $[(2X - 5Y) - 3]^2$

20. $[(5X - 2Y) - 8]^2$

21. $[(5X + 4Y) - 6]^2$

22. $[(3X-5Y) - 4][(3X-5Y) + 4]$

23. $[(2X-5Y) - 3][(2X-5Y) + 3]$

24. $[(2X+7Y) - 5][(2X+7Y) + 5]$

25. $[(3X+8Y) - 7][(3X+8Y) + 7]$

26. $(X + Y)^3 = (X + Y)\ (X + Y)\ (X + Y)$

$= (X + Y)\ (X^2 + 2XY + Y^2)$

$=$

$=$

Now consider the problems **$(X + Y)^3$, $(X - Y)^3$, $(X + Y)^4$, $(X - Y)^4$,** etc. These are **binomials** raised to a **power.** The general case, **$(X + Y)^n$**, is called the **Binomial Theorem** or **Binomial Expansion.** (**Pascal's Triangle**, the concept of the next two pages may be helpful. If you find them more confusing than helpful, and you cannot find someone to convince you of the simplicity of the concept, please continue to the next section.)

The following pattern can easily be developed:

$(X + Y)^0 = 1$

$(X + Y)^1 = (1)X + (1)Y$

$(X + Y)^2 = (1)X^2 + (2)XY + (1)Y^2$

$(X + Y)^3 = (1)X^3 + (3)X^2Y + (3)XY^2 + (1)Y^3$

1. $(X + Y)^4 = 1X^4 + \_\_\_ x^3y + \_\_\_ x^2y^2 + \_\_\_ xy^3 + 1Y^4$

2. $(X + Y)^5 = \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_$

This is called **Pascal's Triangle**, named after the French mathematician **Blaise Pascal** (1623-1662). Notice the pattern of **1**s going down both sides of the "triangle." This pattern of **1**s continues as the triangle continues for higher powers of $(X + Y)^n$. The **numbers** inside the triangle can be obtained from the previous row, by adding the two numbers that are circled above the number. For examples, **1 + 1 = 2; 1 + 2 = 3;** and **2 + 1 = 3.** Now, complete the numbers in the $(X + Y)^4$ row: **1 + 3 =** ____; **3 + 3 =** ____; and **3 + 1 =** ____. Now, notice that moving from left to right on the "triangle," the power of X begins with the power of the binomial, and it decreases with each term, left to right. At the same time, the power of Y begins with "no Ys" in the first term, and, moving left to right, the Y power increases until it reaches the power of the binomial in the last term of the expansion. If you have not done so, complete the pattern of numbers above for $(X + Y)^5$.

Now complete the following, by continuing the triangle from the previous page:

$(X + Y)^0 = 1$

$(X + Y)^1 = 1X + 1Y$

$(X + Y)^2 = 1X^2 + 2XY + 1Y^2$

$(X + Y)^3 = 1X^3 + 3X^2Y + 3XY^2 + 1Y^3$

1. $(X + Y)^4 = 1X^4$ ____ ____ ____ $1Y^4$

2. $(X + Y)^5 =$ ____ ____ ____ ____ ____ ____

3. $(X + Y)^6 =$ ____ ____ ____ ____ ____ ____ ____

4. $(X + Y)^7 =$ ____ ____ ____ ____ ____ ____ ____ ____

5. $(X + Y)^8 =$ ____ ____ ____ ____ ____ ____ ____ ____ ____

What do you think would be the effect of having **$(X - Y)^n$?**

Answer: Instead of having **Y raised to a variety of powers**, you will now have **(-Y) raised to a variety of powers.** The **net effect of the $(X - Y)^n$ is that the signs will alternate.** For examples,

$(X - Y)^2 = X^2 - 2XY + Y^2$; $(X - Y)^3 = X^3 - 3X^2Y + 3XY^2 - Y^3$.

Now, use the numbers from the previous exercises and alternate the signs to complete the following:

6. $(X - Y)^4 =$ ____ ____ ____ ____ ____

7. $(X - Y)^5 =$ ____ ____ ____ ____ ____ ____

8. $(X - Y)^6 =$ ____ ____ ____ ____ ____ ____ ____

ANSWERS 1.08

p. 98-100:

1. $X^2+7X+12$; **2.** $Y^2+7Y+12$; **3.** $\$^2+7\$+12$; **4.** $\pi^2+7\pi+12$;
5. $(Junk)^2+7(Junk)+12$; **6.** $(Junk)^2+13(Junk)+42$; **7.** $(Junk)^2+Junk-12$;
8. $(Junk)^2-Junk-12$; **9.** $(Junk)^2-13(Junk)+42$;
10. $X^2+2XY+Y^2-7X-7Y+12$; **11.** $X^2+2XY+Y^2+3X+3Y-28$;
12. $4X^2+4XY+Y^2-6X-3Y-28$; **13.** $X^2-6XY+9Y^2-14X+42Y+48$;
14. $4X^2-12XY+9Y^{2+}8X-12Y-32$; **15.** $9X^2-12XY+4Y^2-12X+8Y-32$;
16. $X^2+2XY+Y^2+8X+8Y+16$; **17.** $X^2+2XY+Y^2-8X-8Y+16$;
18. $9X^2-30XY+25Y^2+24X-40Y+16$; **19.** $4X^2-20XY+25Y^2-12X+30Y+9$;
20. $25X^2-20XY+4Y^2-80X+32Y+64$; **21.** $25X^2+40XY+16Y^2-60X-48Y+36$;
22. $9X^2-30XY+25Y^2-16$; **23.** $4X^2-20XY+25Y^2-9$; **24.** $4X^2+28XY+49Y^2-25$;
25. $9X^2+48XY+64Y^2-49$; **26.** $X^3+3X^2Y+3XY^2+Y^3$.

p.101-102:

1. $X^4+4X^3Y+6X^2Y^2+4XY^3+Y^4$ 2. $X^5+5X^4Y+10X^3Y^2+10X^2Y^3+5XY^4+Y^5$;
3. $X^6+6X^5Y+15X^4Y^2+20X^3Y^3+15X^2Y^4+6XY^5+Y^6$;
4. $X^7+7X^6Y+21X^5Y^2+35X^4Y^3+35X^3Y^4+21X^2Y^5+7XY^6+Y^7$;
5. $X^8+8X^7Y+28X^6Y^2+56X^5Y^3+70X^4Y^4+56X^3Y^5+28X^2Y^6+8XY^7+Y^8$;
6. $X^4-4X^3Y+6X^2Y^2-4XY^3+Y^4$ 7. $X^5-5X^4Y+10X^3Y^2-10X^2Y^3+5XY^4-Y^5$;
8. $X^6-6X^5Y+15X^4Y^2-20X^3Y^3+15X^2Y^4-6XY^5+Y^6$;